On a Combinatorial Problem of Existing Matchings

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Abstract

In this paper we prove a classic combinatorial result on matchings.

Theorem. Suppose that, in a class, any boy knows at least 1 girl, and there are n students in total. Prove that there exists a group of at least $\frac{n}{2}$ students such that any boy in this group knows an odd number of girls in the same group. *(a classic problem from 1999)*

Proof. Let B be the set of boys and let G be the set of girls. Number the boys from 1 to $|B|$, and let β_i be the number of girls that the boy numbered i knows. Now we consider N, the number of distinct pairs (b, S), where b is some boy and $S \subseteq G$ is a subset of the girls such that b knows an odd number of girls in S. We count N in two different ways.

Fix some boy b, and suppose that his number is k, so that he knows β_k girls. We find the number of was to construct S. Let us write

$$S = S_1 \cup S_2,$$

where S_1 contains exactly the girls that b knows and S_2 contains exactly those that he does not know.

Since S must contain an odd number of girls that b knows, we must have that $|S_1|$ is odd. There are β_k girls that b knows, giving

$$\sum_{1 \le i \le \beta_i, 2 \nmid i} \binom{\beta_k}{i} = 2^{\beta_k - 1}$$

ways to construct S_1 with odd cardinality. Note that the above equality is due to a well-known combinatorial identity.

In addition, b can pick any girls that he does not know to be in S in order to construct S_2. Note that there are $|G| - \beta_k$ such girls, and since b has no restriction on how he can choose such a subset, he has a total of $2^{|G| - \beta_k}$ options. Thus b has $2^{\beta_k - 1}$ choices for S_1 and $2^{|G| - \beta_k}$ options for S_2; since S_1 and S_2 are disjoint sets, we have a total of

$$(2^{\beta_k - 1})(2^{|G| - \beta_k}) = 2^{|G| - 1}$$

ways to construct $S = S_1 \cup S_2$.

It follows that there are $2^{|G|-1}$ pairs (b, S) for some fixed b and associated k. Since k ranges across $[|B|]$, so that there are $|B|$ boys in total, we sum the above to get

$$\begin{aligned} N &= \sum_{i=1}^{|B|} 2^{|G|-1} \\ &= |B| 2^{|G|-1}. \end{aligned}$$

Now we count the same N, but this time by fixing $S \subseteq G$. For this S, let σ_S equal the number of boys knowing an odd number of girls in S. Clearly, it follows that there are σ_S options for a pair (b, S) of the desired form, where S is fixed. Therefore,

$$N = \sum_{S \subseteq G} \sigma_S.$$

Now, note that

$$\begin{aligned}\sum_{S \subseteq G} |S| &= \sum_{i=0}^{|G|} i \binom{|G|}{i} \\ &= |G| 2^{|G|-1},\end{aligned}$$

where the last line is due to another well-known combinatorial identity. Hence, putting everything together, we get

$$\begin{aligned}\sum_{S \subseteq G} (\sigma_S + |S|) &= \sum_{S \subseteq G} \sigma_S + \sum_{S \subseteq G} |S| \\ &= N + |G| 2^{|G|-1} \\ &= |B| 2^{|G|-1} + |G| 2^{|G|-1} \\ &= (|B| + |G|) 2^{|G|-1} \\ &= n 2^{|G|-1},\end{aligned}$$

since $|B| + |G| = n$ by definition.

Since our sum runs across all $2^{|G|}$ subsets of G, using the Pigeonhole Principle we find that there must exist some subset T such that

$$\begin{aligned}\sigma_T + |T| &\geq \frac{\sum_{S \subseteq G} (\sigma_S + |S|)}{2^{|G|}} \\ &= \frac{n 2^{|G|-1}}{2^{|G|}} \\ &= \frac{n}{2}.\end{aligned}$$

This implies that, for this particular choice of $T \subseteq G$, there must exist σ_T boys such that these $|T|$ girls and σ_T boys form a group with size at least $\frac{n}{2}$ such that each boy in this group knows an odd number of girls in the same group, by definition of σ_T. But this is what we wanted to prove, so we are done. ■