FOUNDATIONS OF DIFFERENTIAL CALCULUS: DOES THE THEORY SATISFY THE CRITERION OF TRUTH?

Temur Z. Kalanov
Home of Physical Problems, Yozuvchilar (Pisatelskaya) 6a,
100128 Tashkent, Uzbekistan
tzk_uz@yahoo.com, t.z.kalanov@mail.ru, t.z.kalanov@rambler.ru

Abstract.
A detailed proof of the incorrectness of the foundations of the differential calculus is proposed. The correct methodological basis for the proof is the unity of formal logic and rational dialectics. The unity of formal logic and rational dialectics is the only correct criterion of truth. The proof leads to the following irrefutable statement: differential calculus represents an incorrect theory in mathematics and physics. The proof of this statement is based on the following irrefutable results: (1) the standard theory of infinitesimals and the theory of limits underlying the differential calculus are incorrect theories. The concepts of "infinitesimal quantity", "movement", "process of tendency", and "limit of tendency" are meaningless concepts in mathematics; (2) the concepts of "increment of argument" and "increment of function" as the starting point of the differential calculus are not defined correctly; (3) the definition of the derivative of a function is an incorrect because the following logical contradiction arises: the increment of the argument is both not equal to zero and equal to zero; (4) the differentials of the argument and the function - as infinitesimal quantities - do not take on numerical values. This means that the differentials of quantities have neither quantitative nor qualitative determinacy; (5) the definition of the total differential of a function of two (many) variables does not satisfy the formal-logical law of the lack (absence) of contradiction; (6) the theory of proportions completely refutes the theory of differential calculus.

Thus, differential calculus does not satisfy the criterion of truth and is not correct scientific (mathematical) theory.

Keywords: general mathematics, foundations of mathematics, differential calculus, integral calculus, methodology of mathematics, philosophy of mathematics, philosophy and mathematics, mathematics education, logic, physics.

MSC: 00A30, 00A35, 03A05, 00A05, 00A69, 00A79, 97I40, 97I50, 97E20, 97E30, 97A99.

Introduction

As is known, differential and integral calculus is a fundamental mathematical theory created and developed by outstanding scientists [1-5]. Differential and integral calculus is based on the consideration of variables and operations on increments of quantities. The central point of the differential calculus is the statement of the existence of a derivative and a differential of a function. The definition of the derivative and differential of a function is based on the theory of limits and the theory of infinitesimal variables [6, 7].

According to the standard definition, the derivative of function is the limit of the ratio of the increment of the function to the increment of the argument under the condition that the increment of the argument tends (moves) to zero [7]. But since the mathematical formalism does not contain any movement (process), then, in practice (in practical training), the process (condition) of the movement of the increment to zero means that the infinitesimal increment of the argument is zero at the end of the process of change.

As a result, the following logical contradiction arises: the increment of the argument is both not equal to zero and equal to zero in the definition and calculation of the derivative. Therefore, the existence of this contradiction leads to the conclusion that the differential and integral calculus is an incorrect theory in mathematics [8-21].

The essence of differential calculus can only be analyzed and understood on the basis of the method of proportions. By definition, a proportion is a linear relationship between the increment of a function and the increment of its argument under the condition that the increment of the argument is not equal to zero.

The purpose of this work is to propose a detailed and irrefutable proof of incorrectness of the basic assertions of the differential calculus. The correct methodological basis for the proof is the unity of formal logic and rational dialectics. The unity of formal logic and rational dialectics is the correct criterion of truth. The mathematical formalism used is the method of proportions. The laws of formal logic used are the law of identity and the law of the lack (absence) of contradiction. The category of rational dialectics used is the concept of measure: measure is a concept designating the dialectical unity of the qualitative and quantitative determinacy of an object. The principle of unity of qualitative and quantitative determinacy of an object is the following statement in mathematics: both sides of a mathematical (quantitative) relationship must have identical qualitative determinacy and belong to the same object.

1. The starting point of the correct theory of variable quantities

1) From the point of view of formal logic, the concept of variable quantity is following.

(a) A concrete quantity designates the essence (essential feature) of a material object and represents the unity of the qualitative and quantitative determinacy (i.e. measure) of the object. The quantitative determinacy of an object is characterized and expressed by numbers that have a dimension (qualitative determinacy). Numbers are the result of measurement of a concrete quantity. Therefore numbers are constant numbers. The numbers are neutral real numbers. Numbers as a result of a measurement are permissible (admissible) values for a concrete quantity.

(b) Number is a numerical measure, a numerical determinacy, a numerical characteristic of a quantity in mathematics. A quantity is called a variable if this quantity can take on different numerical values. The set of numerical values of a variable is called the region of permissible (admissible) values of the variable. Different permissible (admissible) values of a given variable can be compared with each other using mathematical symbols of comparison. From the point of view of formal logic, numbers that belong to the region of permissible (admissible) values of a quantity cannot be compared with numbers that do not belong to the region of permissible (admissible) values of this quantity.

(c) A variable is designated in mathematical analysis by a letter, such as x, y. The numerical values of a variable are designated as follows: $x_0, x_1, x_2, ...$ and $y_0, y_1, y_2, ...$. The numbers that belong to the range of permissible values of a variable can be compared with each other and ordered, for example, as follows: $0 < x_0 < x_1 < x_2, ...,$ $0 < y_0 < y_1 < y_2, ...$. The number 0 is the reference point (starting point, initial point) for numbers. The number 0 is denominate (concrete) number; it has dimension: for example, $0\,kg, 0\,m, 0\,s$. The relationship between the values of a variable quantity x and the values of a constant quantity is as follows:

$$0 \leq x \leq x_0; \;\; x_0 \leq x \leq x_1; \;\; x_1 \leq x \leq x_2; \; ... \; ,$$

$$0 \leq y \leq y_0; \;\; y_0 \leq y \leq y_1; \;\; y_1 \leq y \leq y_2; \; ... \; .$$

The difference between the values of a variable and the value of a constant is determined by the condition of the problem and is a conditional difference, because the numerical values of the quantities are constant numbers. In accordance with practice, there are only different (various) constant numbers. The difference between the values of a variable is the difference between different constant numbers. Variable (non-constant) numbers do not exist in practice.

(d) A variable quantity can continuously possess the various permissible numerical values in a process realized, for example, in a computer. These different values of a variable are different constant numbers, but not variable numbers. If the process of quantitative change of the quantity has an end, then the process of quantitative change of the quantity can be illustrated by the following scheme: $x \to a$ where $a = const$ is the numerical value that the variable x takes on at the end of the process of quantitative change. The symbol "$\to$" replaces the words "tendency process; process of movement". This symbol is not a symbol of a mathematical operation, because the mathematical formalism does not contain the process of change of numbers. In other words, in mathematics, the symbol "$\to$" means the mental process of the imaginary tendency (movement) of the numerical values of a variable to a constant number which corresponds to the end of the process. From a mathematical and practical point of view, the value $x = a$ is essential value in the numerical sequence $x \to a$, and the values $x \neq a$ are non-essential value in the numerical sequence $x \to a$.

(e) From the standard point of view, the symbolic expression $\lim_{x \to a} x = a$ means that $x = a$ *under* $x \to a$ where the constant number $x = a$ is the only final value of the variable in a sequence of permissible numerical values (i.e., in a permissible numerical sequence). If the process of change of the quantity x is not completed (finished), then the continuing process is designated as follows: $x \to a,\ \lim_{x \to a} x$. But symbolic expressions $x \to a,\ \lim_{x \to a} x$ have no mathematical meaning. In terms of formal logic, this means that one must use the mathematical expression $x = a$ instead of the non-mathematical expression $\lim_{x \to a} x = a$.

(f) According to the standard definition [7], a variable quantity x is called infinitesimal quantity if the variable quantity x is an infinitely decreasing quantity over time t. The term "infinitely" is essential and designates the process $\lim_{x \to 0} x$ that is characterized by duration (by time). The logical formulation of this condition is the following: $x \to 0$, $\Pr ocess = \lim_{x \to 0} x$.

If the process of decreasing the quantity x is not finished (completed) (i.e., if the final value of the quantity x is not reached), then this is designated as follows: $x \to 0$, $\Pr ocess = \lim_{x \to 0} x$ where $\lim_{x \to 0} x$ does not possess numerical values. If the process of decreasing the quantity x is finished (completed) (i.e., if the final value is reached), then this is designated as follows: $x \to 0$, $\lim_{x \to 0} x = 0$ where the constant number 0 is the only final (boundary, limit) value of the infinitesimal (decreasing) quantity.

If $x = 0$, then the following formal-logical contradiction arises: the number 0 is both the value of the infinitesimal (decreasing) quantity x, and a constant number. But, in accordance with practice, the number 0 is a constant number. Therefore, no constant quantity is an infinitesimal quantity. No number (for example, the constant number 0) is the value of an infinitesimal quantity. For example, numbers $0{,}1;\ 0{,}01;\ 0{,}001;\ 0{,}0001$ are not values of an infinitesimal quantity. Variable numbers do not exist in practice. Consequently, an infinitesimal (decreasing) quantity does not exist in practice.

g) According to the standard definition [6, 7], a variable quantity x is called an infinitesimal quantity (i.e., an infinitely decreasing quantity) if the condition $|x| < \varepsilon$ is satisfied where ε is any arbitrarily small positive constant number. But, in this case, a formal-logical contradiction arises. Really, the permissible values of a variable quantity are the set of constant numbers.

If the values of an infinitesimal (an infinitely decreasing) variable quantity are not constant numbers, then the relationship $|x| < \varepsilon$ means the following formal-logical contradiction:

"A non-constant number $|x|$ is a constant number ε".

In other words, if the values of an infinitesimal quantity $|x|$ are not constant numbers, then the relationship $|x| < \varepsilon$ and modulus $|x|$ have no mathematical (quantitative) meaning. The relationship $|x| < \varepsilon$ has a mathematical (quantitative) meaning if the values of the quantity $|x|$ are constant numbers. If the values of an infinitesimal quantity $|x|$ are not constant numbers (i.e., if the process of change has not reached the limit 0), then the infinitesimal variable $|x|$ does not take on numerical values. If an infinitesimal quantity $|x|$ does not take on numerical values, then an infinitesimal variable does not exist in mathematics.

In addition, the standard statement that the number 0 is both a constant number and the value of infinitesimal quantity represents a formal-logical contradiction. Really, according to practice, the number 0 is a constant number. According to formal logic, the constant number 0 is not "a non-constant number". Consequently, an infinitesimal quantity $|x|$ cannot take on the value 0: the constant number 0 is not a permissible value for an infinitesimal quantity $|x|$; the concept of "infinitesimal quantity" is destroyed if $|x| = 0$. This means that an infinitesimal quantity cannot exist in mathematics because an infinitesimal quantity $|x|$ cannot take on the value 0.

For example, if constant numbers $0{,}1;\ 0{,}01;\ 0{,}001;\ 0{,}0001$ are not the values of an infinitesimal (an infinitely decreasing) quantity x, then the process $0{,}1 \to 0{,}01 \to 0{,}001 \to 0{,}0001 \to ... \to 0$ represents an essential feature of the concept of "infinitesimal (infinitely decreasing) quantity x". If the process is interrupted at times $t_1,\ t_2,\ t_3,\ t_4$, then the corresponding constant numbers $0{,}1;\ 0{,}01;\ 0{,}001;\ 0{,}0001$ become permissible values of the variable quantity x. The infinitesimal quantity x turns into the variable quantity x. In this case, the concept of "infinitesimal (infinitely decreasing) quantity x" is destroyed (i.e., this concept is exterminated) at times $t_1,\ t_2,\ t_3,\ t_4$. This means that an infinitesimal quantity x does not take on numerical values if the process $0{,}1 \to 0{,}01 \to 0{,}001 \to 0{,}0001 \to ...$ is continued.

Thus, the numerical values of a variable quantity are constant numbers. The standard concepts of "infinitesimal quantity x" and "infinitely large quantity $1/x$" are erroneous concepts because the infinitesimal quantity x and the infinitely large quantity $1/x$ do not take on numerical values. Comparison of infinitesimal quantities and comparison of infinitely large quantities are meaningless operations.

2) Correct definition of increments of variable quantities

(a) If x is a variable that takes on numerical values $0,\ x_0,\ x_1,\ x_2,\ ...$ in the region of permissible neutral real numbers, then the relationships $\Delta_{1,0}x = x_1 - x_0,\ \Delta_{2,1}x = x_2 - x_1,\ \dots$ define the increments of the numerical values of the quantities $x_0,\ x_1,\ \dots$, respectively. The increments $\Delta_{\text{var},0}\,x = x - x_0$, $\Delta_{\text{var},1}\,x = x - x_1$, $\Delta_{\text{var},2}\,x = x - x_2$, … of the numbers x_0, x_1, x_2, … are variables if x is a variable. The following statements are true:

$$\Delta x = x_1 - x_0 = (nonsense), \quad \Delta x = x_2 - x_1 = (nonsense),$$
$$\Delta x = x - x = (nonsense), \quad x + \Delta x = (nonsense).$$

(b) By definition, a proportion is a linear relationship between the increment of a function and the increment of its argument, provided that the increment of the argument is not zero. If $y = f(x)$ is an function of argument x, then the definition of the increment of the function is the following proportion:

$$\frac{f(x) - f(x_1)}{f(x_1)} = \frac{x - x_1}{x_1}, \quad \frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x_1)}{x_1}$$

under the condition $\Delta_{\text{var},1}\,x = x - x_1 \neq 0$

or in the following designations (notations):

$$\Delta_{\text{var},1}\,f(x) = \frac{f(x_1)}{x_1}\,\Delta_{\text{var},1}\,x, \quad \Delta_{\text{var},1}\,f(x) \equiv f(x) - f(x_1), \quad \frac{\Delta_{\text{var},1}\,f(x)}{\Delta_{\text{var},1}\,x} = \frac{f(x_1)}{x_1},$$

under the condition $\Delta_{\text{var},1}\,x \neq 0$.

(c) The geometric meaning (interpretation) of the proportion is as follows. The proportion is a linear approximation of the function $y = f(x)$. Therefore, the approximation of the graph of the function $y = f(x)$ represents a broken line segments in the coordinate system XOY (Figure 1):

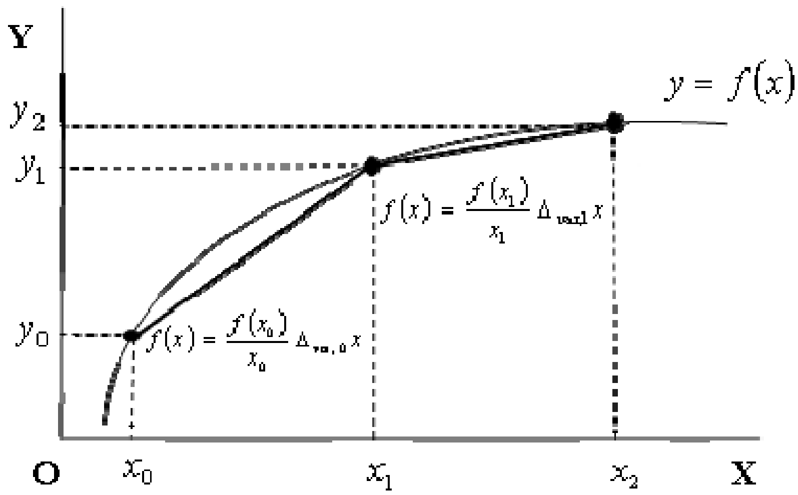

Figure 1. Geometric interpretation of proportion. The broken line, obtained using the method of proportion, approximates the graph of the function $y = f(x)$. The quantities $f(x_1) = \frac{f(x_0)}{x_0}\Delta_{1,0}x$ and $f(x_2) = \frac{f(x_1)}{x_1}\Delta_{2,1}x$ determine the positions of the vertices of the broken line in the coordinate system XOY.

<u>*Remark*</u>: The quantity $\frac{\Delta_{\text{var},1} f(x)}{\Delta_{\text{var},1} x} = \frac{f(x_1)}{x_1}$ is a constant. The quantity $\frac{\Delta_{\text{var},1} y}{\Delta_{\text{var},1} x}$ is not a definition of the quantity of any angle, because one did not construct a right-angled triangle.

(d) If $\varphi[f(x)]$ is a function of the function $f(x)$, then the proportion has the following form:

$$\frac{\varphi[f(x)] - \varphi[f(x_1)]}{\varphi[f(x_1)]} = \frac{f(x) - f(x_1)}{f(x_1)}$$

<u>*Explanation*</u>: (*) In physics, a function $f(x)$ can represent physical quantities: distance, speed, acceleration, etc. (**) Higher-order increments (i.e., increments of increments) do not exist because the quantity $\frac{\Delta_{\text{var},1} f(x)}{\Delta_{\text{var},1} x} = \frac{f(x_1)}{x_1}$ is a constant.

(e) If $f(x)$ is an unknown function, then the relationship $f(x) = f(x_1) + \Delta_{\text{var},1} f(x)$ is equation in $f(x)$; quantities x_1 and $f(x_1)$ are the boundary conditions. This equation cannot contain additional terms. If the equation contained additional terms, then such an equation would contradict to the proportion

$$\frac{\Delta_{\text{var},1} f(x)}{\Delta_{\text{var},1} x} = \frac{f(x_1)}{x_1}.$$

(f) In the case of a function $f(x, y)$ of two mutually independent variables x and y, the partial increments of the function are the following proportions:

$$\Delta_{\text{var},1} f(x, y_1) \equiv f(x, y_1) - f(x_1, y_1), \quad \frac{\Delta_{\text{var},1} f(x, y_1)}{\Delta_{\text{var},1} x} = \frac{f(x_1, y_1)}{x_1}, \quad \Delta_{\text{var},1} x \neq 0;$$

$$\Delta_{\text{var},1} f(x_1, y) \equiv f(x_1, y) - f(x_1, y_1), \quad \frac{\Delta_{\text{var},1} f(x_1, y)}{\Delta_{\text{var},1} y} = \frac{f(x_1, y_1)}{y_1}, \quad \Delta_{\text{var},1} y \neq 0.$$

The expression $\Delta_{\text{var},1} f(x, y_1) \equiv f(x, y_1) - f(x_1, y_1)$ means that the partial increment $\Delta_{\text{var},1} f(x, y_1)$ is an ordinary increment where $y_1 = const$ is a parameter. Also, the expression $\Delta_{\text{var},1} f(x_1, y) \equiv f(x_1, y) - f(x_1, y_1)$ means that the partial increment $\Delta_{\text{var},1} f(x_1, y)$ is an ordinary increment where $x_1 = const$ is a parameter.

(g) If the definition of a total increment of the function $f(x, y)$ is an expression $\Delta_{var,1} f(x, y) \equiv \Delta_{var,1} f(x, y_1) + \Delta_{var,1} f(x_1, y)$, then this expression contains the following formal-logical contradiction:

"the variable quantity x is a constant",
"the constant x_1 is a variable quantity";
"the variable quantity y is a constant",
"the constant y_1 is a variable quantity".

That is, in the definition of the total increment of a function, quantity x is both a variable and a constant; quantity y is both a variable and a constant. This fact means a violation of the formal-logical law of the lack (absence) of contradiction. The law of the lack (absence) of contradiction states the following:

(variable quantity) $\neq$ (constant).

2. Critical analysis of the starting point of the differential calculus

As is known [8-21], the differential calculus is based on the following contradictory definitions:

$$\frac{\Delta y}{\Delta x} \equiv \frac{f(x + \Delta x)}{\Delta x} \quad \text{under} \quad \Delta x \neq 0,$$

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{under} \quad \lim_{\Delta x \to 0} \Delta x = 0,$$

$$\frac{dy}{dx} \equiv y', \quad dy = y' dx$$

where the increment Δx is not defined; $\lim_{\Delta x \to 0} \Delta x = 0$, $\Delta x = 0$ in practical calculations, practical applications.

(a) The first objection is that the variable increment Δx is not defined. The correct definition of the quantity Δx has the following form: $\Delta x \equiv x - x = 0$.

(b) The second objection is the following:

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \neq 0}.$$

This error has the form $0 \neq \Delta x = 0$. It represents a violation of the formal-logical law of identity and the law of the lack (absence) of contradiction.

Example.

If the function $y = 3x^2 + 5$ is given, then the standard calculation of the derivative is performed as follows:

$$y + \Delta y = 3x^2 + 6x \cdot \Delta x + 3(\Delta x)^2 + 5.$$

$$y + \Delta y - y = \Delta y = 6x \cdot \Delta x + 3(\Delta x)^2,$$

$$\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x \neq 0} = [6x + 3 \cdot \Delta x]_{\Delta x \neq 0},$$

$$\frac{dy}{dx} \equiv \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x \neq 0} = \lim_{\Delta x \to 0}[6x + 3 \cdot \Delta x]_{\Delta x \neq 0} = [6x + 0]_{\Delta x \neq 0}.$$

Thus, the formal-logical error has the form $0 \neq \Delta x = 0$.

(*Remark*. Formal-logical errors in mathematics arise, particularly, because mathematicians reason as follows: "First we suppose (assume) that $\Delta x \neq 0$. Thereafter, we suppose (assume) that $\Delta x = 0$ in the same expression". Such a fallacious (vicious) way of reasoning leads to a logical error: the quantity Δx is both $\Delta x \neq 0$ and $\Delta x = 0$ in the same expression).

(c) The third objection is the following definitions as a consequence of the theory of infinitesimal quantities:

$$\left(\frac{dy}{dx}\right)_{(not\ fraction)} \equiv \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)_{(fraction)} \neq \frac{\lim_{\Delta x \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x};$$

$$\left(\frac{dy}{dx}\right)_{(fraction)} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)_{(fraction)} = \frac{\lim_{\Delta x \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x},$$

$$\left(\frac{dy}{dx}\right)_{(fraction)} \equiv y', \ dy = y'dx, \ dx = \lim_{\Delta x \to 0} \Delta x, \ dy = \lim_{\Delta x \to 0} \Delta y.$$

$$\left(\frac{dy}{dx}\right)_{(not\ fraction)} = \left(\frac{dy}{dx}\right)_{(fraction)},$$

where infinitesimal quantities dx and dy do not reach the limit 0.

Really, relationship $\left(\frac{dy}{dx}\right)_{(not\ fraction)} = \left(\frac{dy}{dx}\right)_{(fraction)}$ is a violation of the formal-logical law of the lack (absence) of contradiction. The law of the lack (absence) of contradiction has the following form:

$$\left(\frac{dy}{dx}\right)_{(not\ fraction)} \neq \left(\frac{dy}{dx}\right)_{(fraction)}.$$

(d) The fourth objection is the following. As was shown above, the theory of infinitesimal quantities is a formal-logical error. Therefore, the following relationships follow from the standard definition of an infinitesimal (infinitely decreasing) quantity: $dx = \lim_{\Delta x \to 0} \Delta x$, $dy = \lim_{\Delta x \to 0} \Delta y$ where dx, dy and $\frac{dy}{dx}$ are infinitesimal quantities. If infinitesimal quantities reach the limit 0, then the standard relationships have the following form:

$$\left(\frac{dy}{dx}\right)_{(not\ fraction)} \equiv \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)_{(not\ fraction)} = 0;$$

$$\left(\frac{dy}{dx}\right)_{(fraction)} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)_{(fraction)} = \frac{\lim\limits_{\Delta x \to 0} \Delta y}{\lim\limits_{\Delta x \to 0} \Delta x} = \frac{0}{0};$$

$$dx = \lim_{\Delta x \to 0} \Delta x = 0, \; dy = \lim_{\Delta x \to 0} \Delta y = 0.$$

In this case, the formal-logical contradiction

$$\left(\frac{dy}{dx}\right)_{(not\ fraction)} = \left(\frac{dy}{dx}\right)_{(fraction)}$$

has the following numerical form: $0 = \frac{0}{0}$.

Consequently, infinitesimal (infinitely decreasing) quantities dx, dy, $\frac{dy}{dx}$ do not possess the numerical values if the infinitesimal quantities do not reach the limit 0. If infinitesimal quantities reach the limit 0, then the standard relationships have the following form: $0 = \frac{0}{0}$.

(e) The fifth objection is the physical and geometric interpretations of the derivative.

(e_1) The physical interpretation of the derivative is the following. If the function $S = f(t)$ has the concrete form $S^{(M)} = V^{(M)} \cdot t$ (where the path length $S^{(M)}$ of the material point M has the dimension "meter", the time t has the dimension "second", the speed $V^{(M)}$ of the material point M has the dimension "meter/second"), then the derivative is $\frac{dS^{(M)}}{dt} = V^{(M)}$. But this derivative does not take on numerical values, because the infinitesimal quantities $dS^{(M)}$ and dt do not take on numerical values (i.e., the quantities $dS^{(M)}$ and dt do not have quantitative determinacy, measure, metric). Therefore, in accordance with the dialectical concept of measure, the quantities $dS^{(M)}$ and dt do not have a qualitative determinacy (dimensions). This means that the physical interpretation of the derivative $\frac{dS^{(M)}}{dt} = V^{(M)}$ is a methodological error;

(e_2) The geometric interpretation of the derivative is the following. If the standard geometric interpretation of the derivative of a function $y = f(x)$ is the relationship $tg\,\alpha = \frac{dy}{dx}$, then this relationship is a formal-logical error because the left side of the relationship $tg\,\alpha = \frac{dy}{dx}$ belongs to a right-angled triangle, and the right side of this relationship does not belong to the right-angled triangle. The proof of the incorrectness of the relationship $tg\,\alpha = \frac{dy}{dx}$ is based on the system approach (system concepts). The proof is the following.

(*) If the following material system is given (ready-built!) in Cartesian coordinates XOY (Figure 2): (1) constructed segment of the line $y = f(x)$; (2) ready-built points A and B on the segment of the line $y = f(x)$; points A and B uniquely (unambiguously) determine the constructed secant $\overline{AB}$; (3) the position of the secant $\overline{AB}$ is determined by the constructed right-angled triangle ΔABC , - then the concluded angle (interior angle) α of the right-angled triangle ΔABC is the angle formed by the secant $\overline{AB}$ and the cathetus (leg) $\overline{AC}$ of the right-angled triangle ΔABC (Figure 2).

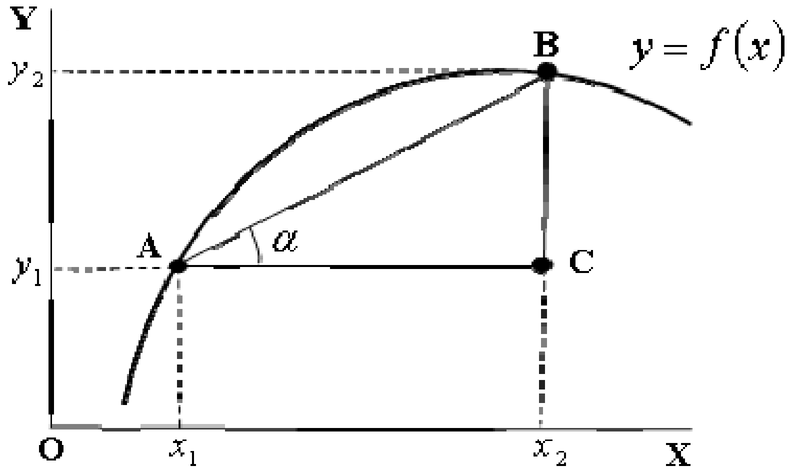

Figure 2. Material system "Segment of the line $y = f(x)$ + right-angled triangle ΔABC" in the Cartesian coordinate system XOY. The secant $\overline{AB}$ is the hypotenuse of the right-angled triangle ΔABC. Quantities x_1, x_2, y_1, y_2 are the abscissas and ordinates of the points A, B, C, respectively.

In this case, the mathematical relationship between lengths of legs of the triangle ΔABC and quantity of the angle α of the triangle ΔABC exist if the points A and B do not coincide: $\Delta x_{2,1} \equiv x_2 - x_1 \neq 0$, $\Delta y_{2,1} \equiv y_2 - y_1 \neq 0$. Then the quantity $\frac{\Delta y_{2,1}}{\Delta x_{2,1}}$ exists. Also, if the points A and B do not coincide, then length of the hypotenuse $\overline{AB}$ is not zero. But if the points A and B coincide (i.e., if length of the hypotenuse $\overline{AB}$ is zero), then the triangle ΔABC, quantity of the angle α, and the quantity $\frac{\Delta y_{2,1}}{\Delta x_{2,1}}$ do not exist.

Consequently, the relationship $tg\,\alpha = \frac{dy}{dx}$ does not exist.

(**) If the following material system is given (ready-built!) in the Cartesian coordinate system (Figure 3): (1) constructed segment of line $y = f(x)$; (2) constructed the point A on the segment of the line $y = f(x)$; points A and D uniquely (unambiguously) determine the constructed secant $\overline{DA}$; (3) the position of the secant $\overline{DA}$ is determined by the constructed right-angled triangle ΔDAE, - then the concluded (interior) angle α of the triangle ΔDAE is the angle formed by the secant $\overline{DA}$ and the cathetus $\overline{DE}$ (Figure 3).

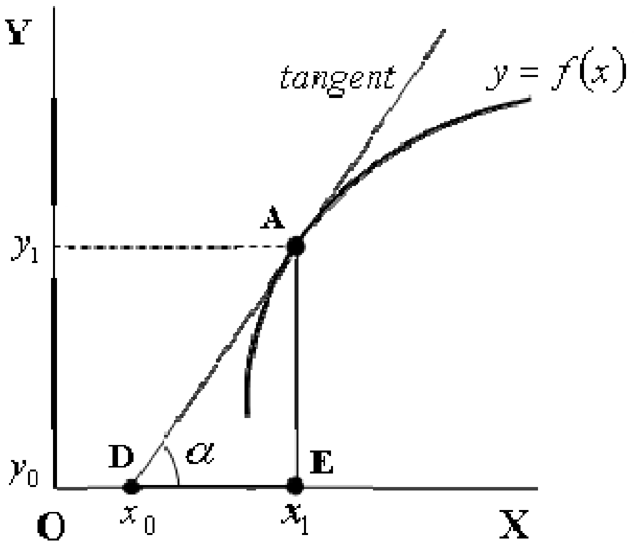

Figure 3. Material system "Segment of line $y = f(x)$ + right-angled triangle ΔDAE" in the Cartesian coordinate system XOY. The tangent $\overline{DA}$ is the hypotenuse of the right-angled triangle ΔDAE. The constants $x_0,\ x_1,\ y_0,\ y_1$ are the abscissas and ordinates of the points D, A, E. The quantities $\Delta x_{1,0} \equiv x_1 - x_0 \neq 0$ and $\Delta y_{1,0} \equiv y_1 - y_0 \neq 0$ determine the lengths of the legs of the triangle ΔDAE.

In this case, the lengths of the legs of the right-angled triangle ΔDAE are the following constants: $\Delta y_{1,0} \equiv y_1 - y_0 \neq 0$, $\Delta x_{1,0} \equiv x_1 - x_0 \neq 0$. Constants $\Delta y_{1,0} \equiv y_1 - y_0 \neq 0$, $\Delta x_{1,0} \equiv x_1 - x_0 \neq 0$ cannot be variables. Then the right-angled triangle ΔDAE and quantities $\frac{\Delta y_{1,0}}{\Delta x_{1,0}}$, $tg\,\alpha$, α exist.

The standard geometric interpretation of the derivative $\frac{dy}{dx}$ is the relationship $tg\,\alpha = \frac{dy}{dx}$. But, according to the standard trigonometric definition, $tg\,\alpha = \frac{\Delta y_{1,0}}{\Delta x_{1,0}}$ in the case of the right-angled triangle ΔDAE. This means that the standard geometric interpretation of the derivative leads to the following contradiction: $\frac{dy}{dx} = \frac{\Delta y_{1,0}}{\Delta x_{1,0}}$, i.e. $dy \equiv \Delta y_{1,0}$, $dx \equiv \Delta x_{1,0}$. This contradiction expresses the following formal-logical error: infinitesimal quantities dy and dx

are constant quantities that take on constant numerical values. Consequently, the relationship $tg\ \alpha = \frac{dy}{dx}$ is an error.

Example. To geometrically interpret a linear function $y = ax$ in the metric coordinate system XOY, one must take into consideration the following definition: the graph of the function $y = ax$ is the locus of material points in the material coordinate system XOY. In this case, the function $y = ax$ will look like (will have form) $y^{(M)} = ax^{(M)}$. The function $y^{(M)} = ax^{(M)}$ is an analytical representation of a material segment of a straight line (graph) in a system XOY. The variable quantities $x^{(M)}$ and $y^{(M)}$ are the coordinates (i.e., the segments of the coordinate scales) of the moving material point M. In other words, the graph of the function $y^{(M)} = ax^{(M)}$ is the locus of the positions of the moving point M in the metric coordinate system XOY. The quantities $x^{(M)}$ and $y^{(M)}$ have both quantitative and qualitative determinacy because they have the dimension "meter". The dimensionless constant a does not determine the quantity of any angle because one did not build a right-angled triangle in the system XOY.

Differentiation of the function $y^{(M)} = ax^{(M)}$ leads to the expression $dy^{(M)} = adx^{(M)}$. In this case, the following contradiction arises (as a formal-logical error): $a = \frac{dy^{(M)}}{dx^{(M)}} = \frac{y^{(M)}}{x^{(M)}}$. $dx^{(M)} = x^{(M)}$, $dy^{(M)} = y^{(M)}$. This formal-logical error is the assertion that "infinitesimal quantities $dx^{(M)}$ and $dy^{(M)}$ are variables $x^{(M)}$ and $y^{(M)}$". This error represent a violation of the formal-logical law of identity:

(infinitesimal variable)=
(infinitesimal variable).

Also, this contradiction is a violation of the formal-logical law of the lack (absence) of contradiction:

(infinitesimal variable) $\neq$ *(non-infinitesimal variable).*

Moreover, $dx^{(M)} \neq dx$ and $dy^{(M)} \neq dy$ because the infinitesimal quantities dx, dy and $dx^{(M)}$, $dy^{(M)}$ have neither quantitative nor qualitative determinacy. Infinitesimal quantities cannot have the index (M) because they cannot belong to the material point M.

Consequently, the relationship $a = \frac{dy}{dx}$ is an error.

From the practical point of view, the existence of the material coordinate system XOY, material points, material line segments, material figures (material triangles) and a measure of material objects negates (denies) the existence of infinitesimal quantities dx and dy.

(f) The sixth objection is the definition of the total differential of a function of several variables. Really, the expression for a total differential is the sum of partial (intermediate) differentials. The partial (intermediate) differential is equal to the product of the corresponding partial derivative and the differential of the corresponding independent variable.

For example:

$$du(x, y) = \left(\frac{\partial u}{\partial x}\right)_{y\,=\,const} dx + \left(\frac{\partial u}{\partial y}\right)_{x\,=\,const} dy\,.$$

The definition of the total differential contains contradictory conditions (statements): "$x = \text{var};\ y = const$" and "$y = \text{var};\ x = const$". Therefore, the definition of the total differential contradicts to the formal-logical law of the lack (absence) of contradiction:

$$(x = \text{var};\ y = const) \neq (y = \text{var};\ x = const).$$

Consequently, the total differential is not the sum of partial differentials.

(g) The seventh objection is the definition of the mixed derivative

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x}\,.\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)_{y\,=\,const} = 0\,.$$

Really, it follows from the expression $\left(\frac{\partial u}{\partial x}\right)_{y\,=\,const}$ that

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)_{y\,=\,const} = 0$$

because the expression $\left(\frac{\partial u}{\partial x}\right)_{y\,=\,const}$ means that $y = const$ in the expression $\left(\frac{\partial u}{\partial x}\right)_{y\,=\,const}$.

(*Note*. Formal-logical errors in mathematics arise, in particular, because mathematicians reason as follows: "First we suppose (consider, assume) that $y = const$. Then we suppose (consider, assume) that $y = \text{var}$ in the same expression". Such a vicious way of reasoning leads to a gross logical error: the quantity y is both $y = const$ and $y = \text{var}$ in the same expression).

(h) The eighth objection is that the symbols "d" and "$\int$" are interpreted as "birth operator" and "destruction (annihilation) operator" of a differential (an infinitesimal quantity):

$$x = x, \quad dx = dx, \quad \int dx = \int dx, \quad x = \int dx\,.$$

Discussion

Thus, the differential and integral calculus, created by eminent scientists, is an erroneous mathematical theory. Moreover, as shown in my papers [8-26], pure mathematics, standard trigonometry, complex number theory, and vector calculus also represent errors in mathematics. Why did the classics of science make errors?

As the history of science shows, outstanding mathematicians and theoretical physicists relied on their intuition (fantasy), but not on a correct methodological basis. They could not find

Therefore, scientists could not correctly, rationally think and create within the framework of the correct methodological basis (the criterion of truth). They could not critically analyze scientific works (papers) because they did not have good sense. (Good sense relies on practice!)

Eminent scientists jumped over the obscure (unclear, doubtful) places of theories because they could not critically analyze the ambiguities, vagueness in detail. Therefore, ambiguities (unclear, doubtful places) remained in their theories. As a result, for example, the theory of relativity, pure mathematics, standard trigonometry, complex number theory, and vector calculus arose, which contain ambiguities (unclear, doubtful places) and errors.

Also, outstanding scientists introduced the idea of mechanical motion into mathematics: the theory of variables, the theory of limits, the theory of infinitesimal and infinite quantities, differential and integral calculus arose. These theories are based on unawareness, incomprehension, lack of understanding that the mathematical formalism does not contain movements (actions).

Actions, operations on mathematical symbols and numbers are performed by people. Mathematicians have not understood that a detailed symbolic designation (definition) of quantities is a requirement of formal logic and rational dialectics. (This requirement is expressed in the necessary condition that a mathematical quantity must represents the unity of qualitative and quantitative determinacy).

For example, correct definitions (designations, notations) of increments are expressions $\Delta_{1,0} x = x_1 - x_0$, $\Delta_{2,1} x = x_2 - x_1$, …, which define increments of numerical values of quantities x_0, x_1, …, respectively. But if one simplifies the designations (definitions) and writes $\Delta x = x_1 - x_0$, $\Delta x = x_2 - x_1$, $\Delta x = x - x$, $x + \Delta x$, then one gets fundamental nonsense.

From this point of view, the differential calculus is based on this nonsense. Another example is the following. Mathematicians have not understood that the vector (i.e., the property of the interaction of material objects) is a physical concept, not a mathematical concept. Therefore, a vector cannot be drawn (i.e., the vector cannot exist) in the geometric coordinate system. That is why, in my works [8, 26], the following statements are proven:

(a) the numbers are neutral numbers; positive and negative numbers do not exist;

(b) pure mathematics, standard trigonometry, complex number theory, and vector calculus represent gross errors.

Thus, mathematicians and physicists did not understand that differentials of variables do not have numerical (quantitative) determinacy. Therefore, differentials of variables do not have dimensions (qualitative determinacy). This means that differential and integral calculus have no scientific and practical meaning.

My 40-year experience of critical analysis of the foundations of theoretical physics and mathematics shows that delusions and errors in science cannot be exterminated, eliminated, abolished. Scientific lie and scientific truth form an inseparable unity (the unity of opposites). This unity is the essence of the inductive way of cognition and development of Mankind.

Conclusion

Thus, the critical analysis of the foundations of differential calculus within the framework of the correct methodological basis leads to the following statement: differential calculus represents a error in mathematics and physics. The proof of this statement is based on the following results:

(1) The standard theory of infinitesimals and the theory of limits underlying the differential calculus are errors. The main error is that infinitesimal (infinitely decreasing) quantities do not take on numerical values in the process of tending to zero. The number “zero” is not a permissible value of infinitesimal quantity. The concepts of “infinitesimal quantity”,

(2) the concepts of “increment of argument” and “increment of function” are the starting point of the differential calculus. The error is that the increment of argument is not defined. An indefinite (undefined, uncertain, ambiguous, undetermined) increment of an argument is a meaningless quantity (concept);

(3) the definition of the derivative of a function is a error. The derivative is the limit of the ratio of the increment of function to the increment of argument under the following conditions: (a) the increment of argument is not equal to zero; (b) the increment of the argument tends to zero and reaches the value “zero”. In this case, the following logical contradiction arises: the increment of the argument is both not equal to zero and equal to zero;

(4) the differentials of the argument and the function - as infinitesimal quantities - do not take on numerical values. This means that the differentials of quantities have neither quantitative nor qualitative determinacy. In this case, the differentials of quantities are meaningless symbols. The geometric and physical interpretations of the derivative are errors;

(5) the definition of the total differential of a function of two (many) variables is a error because the definition contains a formal-logical contradiction, i.e. the definition as the sum of partial differentials does not satisfy the formal-logical law of the lack (absence) of contradiction;

(6) the theory of proportions completely refutes the theory of differential calculus.

References

[1] C.B. Boyer, “A history of mathematics” (Second ed.). John Wiley & Sons, Inc. ISBN 0-471-54397-7. (1991).

[2] R. Nagel (ed.), “Encyclopedia of Science”, 2nd Ed. The Gale Group. (2002).

[3] N. Bourbaki. “Foundations of Mathematics”. Springer, (1998).

[4] W.B. Ewald, B. “From Kant to Hilbert: A Source Book in the Foundations of Mathematics”. Vol. 1. Oxford University Press. ISBN 9780198505358. (2020).

[5] E. Madelung. “Die Mathematischen Hilfsmittel Des Physikers”. Berlin, Gottingen, Heidelberg: Springer-Verlag, (1957).

[6] V.I. Smirnov. Course of higher mathematics, V. 1. Moscow, Science, 1974.

[7] N.N. Luzin. Differential calculus. Moscow, Science, 1952.

[8] T.Z. Kalanov, “Critical analysis of the foundations of differential and integral calculus”. MCMS (Ada Lovelace Publications), (2011), pp. 34-40.

[9] T.Z. Kalanov, “Logical analysis of the foundations of differential and integral calculus”. Indian Journal of Science and Technology, Vol. 4, No. 12 (2011), pp. 1786-1789.

[10] T.Z. Kalanov, “Logical analysis of the foundations of differential and integral calculus”. Bulletin of Pure and Applied Sciences, Vol. 30 E (Math.& Stat.), No. 2 (2011), pp. 327-334.

[11] T.Z. Kalanov, “Critical analysis of the foundations of differential and integral calculus”. International Journal of Science and Technology, Vol. 1, No. 2 (2012), pp.80-84.

[12] T.Z. Kalanov, “On rationalization of the foundations of differential calculus”. Bulletin of Pure and Applied Sciences, Vol. 31 E (Math. & Stat.), No. 1 (2012), pp. 1-7.

[13] T.Z. Kalanov, “Critical analysis of the mathematical formalism of theoretical physics. I. Foundations of differential and integral calculus”. Bulletin of the Amer. Phys. Soc., (April Meeting), Vol. 58, No. 4 (2013).

[14] T.Z. Kalanov. “On the formal–logical analysis of the foundations of mathematics applied to problems in physics”. Aryabhatta Journal of Mathematics & Informatics, Vol. 7, No. 1 (2015), pp. 1-2.

[15] T.Z. Kalanov. “On the formal-logical analysis of the foundations of mathematics applied to problems in physics”. Bulletin of the Amer. Phys. Soc., (April Meeting), (2016).

[16] T.Z. Kalanov. "Critical analysis of the foundations of pure mathematics". Mathematics and Statistics (CRESCO, http://crescopublications.org), Vol. 2, No. 1 (2016), pp. 2-14.
[17] T.Z. Kalanov. "Critical analysis of the foundations of pure mathematics". International Journal for Research in Mathematics and Mathematical Sciences, Vol. 2, No. 2 (2016), pp. 15-33.
[18] T.Z. Kalanov. "Critical analysis of the foundations of pure mathematics". Aryabhatta Journal of Mathematics & Informatics, Vol. 8, No. 1 (2016), pp. 1-14 (Article Number: MSOA-2-005).
[19] T.Z. Kalanov. "Critical Analysis of the Foundations of Pure Mathematics". Philosophy of Mathematics Education Journal, ISSN 1465-2978 (Online). Editor: Paul Ernest), No. 30 (October 2016).
[20] T.Z. Kalanov. "On the formal–logical analysis of the foundations of mathematics applied to problems in physics". Asian Journal of Fuzzy and Applied Mathematics, Vol. 5, No. 2 (2017), pp. 48-49.
[21] T.Z. Kalanov. The critical analysis of the foundations of mathematics. Mathematics: The Art of Scientific Delusion. LAP LAMBERT Academic Publishing (2017-12-05). ISBN-10: 620208099X.
[22] T.Z. Kalanov, "Formal-logical analysis of the starting point of mathematical logic". Aryabhatta Journal of Mathematics & Informatics, Vol.13, No. 1, (2021), pp. 01-14.
[23] T.Z. Kalanov, "On the problem of axiomatization of geometry". Aryabatta Journal of Mathematics & Informatics, Vol.13, No.2, (2021), pp. 151-166.
[24] T.Z. Kalanov, "On the problem of axiomatization of geometry". Chemistry Biology and Physical Sciences Academic, 3(1), (2021), pp. 8–25.
[25] T.Z. Kalanov, "On fundamental errors in trigonometry". Bulletin of Pure and Applied Sciences (Section - E - Mathematics & Statistics),Vol. 41E, No.1, (2022), pp. 16-33.
[26] T.Z. Kalanov, "Theory of complex numbers: gross error in mathematics and physics". Bulletin of Pure and Applied Sciences (Section - E - Mathematics & Statistics), Vol. 41E, No.1, (2022), pp. 61-68.