The extension of the Riemann's zeta function

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Abstract :

Prime numbers [See 1-7] are used especially in information technology, such as public-key cryptography , and recall that the distribution of prime numbers is closely related to the non-trivial zeros of the zeta function therefore related to the Riemann hypothesis.

Here I introduce the function Ⓢ : $(X, z) \longmapsto \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$ which is a generalization of the function ζ of Riemann that I will use to prove the Riemann hypothesis.

Keywords : Prime Number,Holomorphic function, the Riemann hypothesis.

In memory of the great professor, the physicist and mathematician, Moshé Flato.

INTRODUCTION AND THE PROOF OF THE RIEMANN HYPOTHESIS

Prime numbers [See 1-7] are used especially in information technology, such as public-key cryptography which relies on factoring large numbers into their prime factors. And in abstract algebra, prime elements and prime ideals give a generalization of prime numbers.

In mathematics, the search for exact formulas giving all the prime numbers, certain families of prime numbers or the n-th prime number has generally proved to be vain, which has led to contenting oneself with approximate formulas [7].

Recall that Mills' Theorem [7] : "There exists a real number A, Mills' constant, such that, for any integer n > 0, the integer part of A^{3^n} is a prime number" was demonstrated in 1947 by mathematician William H. Mills [7], assuming the Riemann hypothesis [7] is true.

Here I introduce the function Ⓢ : $(X, z) \longmapsto \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$ which is a generalization of the function ζ of Riemann that I will use to prove the Riemann hypothesis.

Theorem 1 *The real part of every nontrivial zero of the Riemann zeta function is 1/2.*

The link between the function ζ and the prime numbers had already been established by Leonhard Euler with the formula [5], valid for $Re(s) > 1$:

$$\zeta(s) \;=\; \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\cdots}$$

where the infinite product is extended to the set $\mathcal{P}$ of prime numbers. This formula is sometimes called the Eulerian product.

And since the Dirichlet eta function can be defined by $\eta(s) = (1 - 2^{1-s})\,\zeta(s)$ where : $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$
We have in particular :

$$\zeta(z) = \frac{1}{1 - 2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

for $0 < Re(z) < 1$,
Let : $s = x + iy$, with $0 < Re(s) < 1$
$\zeta(s)\zeta(\overline{s}) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} \frac{1}{1-p^{-\overline{s}}} = \prod_{p\in\mathcal{P}} \frac{1}{(1-e^{-xln(p)}cos(yln(p)))^2 + (e^{-xln(p)}sin(yln(p)))^2}$
But : $\prod_{p\in\mathcal{P}} \frac{1}{(1-e^{-xln(p)}cos(yln(p)))^2 + (e^{-xln(p)}sin(yln(p)))^2} \geq \prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-xln(p)})^2 + (e^{-xln(p)})^2}$
If $\zeta(s) = 0$, then $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-xln(p)})^2 + (e^{-xln(p)})^2} = 0$ and since the non-trivial zeros of ζ are symmetric with respect to the line $X = \frac{1}{2}$ because the zeta function satisfies the functional equation [7] : $\zeta(s) = 2^s \pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(1 - s)\ \zeta(1 - s)$
then $x = \frac{1}{2} + \alpha$, and if $s' = \frac{1}{2} - \alpha + iy$, then $\zeta(s') = 0$
But the function $\frac{1}{(1+e^{-tln(p)})^2 + (e^{-tln(p)})^2}$ is increasing in $[0, 1]$, so $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-tln(p)})^2 + (e^{-tln(p)})^2} = 0\ \forall t \in [\frac{1}{2} - \alpha, \frac{1}{2} + \alpha]$.
As $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-zln(p)})^2 + (e^{-zln(p)})^2}$ is holomorphic : because :
$\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-zln(p)})^2 + (e^{-zln(p)})^2} = \prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z} \frac{1}{1-B/p^z}$ with $A = i - 1$ and $B = -i - 1$, and both $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ and $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ are holomorphic in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$ as we have :

$$\prod_{p\in\mathcal{P}} \frac{1}{1 - A/p^z} = \prod_{p\in\mathcal{P}} (1 + f_p(z))$$

with $f_p(z) = \frac{1}{(p^z/A) - 1}$

$$\mid f_p(z) \mid \leq \frac{1}{\mid p^z/A \mid -1} = \frac{1}{(p^{\Re(z)}/\sqrt{2}) - 1} \leq k \frac{1}{p^{\frac{1}{2}}}$$

where k is a positive real constant.

So :

$$| \sum_{p\in\mathcal{P},p=N}^{\infty} f_p(z) | \leq k \mid \sum_{n=N}^{\infty} \frac{1}{n^{\frac{1}{2}}} | = k \mid \zeta_N(\frac{1}{2}) |$$

.

But(see Lemma 1 [5]) : $\zeta_N(\frac{1}{2}) = o_N(1)$

We deduce that the series $\sum_p | f_p |$ converges normally on any compact of $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$ and consequently $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ is holomorphic in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$. In the same way $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ is holomorphic in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$

If $\alpha \neq 0$, then the holomorphic function $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-zln(p)})^2+(e^{-zln(p)})^2}$ will be null (because null on $]\frac{1}{2}, \frac{1}{2}+\alpha]$), and it follows that $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ or $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ is null in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$. Let's show that this is impossible :

If $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z} = \prod_{p\in\mathcal{P}}(1 + f_p(z)) = 0$ with $f_p(z) = \frac{1}{(p^z/A)-1}$ $\forall z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$. So for the same reason as above, the application :

Ⓢ : $X \longmapsto \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$ is holomorphic in the open quasi-disc $\mathcal{D} = \{X \in \mathbb{C}, 0 <| X |< \sqrt{2}\}$ with $z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$ (here z is fixed)

Let's extend the function Ⓢ by setting :

For $z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \gneqq \frac{1}{2}\}$ and $\forall s \in \mathbb{R},$ with $s \leq 0$, such as $\Re(s+z) \geq 0$

Ⓢ$(C/q^s) = \prod_{p\in\mathcal{P}} \frac{1}{1-C/(q^s p^z)}$ (where q is a prime number, and C is such that $| C |= \sqrt{2}$)

In particular we have :

Ⓢ$(A/q^s) = \prod_{p\in\mathcal{P}} \frac{1}{1-A/(q^s p^z)}$ (where q is a prime number)

But for $z \in \{z \in \mathbb{R}\backslash\{1\}, z \gneqq \frac{1}{2}\}$ we have :

$$\prod_{p\in\mathcal{P}} | \frac{1}{1-A/(q^s p^z)} | \leq \prod_{p\in\mathcal{P}} | \frac{1}{1-A/(p^z)} |$$

It follows that :

$$Ⓢ(A/q^s) = 0$$

So :

$$Ⓢ(X) = 0, \forall X \in \mathcal{D}$$

And consequently :

$$Ⓢ(1)(z) = \zeta(z) = 0$$

$\forall z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \gneqq \frac{1}{2}\}$
which is absurd, so $\alpha = 0$, hence the Riemann hypothesis.

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