The Expanding Universe

Derivation and solution of the Friedmann expansion equations

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Abstract

In this work, the Friedmann equations, which represent the fundamental equations of cosmological models, are derived using a Newtonian and a relativistic approach by solving Einstein's field equations in a high level of detail. The space-time geometry in the form of the Friedmann-Robertson-Walker metric is derived and the calculations of the Christoffel symbols, the Ricci tensor and Ricci scalar, as well as the solution of the field equations are described in detail. The energy-momentum tensor assumes that matter in the universe behaves like an ideal fluid.

The relationship between the different densities in the universe and the scale factor and the resulting three phases in the evolutionary history of the universe are explained. The time-varying ratio of matter density to vacuum density in the universe eventually led to the reversal of expansion, i.e., the change from a decelerated to an accelerated expansion of space. With the help of the second Friedmann equation and an equation for the expansion force, it is demonstrated at which density ratio and at what time this occurred.

Assuming a flat universe and neglecting the radiation density, the Friedmann equation is solved and equations for the scale factor and the Hubble parameter are derived.

Equations are derived to determine the cosmological horizons, the Hubble radius, and the worldlines of photons (light cones) and of stationary objects moving only within the Hubble flow. Using example calculations and their representations in space-time diagrams, the interrelations of these quantities are particularly elaborated.

Keywords— Cosmology, Friedmann equations, Einstein‘s field equations, FRW metric, scale factor, Hubble radius, event horizon, particle horizon, light cones, worldlines

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1. Introduction

To describe the dynamics of the expanding universe on large scales, this work derives the two Friedmann equations (FE) and in particular solves the first FE and numerically evaluates the scale factor that describes the spatial expansion of the universe. Using the time-dependent scale factor, distances (horizons), light cones, and world lines are calculated and depicted in space-time diagrams.

In principle, only the path via a suitable metric—typically, the Friedmann-Robertson-Walker metric (FRW metric) is employed, as is the case in this study—and solving Einstein's field equations leads to the complete Friedmann equations, i.e. taking into account relativistic considerations. But the basic structure of the FE can already be derived by examining the energy of a test mass under the influence of Newton's law of gravity. For this purpose, initially as an introduction, Newton's law of gravitation, the gravitational potential, and the Poisson equation are examined in a bit more detail.

Both approaches are based on the cosmological principle, which is briefly described in the next chapter.

As early as the 1920s, Hubble observed the expansion of the universe (red shift of the spectral lines of observed objects) through his observations and summarized his results in 1929 in the Hubble diagram named after him. These findings are also briefly summarized.

1.1 The cosmological principle

A fundamental concept in cosmology is the cosmological principle. It is assumed that the universe is homogeneous and isotropic on large scales.

Homogeneity

The distribution of mass and energy is, on average, uniform across space, i.e., that from any position in the universe we see a similar distribution of mass and energy.

Isotropy

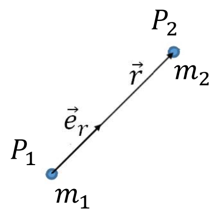
The universe looks the same in all directions, i.e., all physical laws and properties do not depend on the direction in which an observer looks.

It should be emphasized again that the cosmological principle is only valid when averaging over large volumes. Of course, on “small” scales we see a lot of structures in the universe with stars, galaxies, galaxy clusters, etc.

With the cosmological principle, changes in the universe are only permitted as long as these changes do not change the spatial arrangement. This means that only changes are permitted that lead to uniform changes in distance and thus also uniform changes in density through expansion or contraction of the space.

1.2 Potential function of the gravitational force and the Poisson equation

We start with Newton's law of gravitation:



$$F_g = \frac{G \cdot m_1 \cdot m_2}{r^2}, \tag{1.1}$$

G: Gravitational constant; m_1, m_2: mass points
r: distance between the mass points

or as a directed force

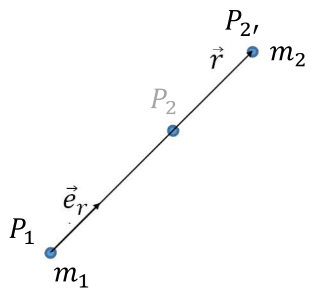
$$\vec{F}_g = -F_g \cdot \vec{e}_r = -\frac{G \cdot m_1 \cdot m_2}{r^2} \cdot \vec{e}_r, \tag{1.2}$$

with the unit vector

$$\vec{e}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}. \tag{1.3}$$

The force that the mass m_1 exerts on m_2 is a "pulling force" and therefore points in the opposite direction of the unit vector. Hence the negative sign in Eq. (1.2).

If we want to move the mass m_2 from the position P_2 to a position $P_{2'}$, a force F_v is required that corresponds in magnitude to F_g, but is directed in the opposite direction:



$$\vec{F}_v = -\vec{F}_g = \frac{G \cdot m_1 \cdot m_2}{r^2} \cdot \vec{e}_r \tag{1.4}$$

The work done here is calculated from

$$W = \int_{P_2}^{P_{2'}} \vec{F}_v \, d\vec{r} \tag{1.5}$$

and corresponds to the difference in potential energy at points P_2 and $P_{2'}$

$$W = \int_{P_2}^{P_{2'}} \vec{F}_v \; d\vec{r} = E_{pot}(P_{2'}) - E_{pot}(P_2). \tag{1.6}$$

If we only consider an infinitesimally small step $d\vec{r}$, we can write Eq. (1.6) in the following form

$$dW = \vec{F}_v \; \cdot d\vec{r} = \begin{pmatrix} F_{v,x} \\ F_{v,y} \\ F_{v,z} \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = dE_{pot}.$$

If we form the dot product $\vec{F}_v \cdot d\vec{r}$ and write the differential dE_{pot} as a change according to the coordinates x, y and z we get

$$F_{v,x}dx + F_{v,y}dy + F_{v,z}dz = \frac{\partial E_{pot}}{\partial x} dx + \frac{\partial E_{pot}}{\partial y} dy + \frac{\partial E_{pot}}{\partial z} dz. \tag{1.7}$$

With Eq. (1.4) we can also write

$$-F_{g,x}dx - F_{g,y}dy - F_{g,z}dz = \frac{\partial E_{pot}}{\partial x} dx + \frac{\partial E_{pot}}{\partial y} dy + \frac{\partial E_{pot}}{\partial z} dz, \tag{1.8}$$

and get for the force components

$$F_{g,x} = -\frac{\partial E_{pot}}{\partial x}; \quad F_{g,y} = -\frac{\partial E_{pot}}{\partial y}; \quad F_{g,z} = -\frac{\partial E_{pot}}{\partial z}. \tag{1.9}$$

This means that the gravitational force $\vec{F}_g$ can be written in the following form:

$$\vec{F}_g = F_{g,x}\vec{e}_x + F_{g,y}\vec{e}_y + F_{g,z}\vec{e}_z = -\left(\frac{\partial E_{pot}}{\partial x}\vec{e}_x + \frac{\partial E_{pot}}{\partial y}\vec{e}_y + \frac{\partial E_{pot}}{\partial z}\vec{e}_z\right),$$

$$\vec{F}_g = -\vec{\nabla}\, E_{pot}\,. \tag{1.10}$$

The gravitational force therefore corresponds to the negative gradient of potential energy.

In the following we want to derive the potential function of the gravitational force. To do this, we first take the gradient of $\frac{Gm_1m_2}{r}$ and consider that $r = \sqrt{x^2 + y^2 + z^2}$.

$$\vec{\nabla}\left(\frac{Gm_1m_2}{r}\right) = Gm_1m_2\vec{\nabla}\left(\frac{1}{r}\right) = Gm_1m_2\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\vec{e}_x + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\vec{e}_y + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\vec{e}_z\right]$$

$$= Gm_1m_2\left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{e}_x - \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{e}_y - \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{e}_z\right]$$

$$= -Gm_1m_2\left(\frac{x}{r^3}\vec{e}_x + \frac{y}{r^3}\vec{e}_y + \frac{z}{r^3}\vec{e}_z\right) = -\frac{Gm_1m_2}{r^3}\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right).$$

With Eq. (1.2)

$$\vec{\nabla}\left(\frac{Gm_1m_2}{r}\right) = -\frac{Gm_1m_2}{r^2}\frac{\vec{r}}{r} = -\frac{Gm_1m_2}{r^2}\vec{e}_r = \vec{F}_g\ . \tag{1.11}$$

According to Eq. (1.10), the gravitational force corresponds to the negative gradient of the potential energy

$$\vec{F}_g = -\vec{\nabla}\,E_{pot} = \vec{\nabla}\left(\frac{Gm_1m_2}{r}\right)\text{or.}\vec{\nabla}\,E_{pot} = \vec{\nabla}\left(-\frac{Gm_1m_2}{r}\right) \tag{1.12}$$

which ultimately leads to the **potential function of the gravitational force**:

$$E_{pot} = -\frac{Gm_1m_2}{r}. \tag{1.13}$$

If we set $m_1 = M$, the mass of a central body (e.g. a planet or a sun), and $m_2 = m$, Eq. (1.12) also be written with the gravitational potential $\Phi(\vec{r})$.

$$\vec{\nabla}\,E_{pot} = m\cdot\vec{\nabla}\left(-\frac{G\cdot M}{r}\right) = m\cdot\vec{\nabla}\Phi(\vec{r}) = -\vec{F}_g\ , \tag{1.14}$$

with $$\Phi(\vec{r}) = -\frac{G\cdot M}{r}. \tag{1.15}$$

This means that the potential energy (see Eq. (1.13)) can also be expressed in the form

$$E_{pot} = m\cdot\Phi(\vec{r})\ . \tag{1.16}$$

Newton's equation of motion in a gravitational field, generated by the mass M, is therefore for a particle of mass m

$$\vec{F} = m \cdot \ddot{\vec{r}} = m \cdot \vec{g} = -m\frac{G \cdot M}{r^2}\vec{e}_r \;; \qquad \vec{g} = -\frac{G \cdot M}{r^2}\vec{e}_r \;. \tag{1.17}$$

With Eq. (1.14)

$$\vec{F} = m \cdot \ddot{\vec{r}} = -m \cdot \vec{\nabla}\left(-\frac{G \cdot M}{r}\right) = -m \cdot \vec{\nabla}\Phi(\vec{r}). \tag{1.18}$$

This allows us to write for the acceleration vector

$$\ddot{\vec{r}} = \dot{\vec{v}} = -\vec{\nabla}\Phi(\vec{r}) \;. \tag{1.19}$$

Assuming that the mass M is a homogeneously distributed mass of a sphere with radius r, i.e.

$$M = \rho \cdot V = \rho \cdot \frac{4}{3} \cdot \pi \cdot r^3, \tag{1.20}$$

the gravitational force field on the spherical surface can, with the Eqs. (1.17) and (1.3), be expressed as:

$$\vec{F} = -m\frac{G \cdot M}{r^2}\vec{e}_r = -m\frac{G \cdot M}{r^3}\vec{r} = -m\frac{G \cdot \rho \cdot \frac{4}{3} \cdot \pi \cdot r^3}{r^3}\vec{r} = -m\frac{4}{3}\pi G \rho \vec{r}.$$

The divergence of this vector is

$$\vec{\nabla} \cdot \vec{F} = -m\frac{4}{3}\pi G \rho \, \vec{\nabla} \cdot \vec{r}.$$

With $\vec{r} = (x, y, z)$ it becomes

$$\vec{\nabla} \cdot \vec{F} = -m\frac{4}{3}\pi G \rho \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = -m\frac{4}{\cancel{3}}\pi G \rho \cancel{3},$$

$$\vec{\nabla} \cdot \vec{F} = -m4\pi G\rho,$$

Let us now replace $\vec{F}$ with Eq. (1.18)

$$\vec{\nabla} \cdot \left(-m\, \vec{\nabla}\Phi(\vec{r})\right) = -m4\pi G\rho,$$

$$-m\, \vec{\nabla} \cdot \vec{\nabla}\Phi(\vec{r}) = -m\, \vec{\nabla}^2\Phi(\vec{r}) = -m4\pi G\rho\,,$$

we finally get the **Poisson equation**

$$\vec{\nabla}^2\Phi(\vec{r}) = \triangle\, \Phi(\vec{r}) = 4\pi G\rho. \tag{1.21}$$

2 Hubble and the expanding Universe

As early as 1929, Edwin Hubble determined a linear relationship between the distance and the escape velocity (i.e. the red shift of spectral lines) through his observations of objects (such as Cepheids) outside the Milky Way and presented it in a diagram, the so-called Hubble diagram. This connection is independent of the viewing direction, i.e., no matter which direction we look or measure, we always find the same linear connection. Since we can assume that our Earth or our galaxy, the Milky Way, does not hold a special position in the universe, we must assume that this relationship between redshift and distance of observed objects is the same from any point in the universe, which corresponds to the basic idea of the cosmological principle. This redshift is primarily caused by the expansion of the universe.

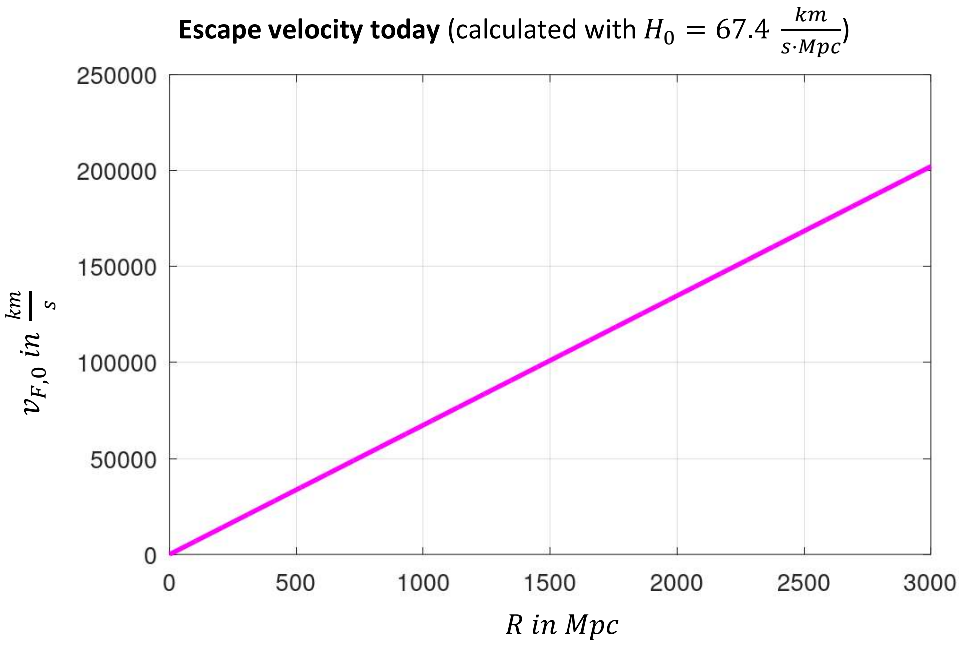
Even though the values measured by Hubble were still very imprecise, it is particularly thanks to him that the idea of an expanding universe was established in science.

Eventually, he was also able to convince Einstein to abandon his static universe - something Einstein had "forced" by adding the cosmological constants to his field equations - and to accept a dynamic universe.

The linear relationship between distance R at the time of light emission and the recessional velocity v_F is described by the Hubble law (see also Fig. 2.1):

$$v_{F,0} = H_0 \cdot R. \tag{2.1}$$

$H_0 \approx 67.4 \pm 0.5\ \frac{km}{s \cdot Mpc}$: Hubble constant (Planck measurements 2018 [1]).



2.1: Hubble diagram

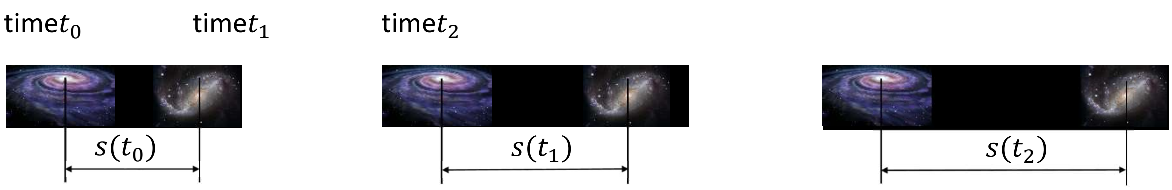
The cosmological redshift is given by the equation (see e.g. [3]):

$$z = \frac{\lambda_0}{\lambda} - 1 \tag{2.2}$$

with the redshifted wavelength λ_0 (value measured today) and the original, non-redshifted wavelength λ.

When we speak of a red shift caused by the expansion of the universe or a recessional velocity of the observed objects, such as galaxies, we mean an expansion of space itself and not a movement of the objects in a static space. Throughout this work peculiar velocities of objects are neglected.

We describe the expansion of space with the help of the scale factor $a(t)$ already mentioned in the introduction. To do this, we look at the increasing distance between two galaxies over time:



$s(t_1)$ is calculated using the scale factor

$$s(t_1) = a(t_1) \cdot s(t_0)\,, \tag{2.3}$$

and $s(t_2)$ accordingly

$$s(t_2) = a(t_2) \cdot s(t_0)\ . \tag{2.4}$$

If one replaces in (2.4) $s(t_0)$ with the relation (2.3), the result is

$$s(t_2) = \frac{a(t_2)}{a(t_1)} s(t_1), \qquad \text{or the other way around} \tag{2.5a}$$

$$s(t_1) = \frac{a(t_1)}{a(t_2)} s(t_2). \tag{2.5b}$$

With $t_1 = t$ and $t_2 = t + \Delta t$ Eq. (2.5a) becomes

$$s(t + \Delta t) = \frac{a(t+\Delta t)}{a(t)} s(t). \tag{2.6}$$

Subtracting $s(t)$ from both sides, rearranging, and then dividing by Δt yields

$$\frac{s(t+\Delta t)-s(t)}{\Delta t} = \frac{a(t+\Delta t)-a(t)}{\Delta t} \cdot \frac{s(t)}{a(t)}\ . \tag{2.7}$$

Now we can take the limit as Δt approaches zero

$$\lim_{\Delta t\to 0}\left(\frac{s(t+\Delta t)-s(t)}{\Delta t}\right) = \lim_{\Delta t\to 0}\left(\frac{a(t+\Delta t)-a(t)}{\Delta t}\right) \cdot \frac{s(t)}{a(t)} \qquad \text{and received}$$

$$\frac{ds}{dt} = \frac{da}{dt} \cdot \frac{s(t)}{a(t)}$$

$$\dot{s}(t) = v_F(t) = \frac{\dot{a}(t)}{a(t)} s(t). \tag{2.8}$$

A comparison with Eq. (2.1) yields the Hubble parameter dependent on the scale factor and thus on time

$$H(t) = \frac{\dot{a}(t)}{a(t)}\,. \tag{2.9}$$

Using $r = a \cdot r_0$ or $a = \frac{r}{r_0}$ and $\dot{a} = \frac{\dot{r}}{r_0}$ we obtain the generalized Hubble law

$$\dot{r}(t) = v_F(t) = H(t) \cdot r(t)\,. \tag{2.10}$$

And with $t = t_0 (i.e., today)$ and thus $\dot{r}(t_0) = v_{F,0}$, $H(t_0) = H_0$, and $r(t_0) = R$, we recover Hubble's law (Eq. (2.1)), $v_{F,0} = H_0 \cdot R$.

If changes in distance due to the expansion of the universe can be calculated according to the equation

$$s = a(t) \cdot s_0 \tag{2.11}$$

then the increase in the wavelength of a light wave must also follow this approach [3], i.e.

$$\lambda = a(t) \cdot \lambda_0. \tag{2.12}$$

Substituting into Eq. (2.2) yields:

$$z = \frac{\lambda_0}{a \cdot \lambda_0} - 1 = \frac{1}{a} - 1 \quad \text{or} \quad a = \frac{1}{z+1} \tag{2.13}$$

The cosmological redshift is therefore also a measure of the expansion of the universe. By definition, the scale factor today is equal to one ($a(t_0) = a_0 = 1$), that is, for the local universe is $z = 0$. If we measure for example $z = 1$ ($a = 0.5$), the photon-emitting object was in a universe only half its current size, with $z = 3$ ($a = 0.25$) the universe had only a quarter of its current size.

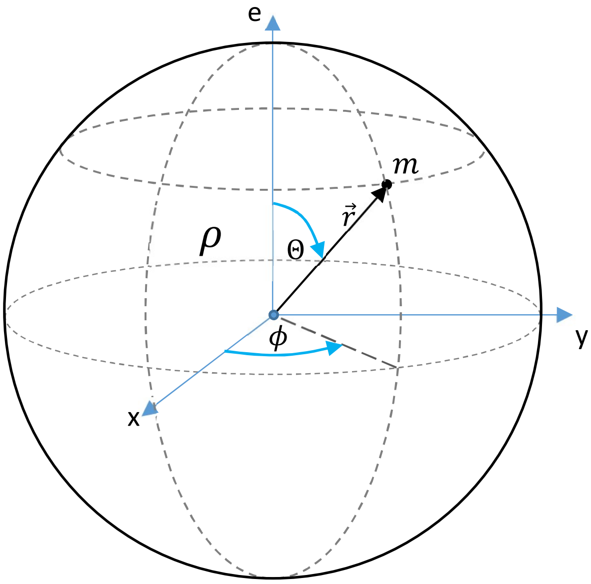
3 Friedmann equations (FE)

As already mentioned in the introduction, two approaches are shown below that can be used to derive the FE. Only with the second approach using the Friedmann-Robertson-Walker metric and the solution of Einstein's field equations can the FE be derived in its full form, i.e., also taking relativistic effects into account. But the basic structure of the FE can be shown with simple energetic considerations in Newton's gravitational field.

3.1 Friedmann equations – Newtonian approach

We consider a spherical model universe (Fig. 3.1), where there is a homogeneous mass distribution ρ. Furthermore, we use a test mass m whose behavior in our model universe is described via the energy balance. The test mass is located at a distance $|\vec{r}| = r$ from the center of the model universe. r is not constant over time and, as shown in Chap. 2, we want to describe this change using the scale factor $a(t)$ (Eq. (2.11)).

$$r = a(t) \cdot r_0\,. \tag{3.1}$$



3.1: Model universe with homogeneous mass distribution

The kinetic energy of the test mass m is (taking into account Eq. (3.1))

$$E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m(\dot{a}(t)\cdot r_0)^2 \ . \tag{3.2}$$

For the potential energy, we obtain using Eq. (1.16), the gravitational potential (Eq. (1.15)) and Eq. (3.1)

$$E_{pot} = m \cdot \Phi(\vec{r}) = -m\frac{G\cdot M}{r} = -m\frac{G\cdot M}{a(t)r_0}. \tag{3.3}$$

With homogeneous mass distribution, we can replace the attracting mass M with

$$M = \rho V = \rho\frac{4}{3}\pi r^3 = \rho\frac{4}{3}\pi(a(t)r_0)^3 \quad \text{and received}$$

$$E_{pot} = -m\ G\ \rho\frac{4}{3}\pi a(t)^2 r_0^2. \tag{3.4}$$

We view our model universe as a closed system, so that the sum of kinetic and potential energy is constant:

$$E_{kin} + E_{pot} = E_g\ , \qquad E_g = const \quad . \tag{3.5}$$

Substituting equations (3.2) and (3.4) into (3.5) gives us

$$\frac{1}{2} m\dot{a}(t)^2 r_0^2 - m\, G\, \rho \frac{4}{3}\pi a(t)^2 r_0^2 = E_g, \tag{3.6}$$

and by rearranging, we finally obtain the first Friedmann equation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\,\pi\, G\, \rho}{3} + \frac{1}{a(t)^2} \cdot \underbrace{\frac{2\, E_g}{m\, r_0^2}}_{=\, const},$$

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\,\pi\, G}{3}\rho(t) + \frac{const}{a(t)^2}\,. \tag{3.7}$$

First Friedmann equation
(Newtonian approach)

In the next chapter, we will see that in relativistic consideration, the constant takes the form $const = -K\, c^2$, with K representing the curvature of space and c representing the speed of light. Furthermore, there is a term missing that includes the cosmological constant Λ.

To answer the question of whether the expansion of the universe proceeds at a constant speed $(F = 0)$, accelerates $(F > 0)$ or decelerates $(F < 0)$, we need an equation for the force within the universe acting on our test mass m

$$F = m \cdot \ddot{r}. \tag{3.8}$$

If we substitute Eq. (2.9) into (2.10) and differentiate w.r.t. time, we obtain the acceleration

$$\ddot{r} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) r + \frac{\dot{a}}{a}\dot{r} = r\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + \frac{\dot{a}}{a}\frac{\dot{a}}{a} r = r\,\frac{\ddot{a}}{a} \quad \text{and} \tag{3.9}$$

substituting into Eq. (3.8), we get for the force

$$F = m \cdot r \cdot \frac{\ddot{a}}{a}\,. \tag{3.10}$$

To calculate the quantity $\frac{\ddot{a}}{a}$, we differentiate the first Friedmann equation (Eq. (3.7)) w.r.t. to time and obtain after some rearrangement:

$$\frac{\ddot{a}}{a} = \frac{4\,\pi\,G}{3}\left(\frac{\dot{\rho}a}{\dot{a}} + 2\rho\right). \tag{3.11}$$

We want to reformulate the term $\frac{\dot{\rho}a}{\dot{a}}$. To do this we use the first law of thermodynamics ($dE = -pdV$) and Einstein's famous energy equation from the special theory of relativity ($E = mc^2$ or $dE = c^2 dm = c^2 d(\rho V)$) and differentiate w.r.t. time:

$$-p\,\frac{dV}{dt} = c^2\,\frac{d}{dt}(\rho V) = c^2\left(\dot{\rho}V + \rho\,\frac{dV}{dt}\right),$$

$$\dot{\rho}V = -\left(\rho + \frac{p}{c^2}\right)\frac{dV}{dt},$$

with $V = \frac{4}{3}\pi r^3$ and $r = a\,r_0$

$$\dot{\rho}\,\frac{4}{3}\pi a^3 r_0^3 = -\left(\rho + \frac{p}{c^2}\right)\frac{4}{3}\pi\,r_0^3\,\frac{d}{dt}(a^3),$$

$$\dot{\rho}a^3 = -\left(\rho + \frac{p}{c^2}\right)3a^2\dot{a},$$

$$\frac{\dot{\rho}a}{\dot{a}} = -3\left(\rho + \frac{p}{c^2}\right). \tag{3.12}$$

Putting Eq. (3.12) into Eq. (3.11) gives us the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G}{3}\left(\rho + \frac{3p}{c^2}\right). \tag{3.13}$$

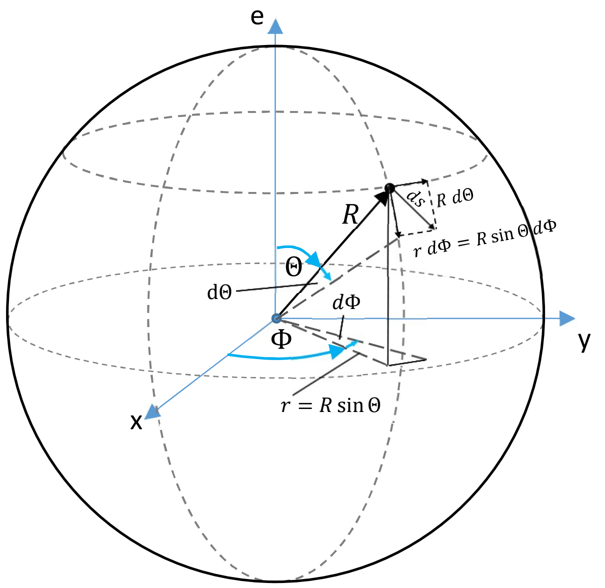
Second Friedmann equation
(Newtonian approach)

Due to the non-relativistic approach, this equation is also incomplete, as we will see in the next chapter. Therefore, we will postpone answering the question of whether the expansion of the universe proceeds at a constant speed ($F = 0$), accelerates ($F > 0$) or decelerates ($F < 0$) until Chapter 4.

3.2 Friedmann equations – relativistic approach

The Friedmann-Robertson-Walker metric (FRW metric) is compatible with the cosmological principle and, when inserted into the Einstein's field equations of the general theory of relativity, leads to the two Friedmann equations.

In order to gain a better understanding of the structure of the FRW metric, it is helpful to derive the metric of surfaces with a constant curvature using the simple example of the sphere (Fig. 3.1). The requirement for constant curvature follows from the cosmological principle of a homogeneous and isotropic universe (see also Chapter 1.1).



3.1: Geometry for calculating the line element ds

The line element ds is calculated according to the geometric relationships shown in Fig. 3.1

$$ds^2 = R^2 d\Theta^2 + R^2 \sin^2\Theta \, d\Phi^2 = R^2(d\Theta^2 + \sin^2\Theta \, d\Phi^2)\,. \qquad (3.14)$$

The metric in matrix form is

$$[g_{ij}] = R^2 \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\Theta \end{bmatrix}. \qquad (3.15)$$

We now aim to describe the sphere as a curved surface in a two-dimensional space with the constant curvature radius R, i.e., we want to express the line element ds in cylindrical coordinates r and Φ. To do this, we first differentiate the term $r = R\ \sin\Theta$ w.r.t. Θ:

$$\frac{dr}{d\Theta} = R\ \cos\Theta,$$ and thus obtain for $d\Theta^2$

$$d\Theta^2 = \frac{dr^2}{R^2\ \cos^2\Theta} = \frac{dr^2}{R^2\ (1-\sin^2\Theta)}\,.$$

Substituting into Eq. (3.14) considering the relationship $r = R\ \sin\Theta$ yields

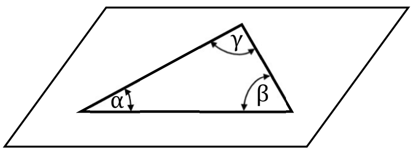
$$ds^2 = R^2\frac{dr^2}{R^2\ (1-\sin^2\Theta)} + r^2 d\Phi^2 = \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2 d\Phi^2. \tag{3.16}$$

For the metric we therefore get

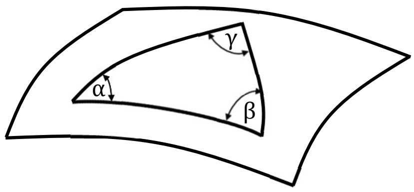
$$[g_{ij}] = \begin{bmatrix} \frac{1}{1-\frac{r^2}{R^2}} & 0 \\ 0 & r^2 \end{bmatrix}\ . \tag{3.17}$$

3.2.1 Measuring curvature

Measuring curvature can be easily explained using the example of a sphere. To do this, let's first consider what happens to a triangle when we go from a plane to a curved surface:

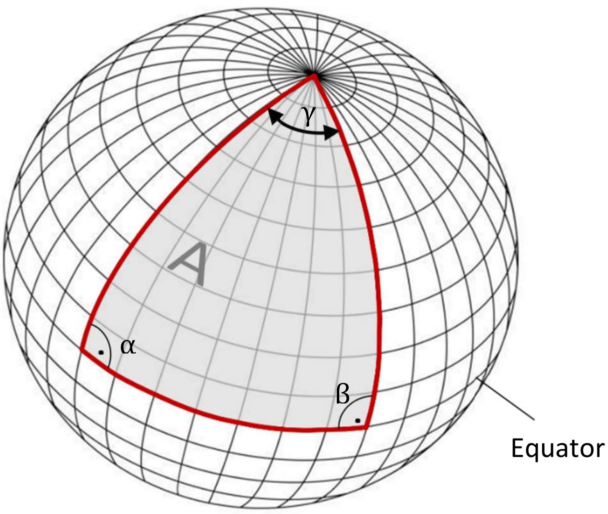


3.2: Sum of angles $\alpha + \beta + \gamma = \pi$



3.3: Sum of angles $\alpha + \beta + \gamma > \pi$

The sum of angles for a triangle in the plane (Fig. 3.2) is π, and for a positively curved surface, such as a sphere, the sum of angles is $> \pi$ (Fig. 3.3). This angle $\delta = \alpha + \beta + \gamma - \pi$ is called the spherical excess for the sphere.



3.4: Triangle on a spherical surface with two right angles, bounded by the equator line and two meridians, with area A and the spherical excess $\delta = \alpha + \beta + \gamma - \pi = \gamma$.

In the special case shown in Fig. 3.4, the spherical excess δ (and thus also the angle γ) behaves with respect to the area A as 2π to half the surface area of the sphere, $\frac{1}{2}4\pi R^2 = 2\pi R^2$, where R is the radius of the sphere.

$$\frac{\delta}{A} = \frac{2\pi}{2\pi R^2} = \frac{1}{R^2}\,. \tag{3.18}$$

This ratio is a measure of the curvature K of two-dimensional surfaces and has the value $\frac{1}{R^2}$ for the sphere.

$$\frac{\delta}{A} = K. \tag{3.19}$$

Thus in Eq. (3.16), the expression $\frac{1}{R^2}$ can be replaced by the general expression of curvature for two-dimensional surfaces, K , obtaining the line element for all two-dimensional geometries with arbitrary curvature radii [4].

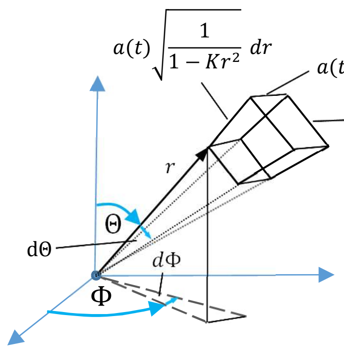
$$ds^2 = \frac{1}{1-Kr^2}dr^2 + r^2 d\Phi^2. \tag{3.20}$$

So, this line element describes all two-dimensional geometries with arbitrary curvature radii. For example, one obtains for $K = \frac{1}{R^2}$ the line element of the sphere (Eq. (3.16)) and for $K = 0$ that of the plane in cylindrical coordinates $ds^2 = dr^2 + r^2 d\Phi^2$.

3.2.2 Friedmann-Robertson-Walker metric and the Friedmann equations

We want to extend the two-dimensional metric with the variables r and Φ (Eq. (3.20)) to three-dimensional geometries by introducing the third spatial coordinate Θ. To do this, we replace $d\Phi^2$ in Eq. (3.20) with the corresponding solid angle $d\Theta^2 + \sin^2\Theta\, d\Phi^2$.

By introducing the angle Θ, we rotate the radius r (see Fig. 3.1) out of the plane and thereby create a volume element, as shown in Fig. 3.5. Multiplying the edge lengths by the scale factor $a(t)$, which describes the expansion of the space over time, leads to the line element of three-dimensional geometries in a dynamic space.



$$ds = \sqrt{a(t)^2\frac{1}{1-Kr^2}dr^2 + a(t)^2r^2d\Theta^2 + a(t)^2r^2\sin^2\Theta\,d\Phi^2}$$

3.5: Components of the line element in three-dimensional space, as an extension of the line element of Eq. (3.20) by the angle Θ and scaled with the scale factor $a(t)$.

$$ds^2 = a(t)^2\left[\frac{1}{1-Kr^2}dr^2 + r^2(d\Theta^2 + \sin^2\Theta\,d\Phi^2)\right]. \tag{3.21}$$

If we expand Eq. (3.21) to include the time element $-c^2dt^2$, we finally obtain the **FRW metric**

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{1}{1-Kr^2}dr^2 + r^2(d\Theta^2 + \sin^2\Theta\,d\Phi^2)\right]. \tag{3.22}$$

$$[g_{\mu\nu}] = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\Theta \end{bmatrix}. \tag{3.23}$$

The curvature K can take the values 0, 1 and -1 (with appropriate rescaling of r) and defines the shape of the universe:

$K = 0$: flat universe

$K = 1$: closed universe

$K = -1$: open universe

(more on this later, see Chapter 4)

It should be noted that a flat universe is not equivalent to a flat spacetime, i.e., to Minkowski spacetime. For this, the scale factor must be constant, so that it can be absorbed into the coordinate r. Only then does the FRW metric Eq. (3.22) become the Minkowski metric:

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\Theta^2 + \sin^2\Theta\,d\Phi^2). \tag{3.24}$$

To derive the FE, we put the FRW metric into **Einstein's field equation**

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\,\pi\,G}{c^4} T_{\mu\nu} \,. \tag{3.25}$$

For the energy-momentum tensor $T^{\mu\nu}$ we assume that the matter in the universe behaves like the particles of an ideal fluid. This means, we have to consider the energy density ρ and pressure p. There are no shear forces present, so the energy-momentum tensor is only occupied with values on the main diagonal [3].

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \tag{3.26}$$

or in component notation

$$T^{\mu\nu} = (\rho + p)\frac{u^\mu u^\nu}{c^2} + p g^{\mu\nu}, \quad \text{with } u^\mu = (c, \mathbf{0}). \tag{3.27}$$

Starting from the Riemann curvature tensor

$$R^\alpha_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\sigma\beta}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\beta} \,, \tag{3.28}$$

with the Christoffel symbols

$$\Gamma^\nu_{\alpha\beta} = \frac{1}{2} g^{\nu\sigma}\left(\partial_\beta g_{\sigma\alpha} + \partial_\alpha g_{\sigma\beta} - \partial_\sigma g_{\alpha\beta}\right) , \tag{3.29}$$

we calculate the Ricci tensor

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = R_{\nu\mu}, \tag{3.30}$$

and the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \,. \tag{3.31}$$

First, we determine the non-zero Christoffel symbols and begin with a few preliminary considerations:

a) The metric tensor is only occupied on the main diagonal, i.e., $g^{\mu\nu}$ or $g_{\mu\nu}$ are only non-zero when $\mu = \nu$.
b) g_{00} is a constant and therefore $\partial_\alpha g_{00} = 0$ for any α.
c) The elements g_{11}, g_{22} and g_{33} are functions of t and r, with g_{33} also depending on Θ. None of the metric elements depend on the coordinate Φ.
d) The Christoffel symbols are symmetrical w.r.t. the lower two indices, i.e., $\Gamma^\nu_{\alpha\beta} = \Gamma^\nu_{\beta\alpha}$

With these findings, the non-zero Christoffel symbols can be determined:

$\underline{\boldsymbol{\nu = 0}}$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} g^{00}\left(\partial_\beta g_{0\alpha} + \partial_\alpha g_{0\beta} - \partial_0 g_{\alpha\beta}\right)$$

$\underline{\alpha = 0, \beta = 0,1,2,3}$

$\Gamma^0_{00} = 0,\ \ \Gamma^0_{01} = 0,\ \ \Gamma^0_{02} = 0,\ \ \Gamma^0_{03} = 0$

$\underline{\alpha = 1, \beta = 0,1,2,3}$

$\boldsymbol{\Gamma^0_{11} \neq 0},\ \ \Gamma^0_{12} = 0,\ \ \Gamma^0_{13} = 0$

$\underline{\alpha = 2, \beta = 0,1,2,3}$

$\boldsymbol{\Gamma^0_{22} \neq 0},\ \ \Gamma^0_{23} = 0$

$\underline{\alpha = 3, \beta = 0,1,2,3}$

$\boldsymbol{\Gamma^0_{33} \neq 0}$

$\underline{\boldsymbol{\nu = 1}}$

$$\Gamma^1_{\alpha\beta} = \frac{1}{2} g^{11}\left(\partial_\beta g_{1\alpha} + \partial_\alpha g_{1\beta} - \partial_1 g_{\alpha\beta}\right)$$

$\underline{\alpha = 0, \beta = 0,1,2,3}$

$\Gamma^1_{00} = 0,\ \ \boldsymbol{\Gamma^1_{01} \neq 0},\ \ \Gamma^1_{02} = 0,\ \ \Gamma^1_{03} = 0$

$\underline{\alpha = 1, \beta = 0,1,2,3}$

$\boldsymbol{\Gamma^1_{11} \neq 0},\ \ \Gamma^1_{12} = 0,\ \ \Gamma^1_{13} = 0$

$\underline{\alpha = 2, \beta = 0,1,2,3}$

$\boldsymbol{\Gamma^1_{22} \neq 0},\ \ \Gamma^1_{23} = 0$

$\underline{\alpha = 3, \beta = 0,1,2,3}$

$\boldsymbol{\Gamma^1_{33} \neq 0}$

$\underline{\boldsymbol{\nu = 2}}$

$$\Gamma^2_{\alpha\beta} = \frac{1}{2} g^{22}\left(\partial_\beta g_{2\alpha} + \partial_\alpha g_{2\beta} - \partial_2 g_{\alpha\beta}\right)$$

$\underline{\alpha = 0, \beta = 0,1,2,3}$

$\Gamma^2_{00} = 0,\ \ \Gamma^2_{01} = 0,\ \ \boldsymbol{\Gamma^2_{02} \neq 0},\ \ \Gamma^2_{03} = 0$

$\underline{\alpha = 1, \beta = 0,1,2,3}$

$\Gamma^2_{11} = 0,\ \ \boldsymbol{\Gamma^2_{12} \neq 0},\ \ \Gamma^2_{13} = 0$

$\underline{\alpha = 2, \beta = 0,1,2,3}$

$\Gamma^2_{22} = 0, \;\; \Gamma^2_{23} = 0$

$\underline{\alpha = 3, \beta = 0,1,2,3}$

$\boldsymbol{\Gamma^2_{33} \neq 0}$

$\underline{\boldsymbol{\nu = 3}}$

$$\Gamma^3_{\alpha\beta} = \frac{1}{2} g^{33}\left(\partial_\beta g_{3\alpha} + \partial_\alpha g_{3\beta} - \partial_3 g_{\alpha\beta}\right)$$

$\underline{\alpha = 0, \beta = 0,1,2,3}$

$\Gamma^3_{00} = 0, \;\; \Gamma^3_{01} = 0, \;\; \Gamma^3_{02} = 0, \;\; \boldsymbol{\Gamma^3_{03} \neq 0}$

$\underline{\alpha = 1, \beta = 0,1,2,3}$

$\Gamma^3_{11} = 0, \;\; \Gamma^3_{12} = 0, \;\; \boldsymbol{\Gamma^3_{13} \neq 0}$

$\underline{\alpha = 2, \beta = 0,1,2,3}$

$\Gamma^3_{22} = 0, \;\; \boldsymbol{\Gamma^3_{23} \neq 0}$

$\underline{\alpha = 3, \beta = 0,1,2,3}$

$\Gamma^3_{33} = 0$

So there are 13 non-zero Christoffel symbols that we now want to determine:

$$\Gamma^0_{11} = \frac{1}{2} g^{00}(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11})$$

$$\Gamma^0_{11} = \frac{1}{2}(-c^2)(-1)\partial_0\left(\frac{a^2}{1-Kr^2}\right) = \frac{1}{2}\frac{1}{c^2}\frac{2a\dot{a}}{1-Kr^2}$$

$$\Gamma^0_{11} = \frac{1}{c^2}\,\frac{a\dot{a}}{1-Kr^2} \tag{3.32}$$

$$\Gamma^0_{22} = \frac{1}{2} g^{00}(\partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22})$$

$$\Gamma^0_{22} = -\frac{1}{2c^2}(-1)\partial_0(a^2 r^2) = \frac{1}{2}\frac{r^2}{c^2}2a\dot{a}$$

$$\Gamma^0_{22} = \frac{r^2 a\dot{a}}{c^2} \tag{3.33}$$

$$\Gamma^0_{33} = \frac{1}{2} g^{00}(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33})$$

$$\Gamma^0_{33} = -\frac{1}{2c^2}(-1)\partial_0(a^2 r^2 \sin^2\Theta) = \frac{1}{2}\frac{r^2 \sin^2\Theta}{c^2} 2a\dot{a}$$

$$\Gamma^0_{33} = \frac{r^2 \sin^2\Theta a\dot{a}}{c^2} \tag{3.34}$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11}(\partial_1 g_{10} + \partial_0 g_{11} - \partial_1 g_{01})$$

$$\Gamma^1_{01} = \frac{1}{2}\frac{1-Kr^2}{a^2}\partial_0\left(\frac{a^2}{1-Kr^2}\right) = \frac{1}{2}\frac{1-Kr^2}{a^2}\frac{2a\dot{a}}{1-Kr^2}$$

$$\Gamma^1_{01} = \frac{\dot{a}}{a} \tag{3.35}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11})$$

$$\Gamma^1_{11} = \frac{1}{2}\frac{1-Kr^2}{a^2}\partial_1\left(\frac{a^2}{1-Kr^2}\right) = \frac{1}{2}\frac{1-Kr^2}{a^2}(-1)\frac{-2Kra^2}{(1-Kr^2)^2}$$

$$\Gamma^1_{11} = \frac{Kr}{1-Kr^2} \tag{3.36}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22})$$

$$\Gamma^1_{22} = -\frac{1}{2}\frac{1-Kr^2}{a^2}\partial_1(a^2 r^2) = -\frac{1}{2}\frac{1-Kr^2}{a^2} 2ra^2$$

$$\Gamma^1_{22} = -r(1-Kr^2) \tag{3.37}$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33})$$

$$\Gamma^1_{33} = -\frac{1}{2}\frac{1-Kr^2}{a^2}\partial_1(a^2 r^2 \sin^2\Theta) = -\frac{1}{2}\frac{1-Kr^2}{a^2} 2ra^2 \sin^2\Theta$$

$$\Gamma^1_{33} = -r(1-Kr^2)\sin^2\Theta \tag{3.38}$$

$$\Gamma^2_{02} = \frac{1}{2} g^{22}(\partial_2 g_{20} + \partial_0 g_{22} - \partial_2 g_{02})$$

$$\Gamma^2_{02} = \frac{1}{2}\,\frac{1}{a^2r^2}\partial_0(a^2r^2) = \frac{1}{2}\,\frac{1}{a^2r^2}2a\dot{a}r^2$$

$$\Gamma^2_{02} = \frac{\dot{a}}{a} \tag{3.39}$$

$$\Gamma^2_{12} = \frac{1}{2}g^{22}(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{12})$$

$$\Gamma^2_{12} = \frac{1}{2}\,\frac{1}{a^2r^2}\partial_1(a^2r^2) = \frac{1}{2}\,\frac{1}{a^2r^2}2a^2r$$

$$\Gamma^2_{12} = \frac{1}{r} \tag{3.40}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33})$$

$$\Gamma^2_{33} = -\frac{1}{2}\,\frac{1}{a^2r^2}\partial_2(a^2r^2\sin^2\Theta) = -\frac{1}{2}\,\frac{1}{a^2r^2}2r^2a^2\sin\Theta\cos\Theta$$

$$\Gamma^2_{33} = -\sin\Theta\cos\Theta \tag{3.41}$$

$$\Gamma^3_{03} = \frac{1}{2}g^{33}(\partial_3 g_{30} + \partial_0 g_{33} - \partial_3 g_{03})$$

$$\Gamma^3_{03} = \frac{1}{2}\,\frac{1}{a^2r^2\sin^2\Theta}\partial_0(a^2r^2\sin^2\Theta) = \frac{1}{2}\,\frac{1}{a^2r^2\sin^2\Theta}2r^2a\dot{a}\sin^2\Theta$$

$$\Gamma^3_{03} = \frac{\dot{a}}{a} \tag{3.42}$$

$$\Gamma^3_{13} = \frac{1}{2}g^{33}(\partial_3 g_{31} + \partial_1 g_{33} - \partial_3 g_{13})$$

$$\Gamma^3_{13} = \frac{1}{2}\,\frac{1}{a^2r^2\sin^2\Theta}\partial_1(a^2r^2\sin^2\Theta) = \frac{1}{2}\,\frac{1}{a^2r^2\sin^2\Theta}2ra^2\sin^2\Theta$$

$$\Gamma^3_{13} = \frac{1}{r} \tag{3.43}$$

$$\Gamma^3_{23} = \frac{1}{2}g^{33}(\partial_3 g_{32} + \partial_2 g_{33} - \partial_3 g_{23})$$

$$\Gamma^3_{23} = \frac{1}{2}\,\frac{1}{a^2r^2\sin^2\Theta}\partial_3(a^2r^2\sin^2\Theta) = \frac{1}{2}\,\frac{1}{a^2r^2\sin^2\Theta}2a^2r^2\sin\Theta\cos\Theta$$

$$\Gamma^3_{23} = \cot\Theta \tag{3.44}$$

With the Christoffel symbols known, we can determine the elements of the Ricci tensor using Eq. (3.28) and (3.30):

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\sigma\alpha}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha}, \tag{3.45}$$

Due to the symmetry of the Ricci tensor (Eq. (3.30)), only 10 of the 16 elements need to be calculated:

$$R_{00}\ ;\ R_{01}\ ; R_{02}\ ; R_{03}\ ; R_{11}\ ; R_{12}\ ; R_{13}\ ; R_{22}\ ; R_{23}\ ; R_{33}$$

If we perform the summation over α and σ in Eq. (3.45), we obtain an equation for calculating the Ricci tensor elements in the form

$$\begin{aligned} R_{\mu\nu} = \partial_0 \Gamma^{0}_{\mu\nu} - \partial_\nu \Gamma^{0}_{\mu 0} &+ \left(\Gamma^{0}_{00}\Gamma^{0}_{\mu\nu} + \Gamma^{0}_{10}\Gamma^{1}_{\mu\nu} + \Gamma^{0}_{20}\Gamma^{2}_{\mu\nu} + \Gamma^{0}_{30}\Gamma^{3}_{\mu\nu}\right) \\ &- \left(\Gamma^{0}_{0\nu}\Gamma^{0}_{\mu 0} + \Gamma^{0}_{1\nu}\Gamma^{1}_{\mu 0} + \Gamma^{0}_{2\nu}\Gamma^{2}_{\mu 0} + \Gamma^{0}_{3\nu}\Gamma^{3}_{\mu 0}\right) \\ + \partial_1 \Gamma^{1}_{\mu\nu} - \partial_\nu \Gamma^{1}_{\mu 1} &+ \left(\Gamma^{1}_{01}\Gamma^{0}_{\mu\nu} + \Gamma^{1}_{11}\Gamma^{1}_{\mu\nu} + \Gamma^{1}_{21}\Gamma^{2}_{\mu\nu} + \Gamma^{1}_{31}\Gamma^{3}_{\mu\nu}\right) \\ &- \left(\Gamma^{1}_{0\nu}\Gamma^{0}_{\mu 1} + \Gamma^{1}_{1\nu}\Gamma^{1}_{\mu 1} + \Gamma^{1}_{2\nu}\Gamma^{2}_{\mu 1} + \Gamma^{1}_{3\nu}\Gamma^{3}_{\mu 1}\right) \\ + \partial_2 \Gamma^{2}_{\mu\nu} - \partial_\nu \Gamma^{2}_{\mu 2} &+ \left(\Gamma^{2}_{02}\Gamma^{0}_{\mu\nu} + \Gamma^{2}_{12}\Gamma^{1}_{\mu\nu} + \Gamma^{2}_{22}\Gamma^{2}_{\mu\nu} + \Gamma^{2}_{32}\Gamma^{3}_{\mu\nu}\right) \\ &- \left(\Gamma^{2}_{0\nu}\Gamma^{0}_{\mu 2} + \Gamma^{2}_{1\nu}\Gamma^{1}_{\mu 2} + \Gamma^{2}_{2\nu}\Gamma^{2}_{\mu 2} + \Gamma^{2}_{3\nu}\Gamma^{3}_{\mu 2}\right) \\ + \partial_3 \Gamma^{3}_{\mu\nu} - \partial_\nu \Gamma^{3}_{\mu 3} &+ \left(\Gamma^{3}_{03}\Gamma^{0}_{\mu\nu} + \Gamma^{3}_{13}\Gamma^{1}_{\mu\nu} + \Gamma^{3}_{23}\Gamma^{2}_{\mu\nu} + \Gamma^{3}_{33}\Gamma^{3}_{\mu\nu}\right) \\ &- \left(\Gamma^{3}_{0\nu}\Gamma^{0}_{\mu 3} + \Gamma^{3}_{1\nu}\Gamma^{1}_{\mu 3} + \Gamma^{3}_{2\nu}\Gamma^{2}_{\mu 3} + \Gamma^{3}_{3\nu}\Gamma^{3}_{\mu 3}\right). \end{aligned} \tag{3.46}$$

Since only 13 Christoffel symbols are non-zero (see (3.32) - (3.44)), many of these terms vanish. Furthermore, we consider that $a = a(t)$, i.e., $\partial_\alpha(a) = 0$ for $\alpha \neq 0$.

$$R_{00} = -\partial_0 \Gamma^{1}_{01} - (\Gamma^{1}_{01})^2 - \partial_0 \Gamma^{2}_{02} - (\Gamma^{2}_{02})^2 - \partial_0 \Gamma^{3}_{03} - (\Gamma^{3}_{03})^2$$

$$R_{00} = -\partial_0(\dot{a}a^{-1}) - \frac{\dot{a}^2}{a^2} - \partial_0(\dot{a}a^{-1}) - \frac{\dot{a}^2}{a^2} - \partial_0(\dot{a}a^{-1}) - \frac{\dot{a}^2}{a^2}$$

$$R_{00} = -3\left[(\ddot{a}a^{-1} - \dot{a}^2a^{-2}) + \frac{\dot{a}^2}{a^2}\right]$$

$$R_{00} = -3\,\frac{\ddot{a}}{a} \tag{3.47}$$

$$R_{01} = \partial_1\Gamma^1_{01} - \partial_1\Gamma^1_{01} + \Gamma^1_{11}\Gamma^1_{01} - \Gamma^1_{11}\Gamma^1_{01} - \partial_1\Gamma^2_{02} + \Gamma^2_{12}\Gamma^1_{01} - \Gamma^2_{12}\Gamma^2_{02}$$
$$-\partial_1\Gamma^3_{03} + \Gamma^3_{13}\Gamma^1_{01} - \Gamma^3_{13}\Gamma^3_{03}$$

$$R_{01} = -\partial_1\left(\frac{\dot{a}}{a}\right) + \frac{1}{r}\frac{\dot{a}}{a} - \frac{1}{r}\frac{\dot{a}}{a} - \partial_1\left(\frac{\dot{a}}{a}\right) + \frac{1}{r}\frac{\dot{a}}{a} - \frac{1}{r}\frac{\dot{a}}{a}$$

$$R_{01} = 0 \tag{3.48}$$

$$R_{02} = -\partial_2\Gamma^1_{01} + \partial_2\Gamma^2_{02} - \partial_2\Gamma^2_{02} - \partial_2\Gamma^3_{03} + \Gamma^3_{23}\Gamma^2_{02} - \Gamma^3_{23}\Gamma^3_{03}$$

$$R_{02} = -\partial_2\left(\frac{\dot{a}}{a}\right) - \partial_2\left(\frac{\dot{a}}{a}\right) + \cot\Theta\,\frac{\dot{a}}{a} - \cot\Theta\,\frac{\dot{a}}{a}$$

$$R_{02} = 0 \tag{3.49}$$

$$R_{03} = -\partial_3\Gamma^1_{01} - \partial_3\Gamma^2_{02} + \partial_3\Gamma^3_{03} - \partial_3\Gamma^3_{03}$$

$$R_{03} = -\partial_3\left(\frac{\dot{a}}{a}\right) - \partial_3\left(\frac{\dot{a}}{a}\right)$$

$$R_{03} = 0 \tag{3.50}$$

$$R_{11} = \partial_0\Gamma^0_{11} - \Gamma^0_{11}\Gamma^1_{01} + \partial_1\Gamma^1_{11} - \partial_1\Gamma^1_{11} + \Gamma^1_{01}\Gamma^0_{11} + (\Gamma^1_{11})^2 - \Gamma^1_{01}\Gamma^0_{11} - (\Gamma^1_{11})^2$$
$$-\partial_1\Gamma^2_{12} + \Gamma^2_{02}\Gamma^0_{11} + \Gamma^2_{12}\Gamma^1_{11} - (\Gamma^2_{12})^2 - \partial_1\Gamma^3_{13} + \Gamma^3_{03}\Gamma^0_{11} + \Gamma^3_{13}\Gamma^1_{11} - (\Gamma^3_{13})^2$$

$$R_{11} = \partial_0\left(\frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2}\right) - \frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2}\frac{\dot{a}}{a} - \partial_1\left(\frac{1}{r}\right) + \frac{\dot{a}}{a}\frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2} + \frac{1}{r}\frac{Kr}{1-Kr^2} - \left(\frac{1}{r}\right)^2$$
$$-\partial_1\left(\frac{1}{r}\right) + \frac{\dot{a}}{a}\frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2} + \frac{1}{r}\frac{Kr}{1-Kr^2} - \left(\frac{1}{r}\right)^2$$

$$R_{11} = \frac{\dot{a}^2 + a\ddot{a}}{c^2(1-Kr^2)} - \frac{\dot{a}^2}{c^2(1-Kr^2)} - (-1)\frac{1}{r^2} + \frac{\dot{a}^2}{c^2(1-Kr^2)} + \frac{2K}{1-Kr^2} - \left(\frac{1}{r}\right)^2 - (-1)\frac{1}{r^2}$$
$$+\frac{\dot{a}^2}{c^2(1-Kr^2)} - \left(\frac{1}{r}\right)^2$$

$$R_{11} = \frac{\dot{a}^2}{c^2(1-Kr^2)} + \frac{a\ddot{a}}{c^2(1-Kr^2)} + \frac{\dot{a}^2}{c^2(1-Kr^2)} + \frac{2K}{1-Kr^2}$$

$$R_{11} = \frac{a\ddot{a}}{c^2(1-Kr^2)} + \frac{2\dot{a}^2}{c^2(1-Kr^2)} + \frac{2K}{1-Kr^2}$$

$$R_{11} = \left(\frac{\ddot{a}}{c^2 a} + \frac{2}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right)g_{11} \tag{3.51}$$

$$R_{12} = -\partial_2\Gamma^1_{11} + \partial_2\Gamma^2_{12} - \partial_2\Gamma^2_{12} - \partial_2\Gamma^3_{13} + \Gamma^3_{23}\Gamma^2_{12} - \Gamma^3_{23}\Gamma^3_{13}$$

$$R_{12} = -\partial_2\left(\frac{Kr}{1-Kr^2}\right) - \partial_2\left(\frac{1}{r}\right) + \cot\Theta\frac{1}{r} - \cot\Theta\frac{1}{r}$$

$$R_{12} = 0 \tag{3.52}$$

$$R_{13} = -\partial_3\Gamma^1_{11} - \partial_3\Gamma^2_{12} + \partial_3\Gamma^3_{13} - \partial_3\Gamma^3_{13}$$

$$R_{13} = -\partial_3\left(\frac{Kr}{1-Kr^2}\right) - \partial_3\left(\frac{1}{r}\right)$$

$$R_{13} = 0 \tag{3.53}$$

$$R_{22} = \partial_0\Gamma^0_{22} - \Gamma^0_{22}\Gamma^2_{02} + \partial_1\Gamma^1_{22} + \Gamma^1_{01}\Gamma^0_{22} + \Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{02}\Gamma^0_{22}$$
$$+\Gamma^2_{12}\Gamma^1_{22} - \Gamma^2_{02}\Gamma^0_{22} - \Gamma^2_{12}\Gamma^1_{22} - \partial_2\Gamma^3_{23} + \Gamma^3_{03}\Gamma^0_{22} + \Gamma^3_{13}\Gamma^1_{22} - (\Gamma^3_{23})^2$$

$$R_{22} = \partial_0\left(\frac{r^2a\dot{a}}{c^2}\right) - \frac{r^2a\dot{a}}{c^2}\frac{\dot{a}}{a} + \partial_1\big(-r(1-Kr^2)\big) + \frac{\dot{a}}{a}\frac{r^2a\dot{a}}{c^2}$$
$$+\frac{Kr}{1-Kr^2}\big(-r(1-Kr^2)\big) - \frac{1}{r}\big(-r(1-Kr^2)\big) - \partial_2(\cot\Theta) + \frac{\dot{a}}{a}\frac{r^2a\dot{a}}{c^2}$$
$$+\frac{1}{r}\big(-r(1-Kr^2)\big) - \cot^2\Theta$$

$$R_{22} = \frac{r^2}{c^2}(\dot{a}^2 + a\ddot{a}) - 1 + 3Kr^2 - Kr^2 + (1+\cot^2\Theta) - \cot^2\Theta + \frac{r^2\dot{a}^2}{c^2}$$

$$R_{22} = \frac{r^2a\ddot{a}}{c^2} + \frac{2r^2\dot{a}^2}{c^2} + 2Kr^2$$

$$R_{22} = \left(\frac{\ddot{a}}{c^2a} + \frac{2}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right)g_{22} \tag{3.54}$$

$$R_{23} = \partial_3\Gamma^3_{23} - \partial_3\Gamma^3_{23}$$

$$R_{23} = 0 \tag{3.55}$$

$$R_{33} = \partial_0\Gamma^0_{33} + \Gamma^0_{33}\Gamma^3_{03} + \partial_1\Gamma^1_{33} + \Gamma^1_{01}\Gamma^0_{33} + \Gamma^1_{11}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^3_{13} + \partial_2\Gamma^2_{33}$$
$$+\Gamma^2_{02}\Gamma^0_{33} + \Gamma^2_{12}\Gamma^1_{33} - \Gamma^2_{33}\Gamma^3_{23} + \Gamma^3_{03}\Gamma^0_{33} + \Gamma^3_{13}\Gamma^1_{33} \pm \Gamma^3_{23}\Gamma^2_{33} - \Gamma^3_{03}\Gamma^0_{33}$$
$$-\Gamma^3_{13}\Gamma^1_{33} - \Gamma^3_{23}\Gamma^2_{33}$$

$$R_{33} = \partial_0\left(\frac{r^2 \sin^2\Theta\, a\dot{a}}{c^2}\right) + \partial_1\left((-r + Kr^3)\sin^2\Theta\right) + \frac{\dot{a}}{a}\frac{r^2\sin^2\Theta\, a\dot{a}}{c^2}$$
$$+\frac{Kr}{1-Kr^2}(-r)(1-Kr^2)\sin^2\Theta - (-r+Kr^3)\sin^2\Theta\ \frac{1}{r}$$
$$+\partial_2(-\sin\Theta\cos\Theta) + \frac{1}{r}(-r+Kr^3)\sin^2\Theta - (-\sin\Theta\cos\Theta)\cot\Theta$$

$$R_{33} = \frac{r^2\sin^2\Theta}{c^2}(\dot{a}^2 + a\ddot{a}) + (-1+3Kr^2)\sin^2\Theta + \frac{r^2\sin^2\Theta\ \dot{a}^2}{c^2} - Kr^2\sin^2\Theta$$
$$-(\cos^2\Theta - \sin^2\Theta) + \sin\Theta\cos\Theta\frac{\cos\Theta}{\sin\Theta}$$

$$R_{33} = \frac{a\ddot{a}\, r^2\sin^2\Theta}{c^2} + \frac{2\,\dot{a}^2\, r^2\sin^2\Theta}{c^2} + 2Kr^2\sin^2\Theta$$

$$R_{33} = \left(\frac{\ddot{a}}{c^2 a} + \frac{2}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right) g_{33} \tag{3.56}$$

So, the **Ricci tensor** has only non-zero elements on the main diagonal:

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{3.57}$$

$$R_{ii} = \left(\frac{\ddot{a}}{c^2 a} + \frac{2}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right) g_{ii}, \quad \text{with} \quad i = 1,2,3 \tag{3.58}$$

With Eq. (3.31) and Eqs. (3.57) and (3.58) we are now able to determine the **Ricci scalar**:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ii}R_{ii},$$
$$R = \frac{1}{c^2}3\frac{\ddot{a}}{a} + g^{ii}\left(\frac{\ddot{a}}{c^2 a} + \frac{2}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right)g_{ii} = \frac{1}{c^2}3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{c^2 a} + \frac{2}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right),$$
$$R = 6\left(\frac{\ddot{a}}{c^2 a} + \frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right) \tag{3.59}$$

To solve the field equations (Eq. (3.25)), we need the contravariant form of the energy-momentum tensor (see Eq. (3.27)):

$T_{\mu\nu} = T^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$. For the main diagonal elements, it becomes

$$T_{00} = T^{00}g_{00}g_{00} = c^4\varrho\ , \tag{3.60}$$

$$T_{ii} = T^{ii}g_{ii}g_{ii} = T^{i}_{\ i}\, g_{ii} = \left[(\rho+p)\frac{u^i u^i}{c^2}g_{ii} + pg^{ii}g_{ii}\right]\, g_{ii}.$$

Taking into account that $u^i = 0$ for $i = 1,2,3$ and $g^{11}g_{11} = g^{22}g_{22} = g^{33}g_{33} = 1$, this becomes

$$T_{ii} = T^i_{\ i}\, g_{ii} = p\, g_{ii}, \text{ for } i = 1,2,3 \tag{3.61}$$

First, we use the time components in Eq. (3.25) to obtain the first FE:

$$R_{00} - \frac{1}{2} g_{00} R + \Lambda g_{00} = \frac{8\,\pi\, G}{c^4} T_{00},$$

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}(-c^2)6\left(\frac{\ddot{a}}{c^2 a} + \frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right) - c^2\Lambda = \frac{8\,\pi\, G}{c^4} c^4 \rho,$$

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{Kc^2}{a^2} - c^2\Lambda = 8\pi G\rho$$

$$\boldsymbol{\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\,\pi\, G}{3}\rho(t) - \frac{Kc^2}{a(t)^2} + \frac{\Lambda\, c^2}{3}}. \quad \textbf{First Friedmann equation} \tag{3.62}$$

The comparison with Eq. (3.7) (Newtonian approach) shows that the constant $const = -Kc^2$ and the term with the cosmological constant is missing in the non-relativistic approach, i.e. in Eq. (3.7).

We will use this equation in the next chapter to derive a solution for the scale factor $a(t)$.

With the spatial components, we get

$$R_{ii} - \frac{1}{2} g_{ii} R + \Lambda g_{ii} = \frac{8\,\pi\, G}{c^4} T_{ii} = \frac{8\,\pi\, G}{c^4} p\, g_{ii},$$

$$\left(\frac{\ddot{a}}{c^2 a} + \frac{2}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2}\right) g_{ii} - 3\left(\frac{\ddot{a}}{c^2 a} + \frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right) g_{ii} + \Lambda g_{ii} = \frac{8\,\pi\, G}{c^4} p\, g_{ii},$$

$$-2\frac{\ddot{a}}{c^2 a} - \frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{K}{a^2} + \Lambda = \frac{8\,\pi\, G}{c^4} p.$$

We see that regardless of which spatial component, i.e., $i = 1, 2\ or\ 3$, is used, we get the same result.

Replacing $\left(\frac{\dot{a}}{a}\right)^2$ with Eq. (3.62) leads, after a few rearrangements, to the second FE:

$$-2\frac{\ddot{a}}{c^2 a} - \frac{1}{c^2}\left(\frac{8\,\pi\, G}{3}\rho(t) - \frac{Kc^2}{a(t)^2} + \frac{\Lambda\, c^2}{3}\right) - \frac{K}{a^2} + \Lambda = \frac{8\,\pi\, G}{c^4} p,$$

$$\frac{\ddot{a}(t)}{a(t)} + \frac{4\,\pi\,G}{3}\rho(t) - \frac{Kc^2}{2a(t)^2} + \frac{\Lambda\,c^2}{6} + \frac{Kc^2}{2a(t)^2} - \frac{\Lambda\,c^2}{2} = -\frac{4\,\pi\,G}{c^2}p$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\,\pi\,G}{3}\left(\rho(t) + 3\,\frac{p}{c^2}\right) + \frac{\Lambda\,c^2}{3}. \qquad \textbf{Second Friedmann equation} \qquad (3.63)$$

The comparison with the Newtonian approach shows that the term with the cosmological constant is missing in Eq. (3.13). With Eq. (3.63) the acceleration of the scale factor can be calculated, what we will use later to calculate the expansion force.

4 Densities in the universe and the resulting dynamics

4.1 Densities

First, let's look at the different components that contribute to the total density in the universe. The density $\rho(t)$ in the FE can be divided into two components:

$$\rho(t) = \rho_m(t) + \rho_r(t), \tag{4.1}$$

ρ_m: Matter density, consisting of baryonic (normal) and cold dark matter.

ρ_r: Radiation density, which was particularly dominant in the early phase of the universe.

Furthermore, we can rewrite the term with the cosmological constant $\frac{\Lambda\,c^2}{3}$ in the so-called vacuum density

$$\rho_v = \frac{\Lambda\,c^2}{8\,\pi\,G} = const \tag{4.2}$$

and thus include it as a further density component in the first FE,

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\,\pi\,G}{3}\rho_t - \frac{Kc^2}{a(t)^2}, \tag{4.3}$$

with a total density of

$$\rho_t = \rho_m + \rho_r + \rho_v. \tag{4.4}$$

Since we assume a flat universe today, i.e., $K = 0$, the question arises as to what value the total density must take in this case. This density is called critical density ρ_c. To derive it, we rearrange Eq. (4.3) as follows

$$\frac{3\,H(t)^2}{8\,\pi\,G} - \rho_t = -\frac{3Kc^2}{8\,\pi\,G\,a(t)^2}. \tag{4.5}$$

The total density must therefore have the value

$$\rho_t = \rho_c = \frac{3\,H(t)^2}{8\,\pi\,G}\,. \tag{4.6}$$

For this density, the left side of Eq. (4.5) equals zero, resulting in curvature $K = 0$, i.e., a flat universe.

With $K = 0$ and today's values, i.e. $t = t_0$, $H(t = t_0) = H_0$ and $a(t = t_0) = a_0 = 1$, Eq. (4.5) becomes

$$\frac{3\,{H_0}^2}{8\,\pi\,G} - \rho_t = 0, \tag{4.7}$$

and thus for the critical density at the present time $\rho_{c,0}$

$$\rho_{c,0} = \frac{3\,{H_0}^2}{8\,\pi\,G}. \tag{4.8}$$

With $H_0 = 67.4\ \frac{km}{s\,Mpc}$ and $G = 6.674 \cdot 10^{-11}\,\frac{m^3}{kg\,s^2}$ we obtain the value for the critical density as:

$\rho_{c,0} = 8.53 \cdot 10^{-27}\,\frac{kg}{m^3}$.

Dividing the matter, radiation, and vacuum densities by the critical density yields the dimensionless quantities:

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c}, \quad \Omega_\Lambda = \frac{\rho_v}{\rho_c} \tag{4.9}$$

and with today's values $\rho_{c,0}$, $\rho_{m,0}$, $\rho_{r,0}$, and $\rho_{v,0} = \rho_v = const$

$$\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{c,0}},\ \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{c,0}},\ \Omega_{\Lambda,0} = \frac{\rho_{v,0}}{\rho_{c,0}} = const. \tag{4.10}$$

The dimensionless total density is

$$\Omega_t = \frac{\rho_t}{\rho_c} = \Omega_m + \Omega_r + \Omega_\Lambda, \tag{4.11}$$

or for $t = t_0$

$$\Omega_{t,0} = \frac{\rho_{t,0}}{\rho_{c,0}} = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0}. \tag{4.12}$$

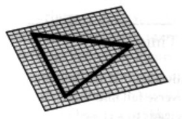
The following values are used ([1],[2]): $\Omega_{m,0} = 0.315,\ \Omega_{\Lambda,0} = 0.685,\ \Omega_{r,0} = 0.000055$

The spatial geometry of the universe is determined by the sum of the matter, radiation, and vacuum densities. We have already encountered the terms "flat", "closed", and "open" universe in Chapter 3 in connection with curvature K, and we want to explain these terms in a bit more detail here (see also [5]):

<u>Flat universe</u>

A flat universe has a flat or Euclidean geometry. The total density equals the critical density. Although the expansion of the universe is slowed down by gravity, it cannot reverse it. Thus, there won't be a so-called Big Crunch. The universe will expand endlessly.

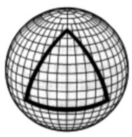
$K = 0,\ \Omega_t = 1,\ \rho_t = \rho_c$ => flat universe (4.13)



<u>Closed universe</u>

A closed universe has a positively curved geometry, like the surface of a three-dimensional sphere. The total density is greater than the critical density. The expansion of the universe is slowed down by gravity and reversed, meaning the universe collapses, the Big Crunch will occur.

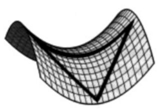
$K = 1,\ \Omega_t > 1,\ \rho_t > \rho_c$ => closed universe (4.14)



<u>Open universe</u>

An open universe has a negatively curved geometry, similar to that of a saddle or hyperbola. The total density is less than the critical density. There is no slowing down of the expansion, i.e., the universe expands infinitely.

$K = -1,\ \Omega_t < 1,\ \rho_t < \rho_c$ => open universe (4.15)



4.2 Relation between density and scale factor and the three phases in the evolution of the universe

The vacuum density is according to Eq. (4.2) a constant over time and therefore independent of the scale factor. However, this does not apply to the matter density ρ_m and the radiation density ρ_r. We want to derive this dependence on $a(t)$ now.

To do this, we will first derive an equation from the combination of the first and second FE that describes mass and energy conservation.

If we multiply the first FE by $a(t)^2$ and then differentiate this equation w.r.t. time, we obtain

$$\dot{a}^2 = \frac{8\,\pi\,G}{3}(\rho_m + \rho_r)a^2 - Kc^2 + \frac{\Lambda\,c^2 a^2}{3},$$

$$2\ddot{a}\dot{a} = \frac{8\,\pi\,G}{3}[(\dot{\rho}_m + \dot{\rho}_r)a^2 + (\rho_m + \rho_r)2a\dot{a}] + \frac{2\Lambda\,c^2 a\dot{a}}{3} \tag{4.16}$$

We multiply the second FE by $2a\dot{a}$:

$$2\ddot{a}\dot{a} = -\frac{8\,\pi\,G}{3}\left[(\rho_m + \rho_r) + 3\frac{p}{c^2}\right]a\dot{a} + \frac{2\Lambda\,c^2 a\dot{a}}{3}. \tag{4.17}$$

If we subtract Eq. (4.17) from (4.16), we get

$$0 = \frac{24\,\pi\,G}{3}(\rho_m + \rho_r)a\dot{a} + \frac{8\,\pi\,G}{3}(\dot{\rho}_m + \dot{\rho}_r)a^2 + \frac{24\,\pi\,G}{3}\frac{p}{c^2}a\dot{a}\,.$$

Multiplying by $\frac{3\,a\,c^2}{8\,\pi\,G}$ and rearranging yields

$$c^2[3(\rho_m + \rho_r)a^2\dot{a} + (\dot{\rho}_m + \dot{\rho}_r)a^3] = -3pa^2\dot{a},$$ which is identical to

$$c^2\frac{d}{dt}[a^3(\rho_m + \rho_r)] = -p\frac{d}{dt}(a^3)\,. \tag{4.18}$$

Eq. (4.18) is the sought energy balance, which states that the change in energy in a comoving volume element (left side) corresponds to the negative product of pressure and volume change.

This is the relativistic version of the 1st law of thermodynamics under constant entropy, which is a good approximation ($Tds = dE + pdV = 0 \quad => \quad dE = -pdV$).

With Eq. (4.18) it is now possible to represent the proportionality between the densities ρ_m and ρ_r and the scale factor. Additionally, we need the so-called equation of state parameter, which describes the ratio of pressure to energy density.

$$\omega = \frac{p}{\rho c^2}\,. \tag{4.19}$$

For matter this ratio is zero, i.e., $p = 0$. For a photon gas (homogeneous radiation) $\omega = \frac{1}{3}$ and therefore $p = \frac{1}{3}\rho_r c^2$ [3].

For a matter-dominated universe, meaning $\rho_r \ll \rho_m$ and $p = 0$, Eq. (4.18) becomes

$$\frac{d}{dt}(a^3\rho_m) = 0\,,$$

$$a^3\rho_m = const,$$

$$\rho_m(a) \sim \frac{1}{a^3}. \tag{4.20}$$

With the known value of the matter density today $\rho_{m,0}$, we obtain an equation for ρ_m as a function of the scale factor:

$$\rho_m(a) = \frac{\rho_{m,0}}{a^3}. \tag{4.21}$$

Dividing Eq. (4.21) by $\rho_{c,0}$ and considering Eq. (4.10) this becomes

$$\frac{\rho_m(a)}{\rho_{c,0}} = \frac{\frac{\rho_{m,0}}{\rho_{c,0}}}{a^3} = \frac{\Omega_{m,0}}{a^3}. \tag{4.22}$$

For a radiation-dominated universe, we have $\rho_m \ll \rho_r$. For the pressure we use the relationship shown above (Eq. (4.19) with $\omega = \frac{1}{3}$).

Substituting into Eq. (4.18) leads to

$$\frac{d}{dt}(a^3\rho_r) = -\frac{\rho_r}{3}\frac{d}{dt}(a^3)\,,$$

$$\frac{d\rho_r}{dt} a^3 + \rho_r \frac{d(a^3)}{dt} = -\frac{\rho_r}{3} \frac{d(a^3)}{dt} ,$$
$$\frac{d\rho_r}{dt} = -\frac{4}{3} \frac{1}{a^3} \rho_r 3a^2 \frac{da}{dt} ,$$
$$\frac{1}{\rho_r} d\rho_r = -4 \frac{1}{a} da ,$$
$$\ln(\rho_r) = -4 \ln(a) = \ln(a^{-4}) ,$$

and thus finally to the sought relationship

$$\rho_r(a) \sim \frac{1}{a^4}. \qquad (4.23)$$

Using the value of today's radiation density $\rho_{r,0}$, we obtain an equation for ρ_r:

$$\rho_r(a) = \frac{\rho_{r,0}}{a^4}. \qquad (4.24)$$

Dividing Eq. (4.24) by $\rho_{c,0}$ and considering Eq. (4.10) again yields

$$\frac{\rho_r(a)}{\rho_{c,0}} = \frac{\frac{\rho_{r,0}}{\rho_{c,0}}}{a^4} = \frac{\Omega_{r,0}}{a^4} . \qquad (4.25)$$

For the sake of completeness, it should be mentioned again that the vacuum density is constant and does not depend on the scale factor (see also Equation (4.2)):

$$\rho_v = \rho_{v,0} = const . \qquad (4.26)$$

One can therefore distinguish three phases in the development of the universe:

1. **Radiation dominated phase**: Shortly after the Big Bang, the radiant energy in the dense and hot universe was very large. Therefore, in this phase, ρ_r is significantly larger than ρ_m and ρ_v, but it decreases rapidly due to proportionality $\frac{1}{a^4}$ (Eq. (4.23)) and asymptotically approaches zero. After only about 10^6 years, the radiation density ρ_r is negligible compared to the matter density ρ_m.

2. **Matter-dominated phase**: Due to proportionality $\frac{1}{a^3}$, ρ_m also decreases over time and approaches zero, but slower than ρ_r. After approximately 10 billion years, the matter density ρ_m falls below the constant vacuum density ρ_v, which can be easily calculated using Eq. (4.22) and the known function $a(t)$.

3. ***Λ*-dominated phase**: The constant vacuum density ρ_v has been dominant for around 4 billion years, although the change to an accelerated expansion of the universe, began earlier, around 6 billion years ago. We will calculate this point later.

If we put the proportionalities given by Eqs. (4.20) and (4.23) as well as the constant vacuum density into the first FE (Eq. (3.62)) one after the other, we can specify for the three phases described above the basic development of the scale factor over time in these phases.

In the radiation-dominated area the FE is also proportional to $\frac{1}{a^4}$:

$$\left(\frac{\dot{a}}{a}\right)^2 \sim \frac{1}{a^4},$$

$$\frac{da}{dt} \sim \frac{1}{a},$$

$$a\,da \sim dt.$$

Integration yields

$$\int a\,da \sim \int dt\,,$$

$$\frac{1}{2}a^2 \sim t\,,$$

$$a(t) \sim \sqrt{t}. \qquad \textbf{Radiation-dominated area} \qquad (4.27)$$

In the matter-dominated area applies

$$\left(\frac{\dot{a}}{a}\right)^2 \sim \frac{1}{a^3},$$

$$\sqrt{a}\,da \sim dt.$$

We integrate again

$$\int \sqrt{a}\,da \sim \int dt\,,$$

$$\frac{2}{3}a^{\frac{3}{2}} \sim t\,,$$

$$a(t) \sim t^{\frac{2}{3}}. \qquad \textbf{Matter-dominated area} \qquad (4.28)$$

In the **Λ**-dominated area, where the constant vacuum density ρ_v dominates, we find

$$\left(\frac{\dot{a}}{a}\right)^2 \sim \frac{\Lambda\, c^2}{3},$$

$$\frac{1}{a}\frac{da}{dt} \sim \sqrt{\frac{\Lambda\, c^2}{3}},$$

$$\frac{1}{a}\, da \sim \sqrt{\frac{\Lambda\, c^2}{3}}\, dt.$$

We integrate and obtain

$$\int \frac{1}{a}\, da \sim \int \sqrt{\frac{\Lambda\, c^2}{3}}\, dt\,,$$

$$\ln(a) \sim \sqrt{\frac{\Lambda\, c^2}{3}}\, t\,,$$

$$a(t) \sim e^{\sqrt{\frac{\Lambda\, c^2}{3}}\, t}. \qquad \boldsymbol{\Lambda}\textbf{-dominated area} \qquad (4.29)$$

4.3 Force of Expansion

Already in Chapter 3, we posed the question of whether the expansion of the universe proceeds at a constant speed ($F = 0$), accelerated ($F > 0$), or decelerated ($F < 0$).

According to the Big Bang theory, after an initial inflation period, the expansion of the universe initially slowed down during the first billion years. Approximately 6 billion years ago, a reversal of the expansion occurred, leading to the change from decelerated to accelerated expansion. We can already surmise that this change is associated with the decrease in the density of matter ρ_m – as per Eq. (4.20) – and thus the increase in the dominance of dark energy, i.e., the constant vacuum energy density ρ_v. We want to take a closer look at this with the help of the second FE.

With Eq. (3.10) we defined already the force that is responsible for the expansion in the universe:

$$F = m \cdot r \cdot \frac{\ddot{a}}{a}.$$

The term $\frac{\ddot{a}}{a}$ represents precisely the second FE. As with the first FE in the form of Eq. (4.3), we want to use the total density $\rho_t = \rho_m + \rho_r + \rho_v$ and also interpret the pressure as a total pressure $p_t = p_m + p_r + p_v$, which represents the three different sources of pressure in the universe, such as radiation, matter, and dark energy.

With this we can write Eq. (3.63) in the form

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\,\pi\,G}{3}\left[\rho_m + \rho_r + \rho_v + \frac{3}{c^2}(p_m + p_r + p_v)\right]. \tag{4.30}$$

Since we have integrated the Λ-term, i.e., the vacuum energy, into ρ_v and p_v, the comparison of Eq. (4.30) with Eq. (3.63) using also Eq. (4.2) leads to

$$-\frac{4\,\pi\,G}{3}\rho_v - \frac{4\,\pi\,G}{c^2}p_v = \frac{\Lambda\,c^2}{3},$$

$$-\rho_v - \frac{3}{c^2}p_v = \frac{\Lambda\,c^2}{4\,\pi\,G} = 2\frac{\Lambda\,c^2}{8\,\pi\,G} = 2\rho_v\,,$$

$$\rho_v = -\frac{p_v}{c^2}. \tag{4.31}$$

Substituting in Eq. (4.30) p_v with $-\rho_v c^2$ and taking into account that radiation pressure and radiation density were only relevant in the early phase of the evolution of the universe, and also neglecting p_m, Eq. (4.30) becomes

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\,\pi\,G}{3}(\rho_m - 2\rho_v)\,. \tag{4.32}$$

Using this in Eq. (3.10) gives us an equation to calculate the force acting in the universe:

$$F = -m \cdot r\,\frac{4\,\pi\,G}{3}(\rho_m - 2\rho_v). \tag{4.33}$$

This equation explains the aforementioned reversal of the expansion. The force F changes from an attractive (negative) to a repulsive (positive) force, causing the accelerated expansion of the universe once

$$\rho_m < |-2\rho_v|. \tag{4.34}$$

What is interesting is the point in time when this reversal occurred, i.e., when was

$$\rho_m = 2\rho_v. \tag{4.35}$$

When we substitute the relationships $\rho_m(a) = \frac{\Omega_{m,0} \cdot \rho_{c,0}}{a^3}$ and $\rho_v = \Omega_{\Lambda,0} \cdot \rho_{c,0}$, derived from equations (4.10), (4.22), and (4.26), into equation (4.35) we obtain

$$\frac{\Omega_{m,0}}{a^3} = 2\,\Omega_{\Lambda,0}\,,$$

$$a = \sqrt[3]{\frac{\Omega_{m,0}}{2\,\Omega_{\Lambda,0}}}. \tag{4.36}$$

With the values based on observations and analyzes of cosmological development (cosmological parameters of the Planck measurements 2018 [1]) for the dimensionless matter density $\Omega_{m,0} = 0.315$ and the vacuum density $\Omega_{\Lambda,0} = 0.685$, we get a scale factor of $a = 0.6126$. In Chapter 5 we will derive a function $t(a)$ (Eq. (5.12)), allowing us to assign a time to this value. Anticipating this result, let us mention here the age of the universe at which the change in expansion occurred:

$$t(a = 0.6126) = 7.7 \cdot 10^9\ years. \tag{4.37}$$

5 Special solution of the Friedmann equation

We start with the first FE in the form of Eq. (4.3) and take into account Eq. (4.4)

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\,\pi\,G}{3}(\rho_r + \rho_m + \rho_v) - \frac{Kc^2}{a(t)^2}\,. \tag{5.1}$$

If we want to use the densities in their dimensionless form, we can rewrite Eq. (5.1) using Eq. (4.9) as follows:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\,\pi\,G}{3}(\Omega_r + \Omega_m + \Omega_\Lambda)\,\rho_c - \frac{Kc^2}{a(t)^2}\,, \tag{5.2}$$

and with Eq. (4.6) this becomes

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = H(t)^2(\Omega_r + \Omega_m + \Omega_\Lambda) - \frac{Kc^2}{a(t)^2}\,. \tag{5.3}$$

To be able to drag the curvature term $-\frac{Kc^2}{a(t)^2}$ into the parenthesis, we define a curvature component of the form

$$\Omega_K \equiv -\frac{Kc^2}{H(t)^2 a(t)^2} \tag{5.4}$$

and thus $$\Omega_K\, H(t)^2 = -\frac{Kc^2}{a(t)^2}\,, \tag{5.5}$$

or for $t = t_0$ (i.e. today) and thus $H(t = t_0) = H_0$and $a(t = t_0) = a_0 = 1$

$$-Kc^2 = \Omega_K\, H_0^2 \quad \text{or} \quad -\frac{Kc^2}{a(t)^2} = \frac{\Omega_K\, H_0^2}{a(t)^2}\,. \tag{5.6}$$

Substituting Eq. (5.5) into (5.3), cancelling $H(t)^2$ on both sides, and considering Eq. (4.11), we obtain the interesting relationship

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_K = \Omega_t + \Omega_K\,. \tag{5.7}$$

From this, we can immediately see that a flat universe, i.e., $K = 0$ (and hence $\Omega_K = 0$) corresponds to the dimensionless total density $\Omega_t = \frac{\rho_t}{\rho_c} = 1$, meaning the total density equals the critical density, $\rho_t = \rho_c$.

If we divide the densities in Eq. (5.1) by $\rho_{c,0}$ (Eq. (4.8)) instead of ρ_c, this leads to

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\,\pi\,G}{3}\left(\frac{\rho_r}{\rho_{c,0}} + \frac{\rho_m}{\rho_{c,0}} + \frac{\rho_v}{\rho_{c,0}}\right)\rho_{c,0} - \frac{Kc^2}{a(t)^2}\,,$$

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = H_0^2\left(\frac{\rho_r}{\rho_{c,0}} + \frac{\rho_m}{\rho_{c,0}} + \frac{\rho_v}{\rho_{c,0}}\right) - \frac{Kc^2}{a(t)^2}\,,$$

and with the Eqs. (4.10), (4.22), (4.25), and (5.6) to

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H(t)^2 = H_0^2\left(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{\Omega_K}{a(t)^2}\right). \tag{5.8}$$

If one assumes a flat universe, i.e., $\Omega_K = 0$, and neglects the radiation density, which is small compared to the other densities and was only relevant in a short phase after the Big Bang, Eq. (5.8) simplifies to

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H_0^2\left(\frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0}\right). \tag{5.9}$$

By further rearranging and integrating this equation, we obtain the function $t(a)$:

$$\frac{1}{a}\frac{da}{dt} = H_0\sqrt{\frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0}}\,, \tag{5.10}$$

$$dt = \frac{1}{H_0}\,\frac{da}{a\cdot\sqrt{\frac{\Omega_{m,0}}{a^3}+\Omega_{\Lambda,0}}}\,,$$

$$t = \frac{1}{H_0}\int_{a=0}^{a}\frac{\sqrt{a}\,da}{\sqrt{\Omega_{m,0}+\Omega_{\Lambda,0}\,a^3}}\,. \tag{5.11}$$

With the substitution $x = a^{\frac{3}{2}}$ and consequently $a^3 = x^2$, $\frac{dx}{da} = \frac{3}{2}\sqrt{a}$, and $da = \frac{2}{3\sqrt{a}}dx$, Eq. (5.11) can be rewritten

$$t = \frac{2}{3H_0\sqrt{\Omega_{\Lambda,0}}}\int_{x=0}^{x}\frac{dx}{\sqrt{\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}+x^2}}\,,$$

with the solution

$$t = \frac{2}{3H_0\sqrt{\Omega_{\Lambda,0}}}\,\mathrm{arsinh}\left(\frac{x}{\sqrt{\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}}}\right),$$

or

$$t(a) = \frac{2}{3H_0\sqrt{\Omega_{\Lambda,0}}}\,\mathrm{arsinh}\left(\sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}}\;a^{\frac{3}{2}}\right). \tag{5.12}$$

We had already used this equation to obtain the result in Eq. (4.37). If we take into account that the scale factor today, i.e., at $t = t_0$, has the value of one, Eq. (5.12) gives us an age of the universe of $t(a_0) = t_0 = 13.796$ billion years.

Solving for the scale factor gives

$$a(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2}\ t\right). \tag{5.13}$$

The Hubble parameter can easily be obtained from the relationship Eq. (2.9), $H(t) = \frac{\dot{a}(t)}{a(t)}$, and Eq. (5.13):

$$H(t) = \frac{1}{a}\frac{da}{dt} = \frac{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3}} \frac{2}{3} \sinh^{-\frac{1}{3}}\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2} t\right) \cdot \cosh\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2} t\right) \cdot \frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2}}{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2} t\right)},$$

$$H(t) = H_0\sqrt{\Omega_{\Lambda,0}}\ \cdot\ \coth\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2} \cdot t\right). \tag{5.14}$$

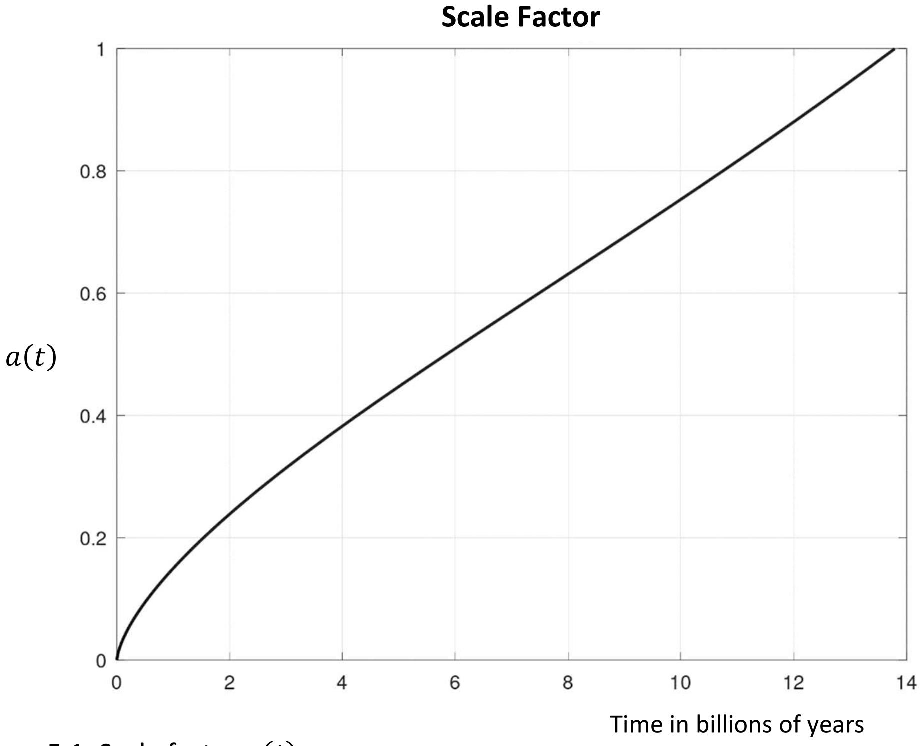
5.1 Scale factor, Hubble radius, horizons, light cones and worldlines

After obtaining a solution for the scale factor and thus also for the Hubble parameter in the previous chapter, we will utilize these results in this chapter to gain a better understanding of the dynamic processes in the expanding universe.

For numerical integrations, derivatives, etc. and all plots, the program **GNU Octave** was utilized.

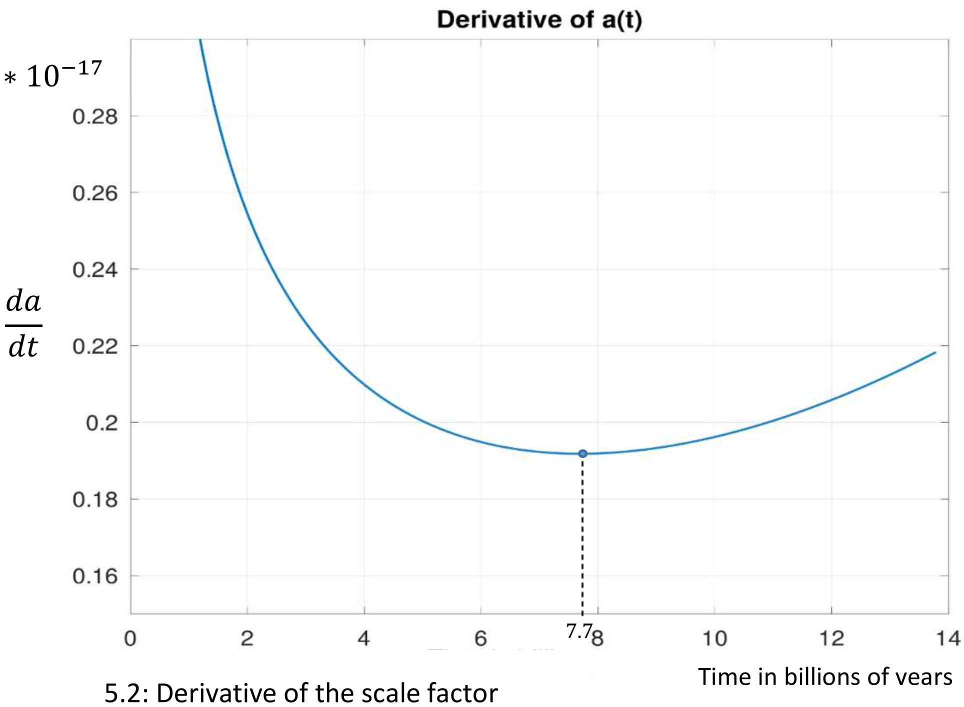
Scale factor

Figure 5.1 shows the scale factor over time (Eq. (5.13)). From the shape of the curve, we can discern that between 6 and 8 billion years after the Big Bang the expansion transitions from a decelerated to an accelerated expansion.



5.1: Scale factor $a(t)$

We already know that this moment occurred $7.7 \cdot 10^9\ years$ after the Big Bang (see Eq. (4.37)). This event becomes more apparent when differentiate the function $a(t)$ w.r.t. time, see Fig. 5.2.



5.2: Derivative of the scale factor

Hubble radius

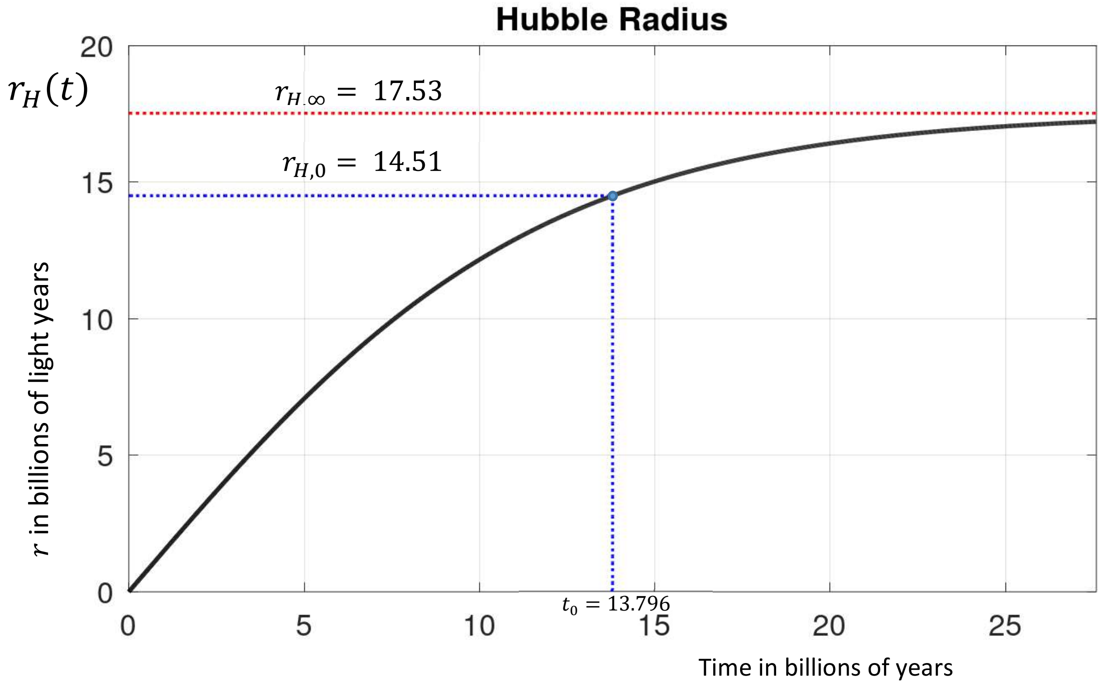
With Eq. (2.10) we found already an equation for the universe's recession velocity. If we want to determine at what distances $r(t)$ the recession velocity was, is and will be equal to the speed of light, we substitute v_F in Eq. (2.10) by the speed of light c and rearrange the equation to the radius. This radius is referred to as the Hubble radius r_H:

$$r_H(t) = \frac{c}{H(t)}. \tag{5.15}$$

Using today's Hubble constant $H_0 \approx 67.4 \frac{km}{s \cdot Mpc}$ (from [1]), this calculates a distance of

$$r_{H,0} = 4464.52\ Mpc\ \widehat{=}\ 14.51 \cdot 10^9\ ly. \tag{5.16}$$

To represent the time evolution of the Hubble radius, we substitute Eq. (5.14) into Eq. (5.15) and obtain the evolution illustrated in Fig. 5.3



5.3: Hubble radius. Also shown are the values of the Hubble radius at the present time, $r_{H,0}$, and the limit $r_{H,\infty}$ towards which the Hubble radius tends.

In Fig. 5.3, the limit $r_{H,\infty}$ is also indicated, towards which the Hubble radius tends. It is obtained from Eq. (5.9), taking into account that as $t \to \infty$, the term $\frac{\Omega_{m,0}}{a^3}$ tends to zero, and thus

$$H_\infty = H_0\sqrt{\Omega_{\Lambda,0}}\ . \tag{5.17}$$

If we substitute Eq. (5.17) into Eq. (5.15), the limit value $r_{H,\infty}$ can be calculated.

$$r_{H,\infty} = \frac{c}{H_0\sqrt{\Omega_{\Lambda,0}}}\ . \tag{5.18}$$

We get the value $r_{H,\infty} = 17.528\ billion\ light - years.$

Since “stationary” objects (i.e., without peculiar velocity) located on the surface of the sphere with the radius $r_H(t)$ move away from us at the speed of light due to the expansion of space, the velocity of objects within the sphere is less than *c* and outside it is greater than *c*. However, under certain conditions, we can still receive information from objects located outside the sphere with the radius

r_H. There is, however, a distance known as the event horizon r_E, from which photons take an infinite amount of time to reach us (i.e., never). We'll soon see how this is calculated.

First, some preliminary considerations about the distances traveled by photons in an expanding universe. If we were dealing with a static universe, such a distance could be calculated using the simple relationship $r = \int_{t=t_1}^{t_2} c\,dt = c \cdot (t_2 - t_1)$. But since we live in an expanding universe, we have to continuously correct the distance $c \cdot dt$ during the integration with the time-dependent scale factor:

$$r_{co} = \int_{t=t_S}^{t_F} \frac{c\,dt}{a(t)}. \tag{5.19}$$

This is the so-called **co**moving distance, a distance that a comoving observer would measure. However, we can use the corresponding scale factor to convert the distance at the times t_S or t_F to a stationary observer (e.g. us). We call this distance the proper distance. The Eq. (5.19) would only provide a proper distance without the need for correction if $t_F = t_0$ (i.e., today) and we wanted to know the distance to us at the current time (the scale factor would then be $a_0 = 1$ and thus would not change the value of the integral).

By scaling the comoving distance by the scale factor, we get the following two distances:

1. **Distance** of an object moving with the expansion of space **at time t_F**, which emitted photons at time t_S that arrive at the stationary observer (us) at time t_F. It can be helpful for understanding to consider the photon-emitting object as stationary (the determination what is at rest and what is in motion is arbitrary anyway). This stationary object emits photons in our direction at time t_S. They arrive at us at the time t_F taking into account the intrinsic velocity of the photons and the expansion of space. The total distance traveled is $r_{F,pr}(t_F)$.

$$r_{F,pr}(t_F) = a(t_F) \cdot r_{co} = a(t_F) \int_{t=t_S}^{t_F} \frac{c\,dt}{a(t)}. \tag{5.20}$$

2. **Distance** of an object **at time t_S** that emitted photons at that time, which arrive at the stationary observer at time t_F.

$$r_{S,pr}(t_S) = a(t_S) \cdot r_{co} = a(t_S) \int_{t=t_S}^{t_F} \frac{c\,dt}{a(t)}. \tag{5.21}$$

If we want to show a progression over time, we need to perform calculations step by step from t_S to t_F using Eqs. (5.19), (5.20), and (5.21) to obtain enough data points for the graphic representation of a temporal progression of the radii:

$$r_{co}(i) = \int_{t_S}^{t(i)} \frac{c\,dt}{a(t)}\,,\; r_{F,pr}(i) = a(t_i) \int_{t_S}^{t(i)} \frac{c\,dt}{a(t)}\,,\; r_{S,pr}(i) = a(t_i) \int_{t(i)}^{t_F} \frac{c\,dt}{a(t)}\,, \tag{5.22}$$

with $i = 1 \ldots i_{max}$, $\Delta t = \frac{t_{max} - t_{min}}{i_{max} - 1}$, $t(i) = t_{min} + \Delta t(i-1)$

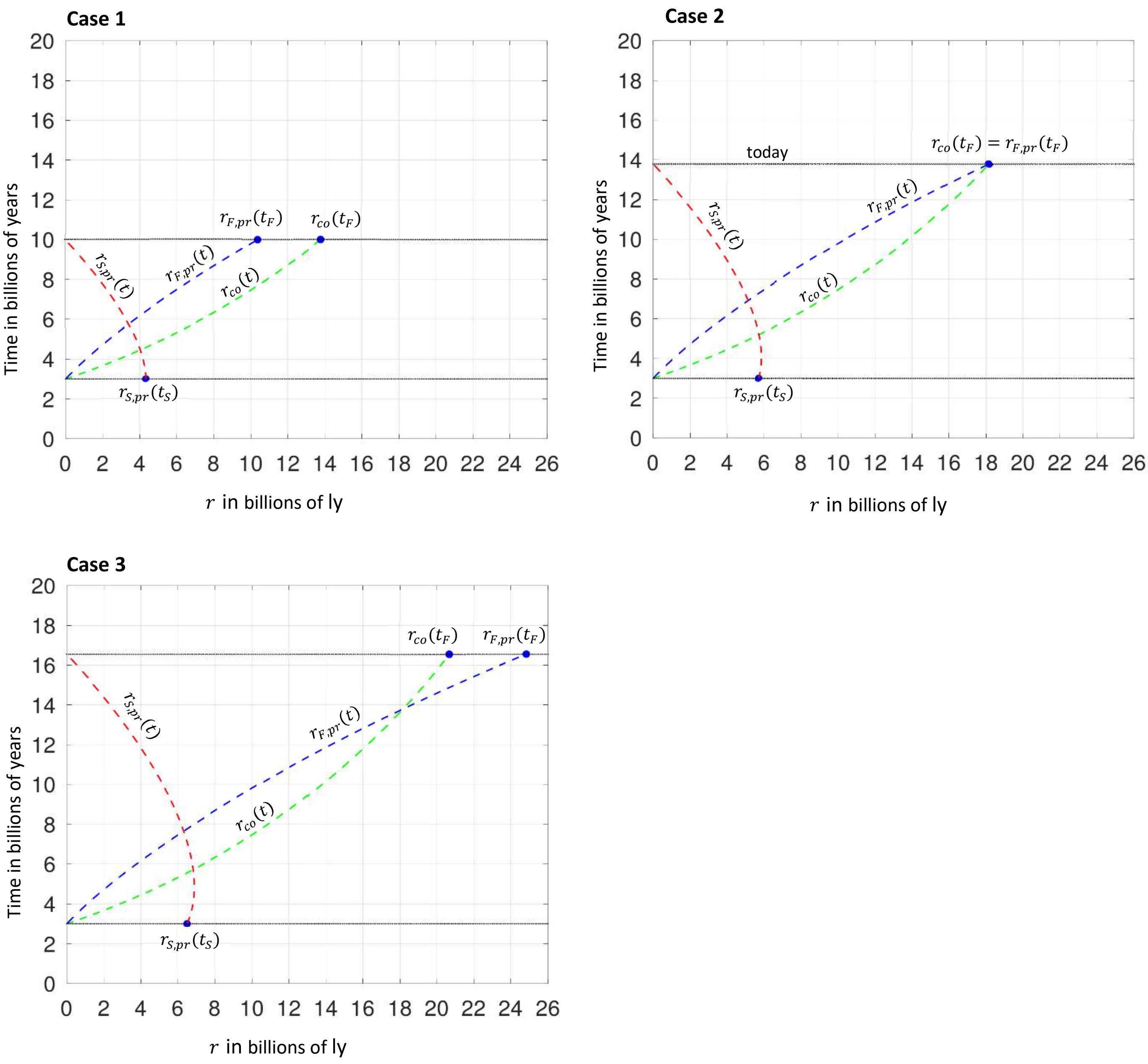
t_{min} and t_{max} must be determined according to the case under investigation.

In Figure 5.4, 3 cases are distinguished (t_S is fixed, t_F varies):

Case 1: $t_F < t_0$: The scale factor $a(t_i) < 1$.

Case 2: $t_F = t_0$: $a(t_i)$ reaches at $t_F = t_0$ the value one, i.e., $a(t_F) = a_0 = 1$, hence at time t_F, as already mentioned above, $r_{co}(t_F) = r_{F,pr}(t_F)$.

Case 3: $t_F > t_0$: $a(t_i) > 1$ after $t_F = t_0$. Hence $r_{F,pr}(t)$ is greater than $r_{co}(t)$ after t_0.



5.4: Distances from the perspective of a moving observer (comoving distance) and a stationary observer (proper distance). There are 3 cases: 1) $t_F < t_0$, 2) $t_F = t_0$ and 3) $t_F > t_0$.

Combining Eqs. (5.20) and (5.21), as already shown in Eqs. (2.5a/b), the distances $r_{F,pr}(t_F)$ and $r_{S,pr}(t_S)$ can be easily converted into each other using the scale factors:

$$r_{S,pr}(t_S) = \frac{a(t_S)}{a(t_F)} r_{F,pr}(t_F), \tag{5.23}$$

$$r_{F,pr}(t_F) = \frac{a(t_F)}{a(t_S)} r_{S,pr}(t_S). \tag{5.24}$$

From Eqs. (5.20) and (5.21), special cases, known as horizons and the past light cone can be derived depending on the choice of integration limits.

<u>Event horizon</u> $\boldsymbol{r_{E,pr}}$: Eq. (5.21) with $t_F = \infty$

The event horizon is the distance of an object at time t_S, that emitted photons at that time, which will never reach us, even after an infinitely long time.

$$r_{E,pr}(t_S) = a(t_S) \int_{t=t_S}^{\infty} \frac{c\,dt}{a(t)}. \tag{5.25}$$

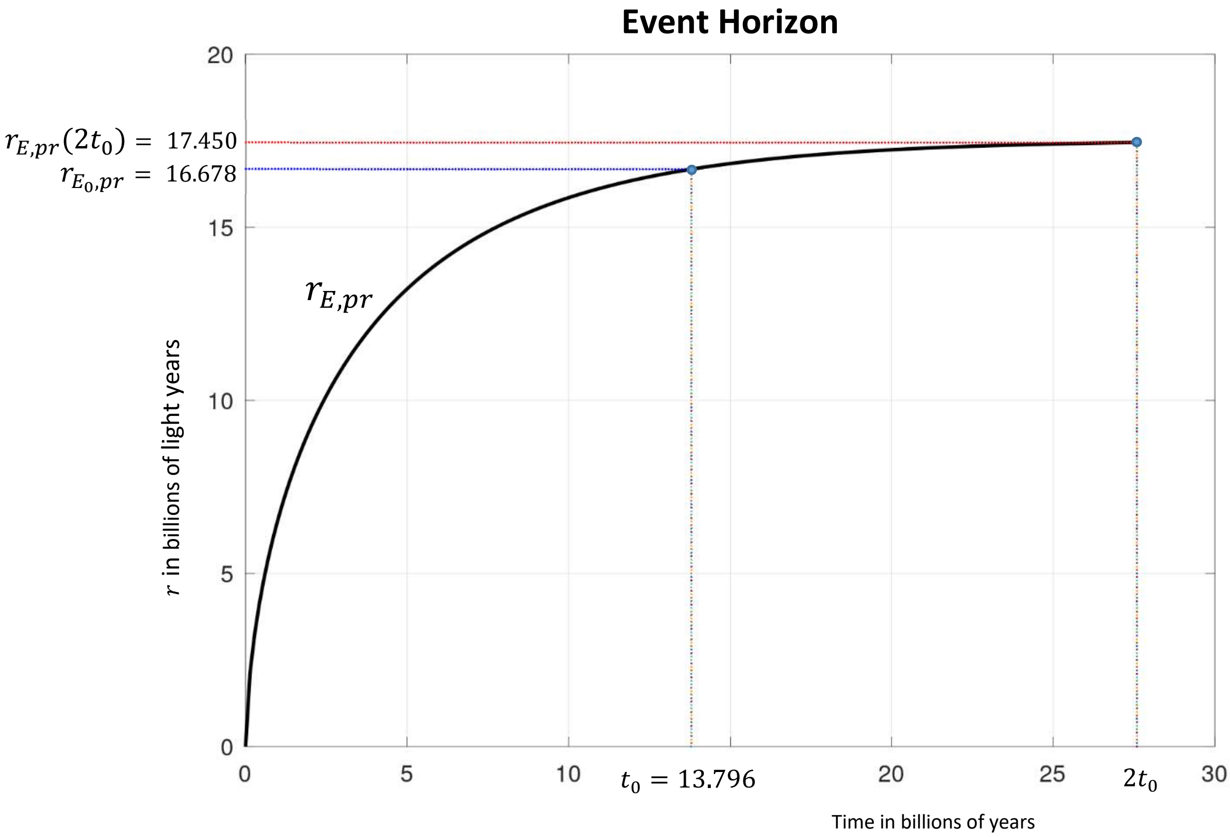
For the distance an object has to us today, we set $t_S = t_0$ and thus $a(t_S) = a_0 = 1$.

$$r_{E_0,pr} = \int_{t=t_0}^{\infty} \frac{c\,dt}{a(t)}. \tag{5.26}$$

This results in a value of $r_{E_0,pr} = 16.678\ billion\ ly$.

From objects located on the surface of this sphere (with us at the center) or further away from us today, we will no longer receive any information.

As shown in Eq. (5.22), we can also calculate and illustrate the temporal evolution of the event horizon. From the trend of the curve in Fig. 5.5, it can already be inferred that the event horizon also converges to a limit value.



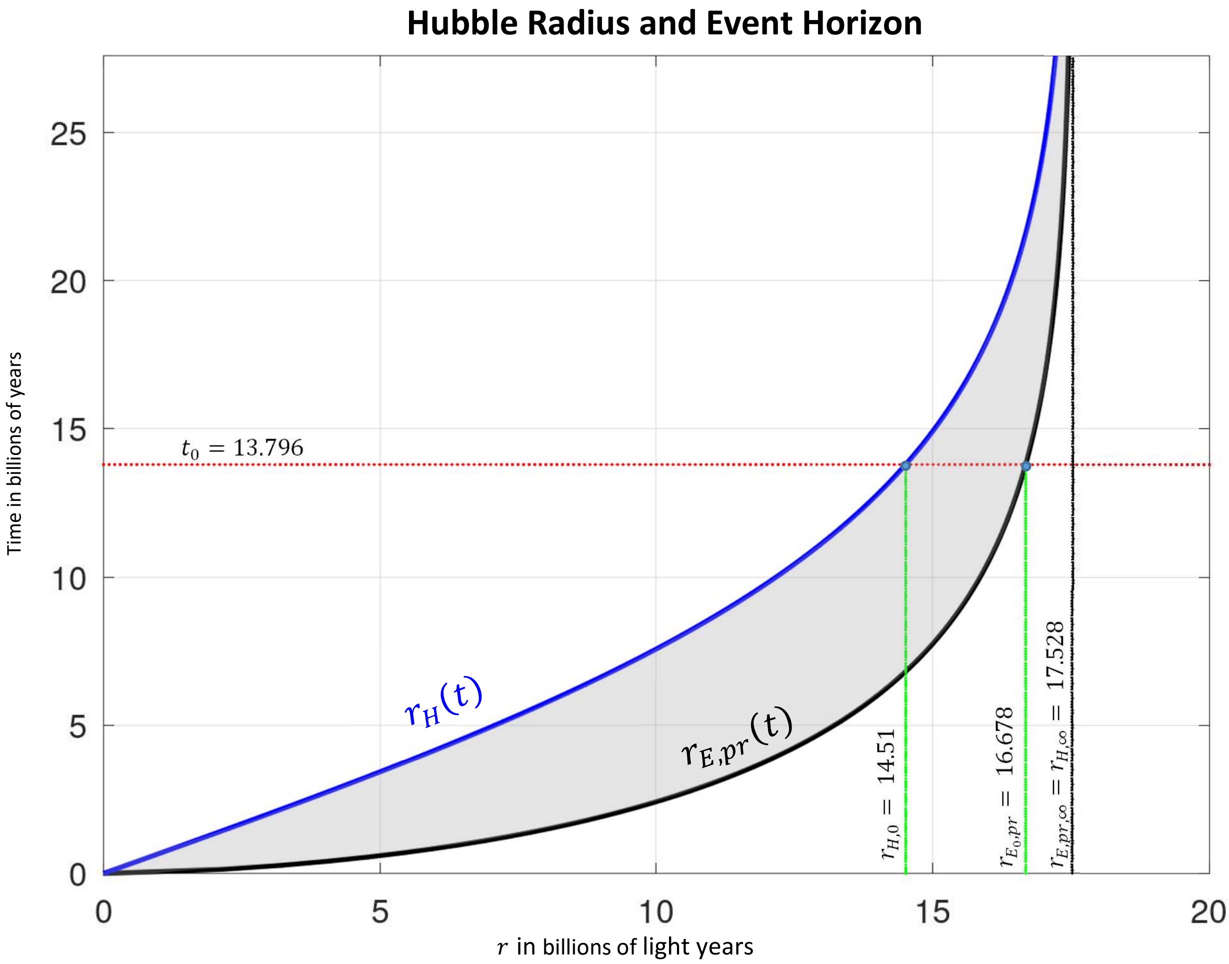
5.5: Event horizon $r_{E,pr}(t_S)$. The values for today and for $2t_0$ are indicated.

In Table 5.1, values for $t_S = t_0$ up to $t_S = 5t_0$ with the respective increase in percentage are listed. We see that the increases rapidly diminish, and $r_{E,pr}$ tends toward the same limit as the Hubble radius $r_{E,pr,\infty} = r_{H,\infty} = 17.528$: billions of light years.

t_S	**rE,pr** (10^9 ly)	**Growth** (%)
t_0	16,678	
$2t_0$	17,450	4,627
$3t_0$	17,521	0.406
$4t_0$	17,528	0.038
$5t_0$	17,528	0.004

5.1: Values for the event horizon $r_{E,pr}(t_S)$ for $t_S = t_0$ up to $t_S = 5t_0$.
The increase is given in percent, which diminishes very quickly, i.e., $r_{E,pr}$ approaches the limit value $r_{E,pr,\infty} = 17.528$ billions of ly.

In Fig. 5.6 the Hubbel radius and the event horizon are shown together in a spacetime diagram, where the time axis is now vertical and the space axis is horizontal.



5.6: Spacetime diagram with the Hubbel radius and the event horizon.

We have already discussed in the context of Eq. (5.15) that under certain conditions, we can receive information from objects located outside the sphere with the radius r_H, i.e., those having a recessional velocity > *c*. This condition can now be easily interpreted from Fig. 5.6: These objects must lie within the gray shaded region shown in the figure. We will see an example of this further below.

Particle horizon $\boldsymbol{r_{P,pr}}$: Eq. (5.20) with $t_S = 0$ (or $t_S = t_{CMB} \approx 380000\ years$)

Note: Often, instead of zero, the lower time limit is chosen as the time of recombination t_{CMB}, which is the moment when the universe became transparent. It is the earliest time from which we receive information in the form of photons. We refer to this photon radiation received today as the Cosmic Microwave Background Radiation (CMB).

The particle horizon is the distance of an "object" at a time t_F, that emitted photons, for example, at the time $t_S = 0$ (theoretically), which arrive at our location at the time t_F.

$$r_{P,pr}(t_F) = a(t_F) \int_{t=0}^{t_F} \frac{c\,dt}{a(t)}\,. \tag{5.27}$$

For the distance that such an "object" has to us today, we set $t_F = t_0$ and thus $a(t_F) = a_0 = 1$.

$$r_{P_0,pr} = \int_{t=0}^{t_0} \frac{c\,dt}{a(t)}. \tag{5.28}$$

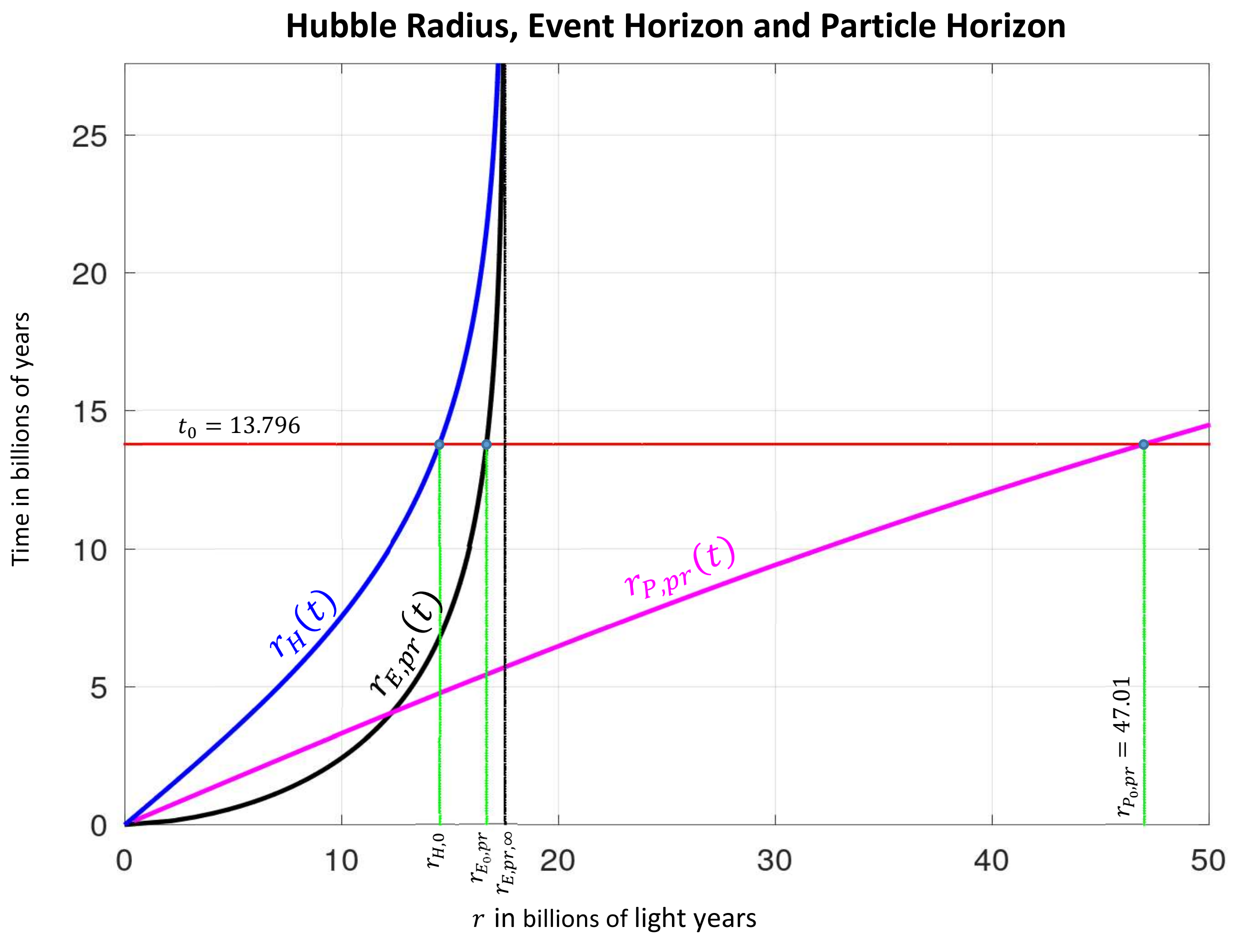
This results in a value of 47.01 billion ly. This distance also defines the limit of our **observable universe**. However, the received information is not current, but as old as the universe itself, which is $t_0 = 13.796\ billion\ years$ old.

For $t_S = t_{CMB} \approx 380000\ years$

$$r_{P_{CMB},pr} = \int_{t=t_{CMB}}^{t_0} \frac{c\,dt}{a(t)} \tag{5.29}$$

We obtain a distance of $r_{P_{CMB},pr} = 45.56\ billion\ ly$

If we include the particle horizon in the space-time diagram of Fig. 5.6, it looks as follows:



5.7: Space-time diagram with the Hubbel radius, the event horizon and the particle horizon

Past light cone $r_{LC,pr}$: Eq. (5.21)

As the name suggests, the past light cone gives us the distance of an object at time t_S, i.e., in the past, that at that time emitted photons which reach the observer at rest at time t_F.

$$r_{LC,pr}(t_S) = a(t_S)\int_{t=t_S}^{t_F}\frac{c\,dt}{a(t)}. \qquad (5.30)$$

Generally, we are interested in the distances of objects in the past, whose photons arrive at our location today, that is, at the current time $t_F = t_0$.

$$r_{LC_0,pr}(t_S) = a(t_S)\int_{t=t_S}^{t_0}\frac{c\,dt}{a(t)}. \qquad (5.31)$$

If we substitute the time of recombination, $t_S = t_{CMB} \approx 380000\ years$, into the equation, we obtain the distance that the photons of the CMB radiation had to "us" at that time (we didn't exist at that time, so it refers to the "stationary" location that we occupy today, from which we determine proper distances in the universe).

$$r_{LC_0,pr}(t_{CMB}) = a(t_{CMB})\int_{t=t_{CMB}}^{t_0}\frac{c\,dt}{a(t)}. \qquad (5.32)$$

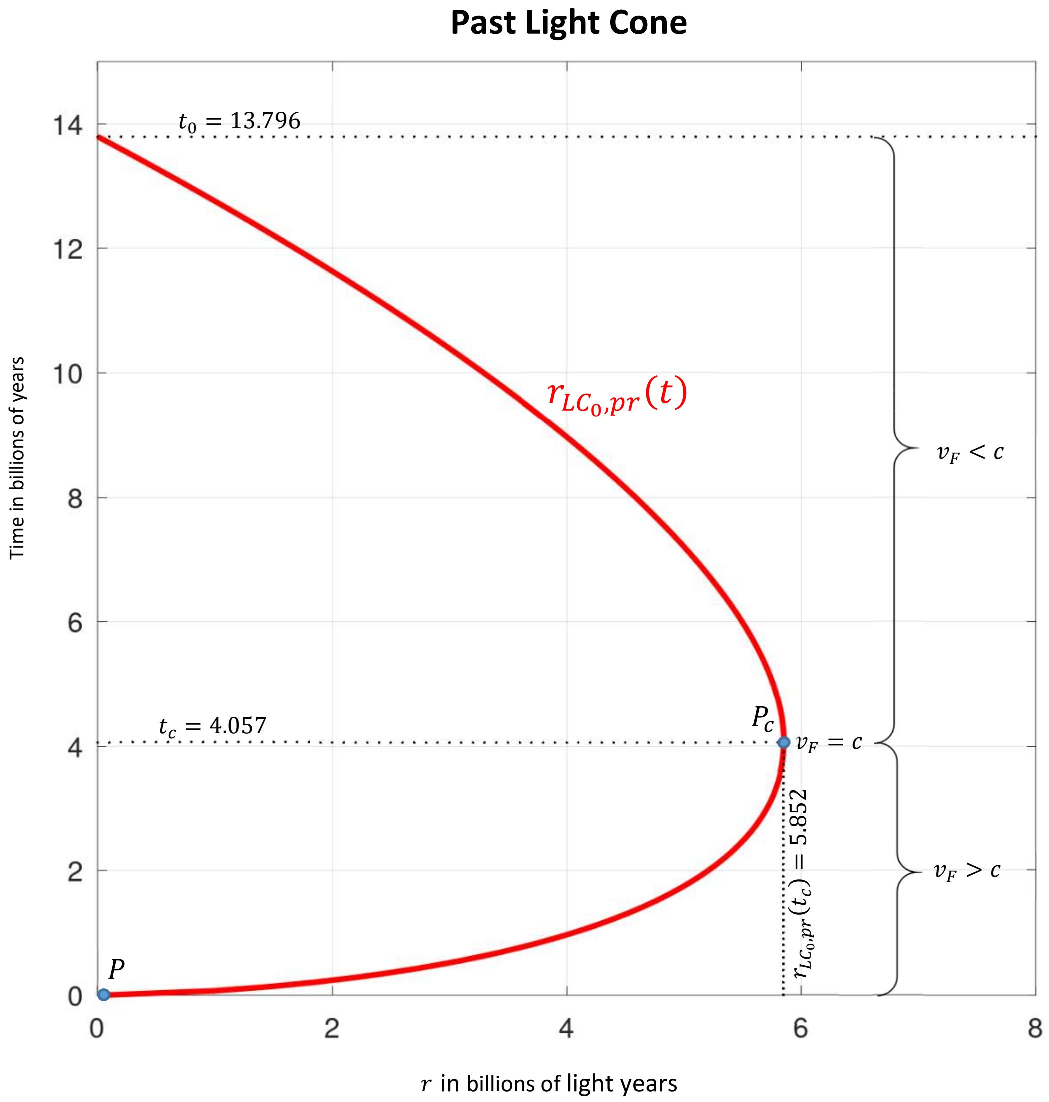
This results in a distance of $r_{LC_0,pr}(t_{CMB}) = 3.6\cdot 10^{-2} billion\ ly$. Since we calculated already the particle horizon for $t_S = t_{CMB} \approx 380000\ years$, using Eq. (5.29), we can also calculate $r_{LC_0,pr}(t_{CMB})$ using Eq. (5.23):

$$r_{LC_0,pr}(t_{CMB}) = \frac{a(t_{CMB})}{a_0} r_{P_{CMB},pr}\ . \qquad (5.33)$$

we calculate $a(t_{CMB})$ with Eq. (5.13): $a(t_{CMB}) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2}\ t_{CMB}\right) = 7.863737\cdot 10^{-4}$ and can therefore confirm the above value.

$$r_{LC_0,pr}(t_{CMB}) = \frac{7.863737\cdot 10^{-4}}{1} 45.56\ billion\ ly = 3.6\cdot 10^{-2} billion\ ly.$$

Fig. 5.8 shows the temporal evolution of the past light cone.



5.8: Spacetime diagram with past light cone $r_{LC_0,pr}(t)$. Starting point is $P\left(r_{LC_0,pr}(t_{CMB}), t_{CMB}\right)$. In point $P_c\left(r_{LC_0,pr}(t_c), t_c\right)$ the expansion velocity of space is equal to the speed of light.

The trajectory of the light cone shows that initially, the photons directed towards the stationary observer, which were only 36 million light-years away from it at the starting point *P*, move away from it due to the high expansion rate of space. Only at the time $t_c = 4.057$ billion years does the direction of motion reverse. At this point, the recession velocity v_F is equal to the speed of light *c*. Subsequently, the velocity of the expansion of space continues to decrease, reaching zero at the location of the stationary observer today (i.e., at the time t_0).

Using Eq. (2.10) in combination with Eq. (5.14), we can calculate the corresponding escape velocity for the radii of the light cone obtained with Eq. (5.30). As already indicated in Fig. 5.8 (point P_c), at $t_c = 4.057$ billion years, it is equal to the speed of light:

$$v_F(t_c = 4.057 \cdot 10^9\, y) = H(t_c) \cdot r(t_c)\,,$$

$$v_F(t_c) = H_0\sqrt{\Omega_{\Lambda,0}} \;\cdot\; coth\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2} \cdot t_c\right) \cdot a(t_c) \int_{t=t_c}^{t_0} \frac{c\,dt}{a(t)}\,,$$

$$v_F(t_c) = 167.080201560149 \frac{km}{s \cdot Mpc} \cdot 5.85223343\ GLj \cdot \frac{306.6013938343741 Mpc}{GLj}$$

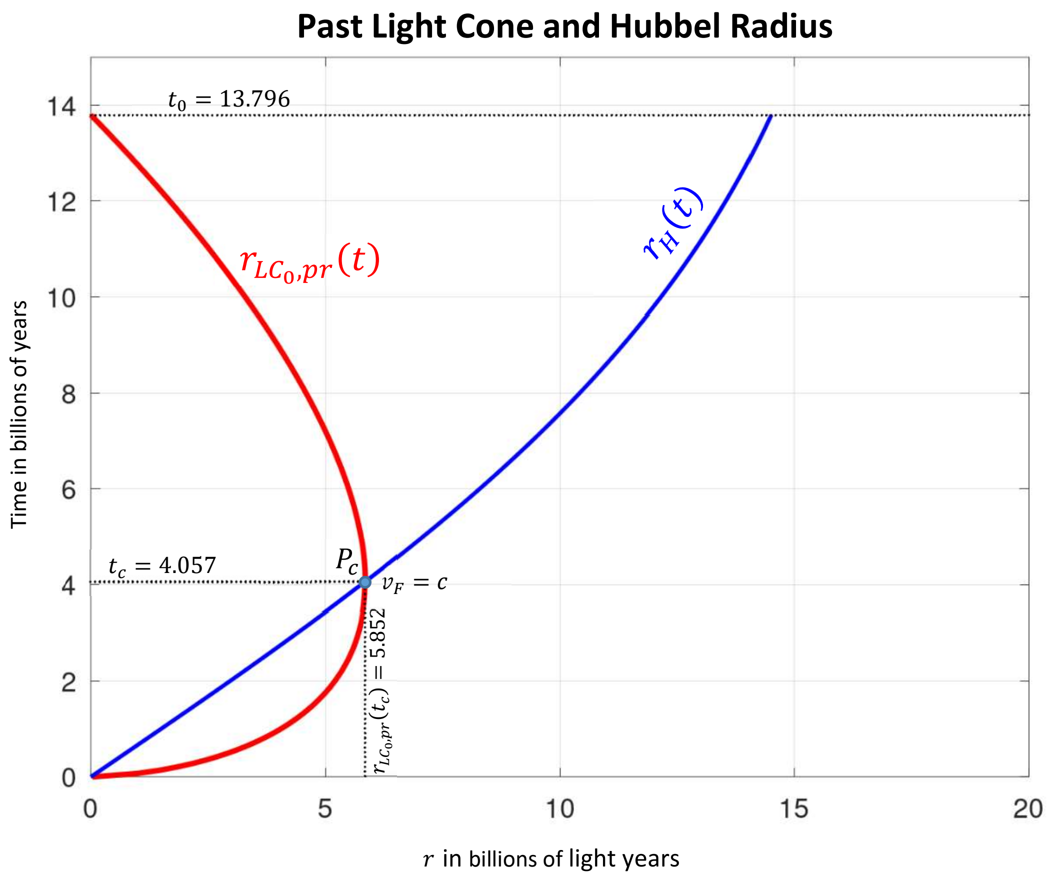
$$v_F(t_c) = 299792.5 \frac{km}{s} = c\ .$$

Let's do the same calculation for the point $P(r_{LC_0,pr}(t_{CMB}), t_{CMB})$:

$$v_F(t_{CMB}) = H_0\sqrt{\Omega_{\Lambda,0}}\ \cdot\ coth\left(\frac{3H_0\sqrt{\Omega_{\Lambda,0}}}{2} \cdot t_{CMB}\right) \cdot r_{LC_0,pr}(t_{CMB}),$$

This results in a velocity of $v_F(t_{CMB}) \approx 63 \cdot c$, i.e., more than **60 times the speed of light**.

With the Hubble radius $r_H(t)$, we defined a distance at which the escape velocity equals the speed of light. If we plot the light cone in a spacetime diagram, these curves intersect at the point P_c, see Fig. 5.9).



5.9: Spacetime diagram with past light cone $r_{LC_0,pr}(t)$ and Hubbel radius $r_H(t)$. The curves intersect at the point $P_c(r_{LC_0,pr}(t_c), t_c)$ where $v_F = c$.

In connection with Fig. 5.6, we have already discussed that we can also receive information about events that occurred in the past and were beyond the Hubble radius but still within the gray shaded area of Fig. 5.6, within a finite time. With the definition of the light cone, we can also numerically capture this situation and represent it in a diagram.

Therefore, if we specify a time t_S of an event in the past and a location $r_{LC,pr}(t_S)$ so that we end up within the gray area of Fig. 5.6, the question now is when the photons emitted in our direction will reach us. Mathematically, this means finding for which t_F the following equation is satisfied:

$$\frac{r_{LC,pr}(t_S)}{a(t_S)} - \int_{t=t_S}^{t_F} \frac{c\ dt}{a(t)} = 0\ . \qquad (5.34)$$

The Eq. (5.34) can be easily solved numerically using, for example, the bisection method.

In Fig. 5.10 the past light cones are shown for the following points (events):

$t_S = 10\ billion\ years, \quad r_{LC,pr}(t_S) = 13\ /\ 14\ /\ 15\ /\ 15.855\ billion\ ly$

All points lie outside the Hubble radius, but still in the gray shaded region between the Hubble radius and the event horizon.

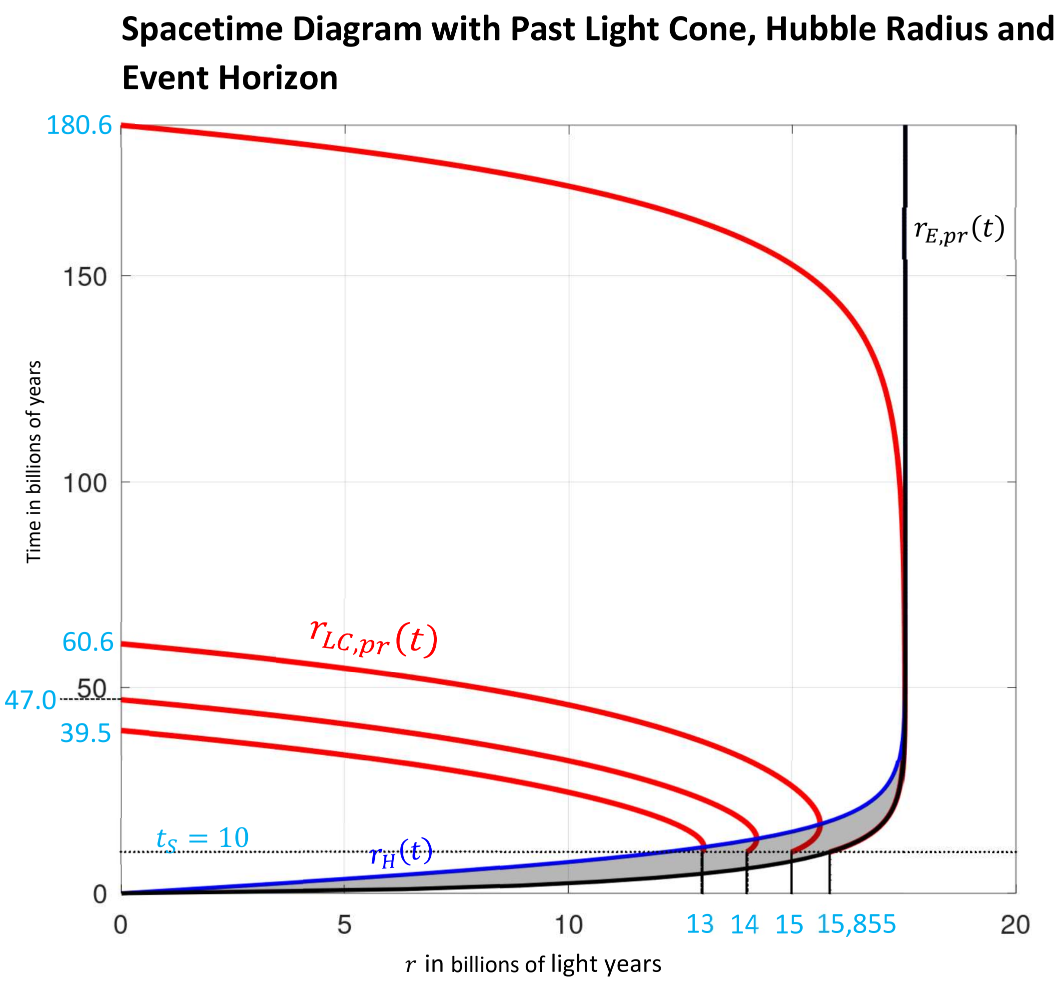
The last point with $r_{LC,pr}(t_S) = 15.855\ billion\ ly$ is very close to the event horizon, $r_{E,pr}(t_S = 10\ billion\ ly) = 15.85564\ billion\ ly$.

We see that the photons from all 4 events reach us in a finite time, although they are outside the Hubble radius (even though these periods are hard to imagine).

We already know that the Hubble radius intersects the light cones at the points where the escape velocity is equal to the speed of light.

As expected, the photons will reach “us” only very far in the future. Already at the first time point, i.e., at $t_F = 39.5\ billion\ years$, we’ll have long since passed the collision with our neighboring galaxy Andromeda, which will take place in around 4 billion years. The Sun will exist only as a black dwarf. Life on Earth will no longer be possible. Whether humanity - wherever it may be - will survive until then is more than questionable.

Especially for the point near the curve of the event horizon, i.e., at $r_{LC,pr}(t_S) = 15.855\ billion\ ly$, we see that the light cone hugs the event horizon and eventually merges into it with closer approach.



5.10: Past light cone for $t_S = 10$ *billion years* and $r_{LC,pr}(t_S) = 13 / 14 / 15 / 15.855$ billion ly

Future Light Cone

So far, we have only considered the past light cone, that is, the worldline of photons that have embarked on their journey towards us in the past and arrive at out location today. Of course, the "journey" of these photons does not end with us; rather, if they have arrived from a positive direction in the spacetime diagram, they will continue their path in the negative direction.

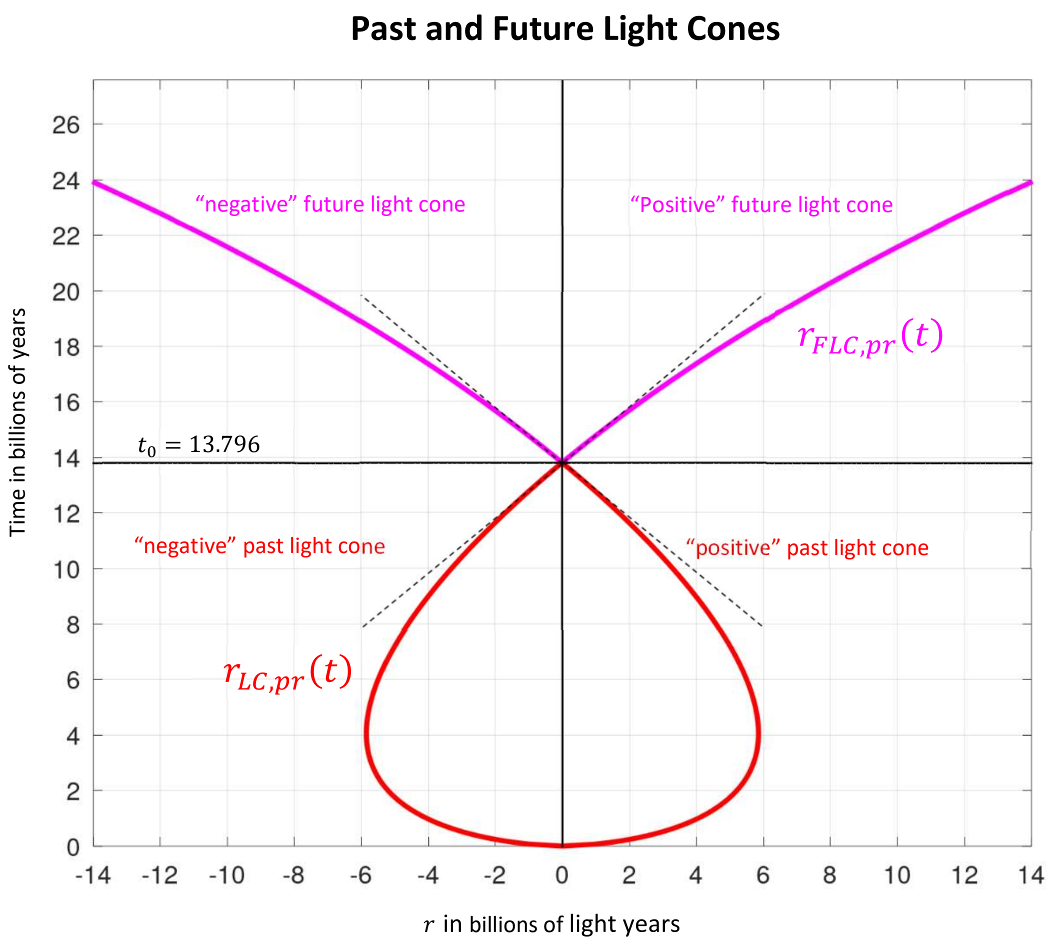
We also already know how photons move in the Hubble flow, i.e., in expanding space: The distance that a photon passing the stationary observer at the time $t_S = t_0$ covers up to a future time t_F, can be calculated using Eq. (5.20).

$$r_{FLC,pr}(t_F) = (\pm)a(t_F)\int_{t=t_0}^{t_F} \frac{c\,dt}{a(t)}. \tag{5.35}$$

If we consider the photon worldline in the form of a future light cone as an extension of a past light cone, then we need the positive sign for a "negative" past light cone and the negative sign accordingly for a "positive" past light cone (see Fig. 5.11).

The future light cone can also be understood as a particle horizon directed into the future, as evidenced by the comparison of equations (5.35) and (5.27).

Figure 5.11 depicts the temporal evolution of past light cones with $t_S = 0$ and $t_F = t_0$, which arrive at our location from both positive and negative directions and then continue along the trajectories of the photon worldlines according to Eq. (5.35).



5.11: Past and future light cones for photons that started at $t_S = 0$ and reaching us today. $r_{FLC,pr}(t)$ describes the future worldline of these photons.

Light cones are also known from special relativity and are represented as bisectors in the so-called Minkowski diagrams of spacetime. Since space expansion is not considered in special relativity, the light cones shown there (dashed lines in Fig. 5.11) are to be understood as an approximation of the future and past light cones depicted here.

Worldlines

With the light cones, we have already learned about the world line of photons. It describes the path of photons through an expanding universe, taking into account their proper velocity, which is the speed of light. Now, let's consider the path of objects at rest, which arises solely due to the expansion of space. These resting objects move through space and time with the Hubble flow, as observed by us, as observers at rest on the time axis. Using Eq. (5.20), we are able to calculate the distance of these objects from us at the present time t_0, provided we know when in the past (at time t_S) these objects emitted photons that arrive at our location today.

$$r_{0,pr}(t_0) = a(t_0) \int_{t=t_S}^{t_0} \frac{c\,dt}{a(t)} = \int_{t=t_S}^{t_0} \frac{c\,dt}{a(t)}. \tag{5.36}$$

This is the value of a particle horizon at the present time. However, the temporal evolution of this particle horizon, starting from t_S (i.e., $r_{P,pr}(t) = a(t) \int_{t=t_S}^{t} \frac{c\,dt}{a(t)}$, $t_S \leq t \leq t_0$) does not represent the worldline of the object under consideration. The two curves only intersect at $t = t_0$ (see Fig. 5.12).

If the distance to a stationary object at a specific point in the past is known to us, then we can use the scale factor to calculate any distance that this object will have to us in the future due to the expansion of space. The same applies, of course, in reverse: if the distance that the object has to us today or at a point in time in the future is known, we can calculate any distance in the past using the scale factor. We have already acquainted with the equations for this with Eqs. (5.23) and (5.24).

Case 1: Distance to a stationary object (without peculiar velocity) at a certain time in the past is known (Eq. (5.24)):

$$r_{wl}(t) = \frac{a(t)}{a(t_S)} r_{S,pr}(t_S)\,, \tag{5.37}$$

Case 2: The distance that a stationary object (without peculiar velocity) has to us today or will have to us at some point in the future is known (Eq. (5.23)):

$$r_{wl}(t) = \frac{a(t)}{a(t_F)} r_{F,pr}(t_F)\,, \tag{5.38}$$

Since both approaches lead to the same worldline, we can equate these equations,

$$\frac{r_{S,pr}(t_S)}{a(t_S)} a(t) = \frac{r_{F,pr}(t_F)}{a(t_F)} a(t)\,, \qquad \text{or}$$

$$\frac{r_{S,pr}(t_S)}{a(t_S)} = \frac{r_{F,pr}(t_F)}{a(t_F)} = const,$$

and realize that for a worldline of an object moving with the Hubble flow, the ratio of the distance to a stationary observer to the scale factor at any arbitrarily chosen time is constant, i.e.,

$$\frac{r_{wl}(t_{rev})}{a(t_{rev})} = r_{wl,co} = const. \tag{5.39}$$

Eq. (5.39) represents the worldline of a comoving object. Later, we will see in a spacetime diagram with an x-axis representing the comoving distance, this world line parallels the time axis. The value of the constant corresponds to the distance of the object to us at the present time and intersects at t_0 with the specific particle horizon (starting at $t_S = t_{rev}$), according to Eq. (5.36).

To get from the comoving distance to the proper distance, we multiply again by the scale factor and obtain for the world line at proper distance:

$$r_{wl,pr}(t) = a(t)\frac{r_{wl}(t_{rev})}{a(t_{rev})}\,. \tag{5.40}$$

$r_{wl}(t_{rev})$: Location of the observed object at time t_{rev}.

If we know the time t_S at which an object emitted photons that reach us today, we have already learned about two ways to calculate the required distance $r_{wl}(t_{rev})$:

1. Past light cone: $t_{rev} = t_S$, Eq. (5.30)

$$r_{LC_0,pr}(t_S) = r_{wl}(t_S) = a(t_S)\int_{t=t_S}^{t_0}\frac{c\,dt}{a(t)} \quad \text{and thus} \tag{5.41}$$

$$r_{wl,pr}(t) = a(t)\cdot a(t_S)\frac{\int_{t=t_S}^{t_0}\frac{c\,dt}{a(t)}}{a(t_S)},$$

$$r_{wl,pr}(t) = a(t)\cdot\int_{t=t_S}^{t_0}\frac{c\,dt}{a(t)} = a(t)\cdot r_{wl,co}\,. \tag{5.42}$$

2. Particle horizon: $t_{rev} = t_0$ and lower integration limit $t = t_S$, Eq. (5.28)

$$r_{P_0,pr} = r_{wl}(t_0) = \int_{t=t_S}^{t_0} \frac{c\,dt}{a(t)} \qquad \text{and thus} \tag{5.43}$$

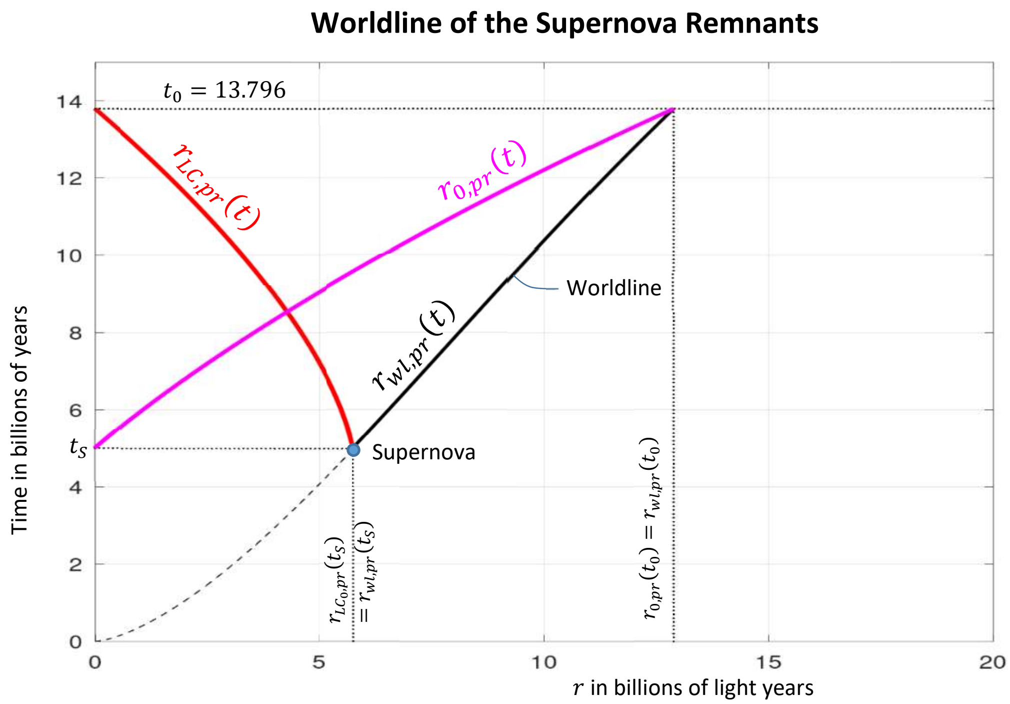
$$r_{wl,pr}(t) = a(t) \cdot \frac{\int_{t=t_S}^{t_0} \frac{c\,dt}{a(t)}}{a_0},$$

$$r_{wl,pr}(t) = a(t) \cdot \int_{t=t_S}^{t_0} \frac{c\,dt}{a(t)} = a(t) \cdot r_{wl,co}\,. \tag{5.44}$$

As expected, both approaches lead to the same result.

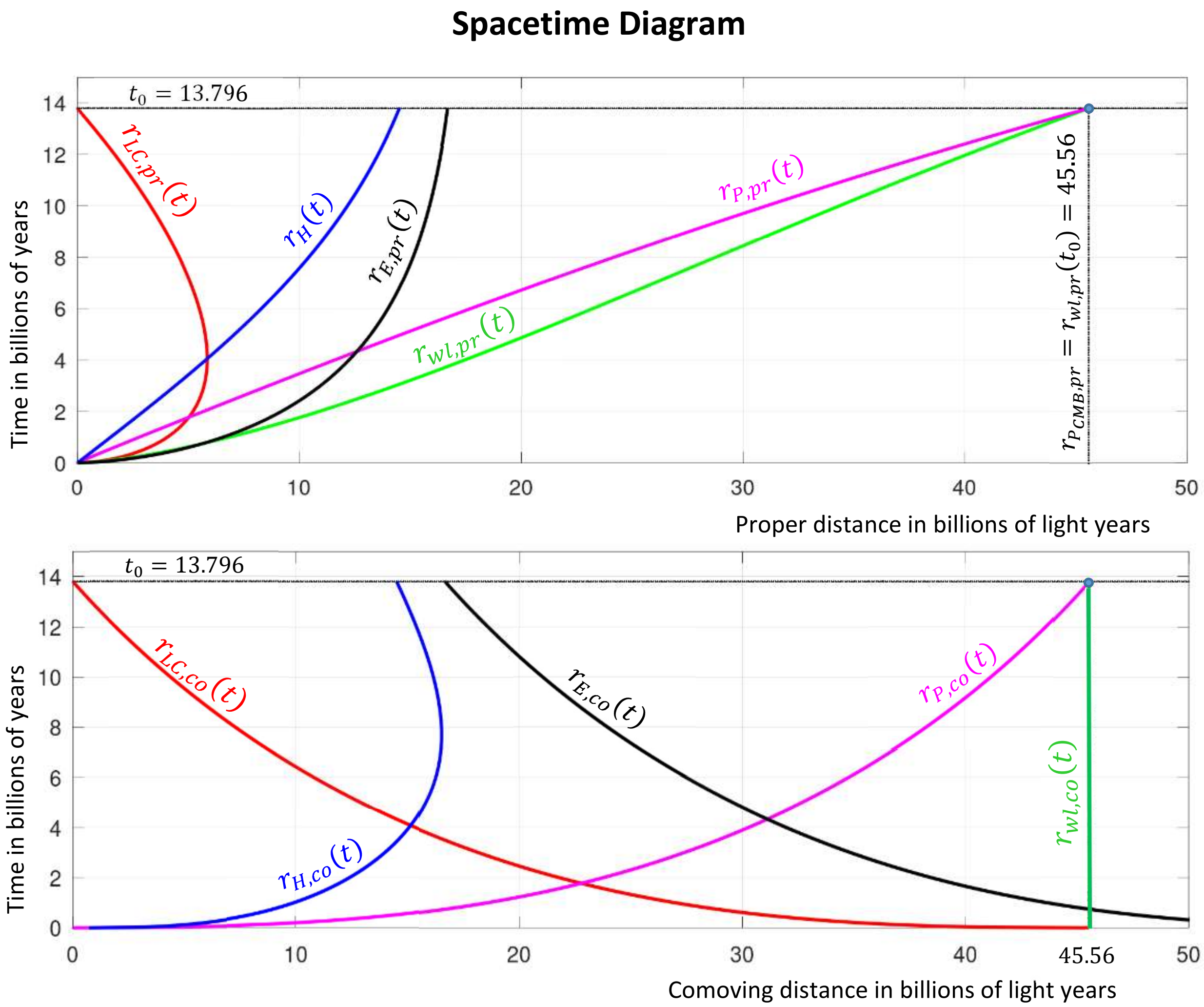
As an example, let's consider that the light from a supernova that occurred at $t_S = 5\ billion\ years$, reaches us today. We calculate the worldline of the supernova remnants using Eq. (5.44).

In Fig. 5.12, the worldline, the past light cone, and the particle horizon for the described event are plotted.



5.12: Worldline of the remnants of the supernova that occurred approximately 8.8 billion years ago. Also shown are the past light cone and the particle horizon.

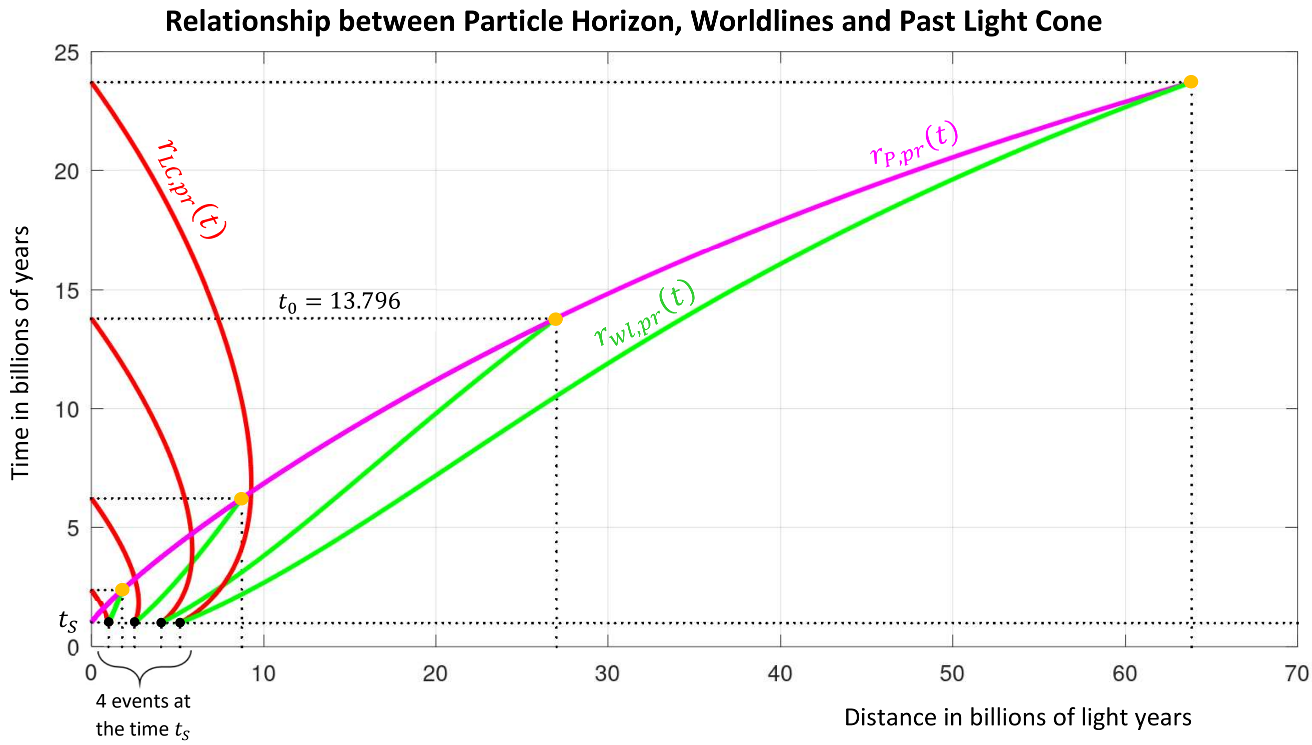
In Fig. 5.13, spacetime diagrams are presented for the time of recombination, $t_S = t_{CMB} \approx 380000\ years$. In the upper image, as usual, the x-axis represents proper distance. In the lower image, the x-axis represents comoving distance. Plotted are the past light cone, the Hubbel radius, the event horizon, the particle horizon and the worldline. The worldline and the particle horizon intersect on the "today line" at a distance of 45.56 billion ly (cf. also Eq. (5.29)). As described earlier, the worldline in the comoving distance diagram is a constant with the value $r_{wl,co} = 45.56\ billion\ ly$.



5.13: Spacetime diagram for the event of recombination, featuring the past light cone, Hubbel radius, event horizon, particle horizon and worldline. Upper representation at proper distance, lower representation in comoving distance.

At this point, an interesting relationship between a particle horizon and worldlines should be noted:

We already know that for an object that emitted photons at a time t_S in the past, which arrive at our location today, the worldline of this object and the particle horizon intersect at t_0. This intersection indicates where the object is located today due to the expansion of space. However, the temporal evolution of this particle horizon for t_S contains more information than just this one point at t_0. If we consider other objects with photon-emitting events that occurred at the same time t_S but at different distances from us in the past, the intersection of the respective world line of these objects with the particle horizon gives us the time when the photons of these events reach us, and where the object are at that time. In figure 5.14 four events are plotted to illustrate this concept. Thus, this one particle horizon "represents" all events occurring at the time t_S.



5.14: Spacetime diagram for various events at a time t_S. The intersections between the particle horizon for this time and the different worldlines of the events indicate when the photons of the events reach us and where the photon-emitting object is located at that time.

6 Literature

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