Integrability of Continuous Functions and Dynamical Systems

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Abstract. This paper consists of three parts: Starting with the field $\mathbb{C}$ of complex numbers, the Banach space $\mathcal{C}(V)$ of continuous, $\mathbb{C}$-valued functions on a simply connected compact region $V \subset \mathbb{C}$ is shown to decompose the topological direct sum of two complementary subspaces: a subspace of integrable and therefore analytic functions, and a subspace of un-integrable and anti-analytic functions. Introducing orientation as a central notion, orientation inversion (parity) turns out to be the complex conjugation, which maps integrable (w.r.t. positive orientation) to un-integrable functions, which are integrable w.r.t. negative orientation, and vice versa. Orientation allows the extension of complex analyticity to $\mathbb{R}^2$, which ends part 1. Part 2 is devoted to the extension of analyticity to multi-dimensions. These results are then applied in part three to continuous mechanical dynamical systems, where it is shown that Hamilton-Jacoby theory yields the unrestricted integrability of any continuous, mechanical dynamical system of either parity and represents their solutions as geodesics of (integrated) action functions of positive/negative parity, i.e.: as fermionic and bosonic solutions.

Part 1. Integrability and orientation of continuous functions in 1 complex and 2 real dimensions

1. Introduction: Preliminaries and problem statement

Let $\mathbb{K}$ stand for either $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{C}$. A function f from V to either $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{C}$ is is called continuous on V, if is well-defined and continuous in an open environment $U \subset \mathbb{K}$ of V. The set of continuous $\mathbb{K}$-valued functions on V then is a Banach space $\mathbb{C}(V, \mathbb{K})$ with the supremum norm $\|\cdot\| : f \mapsto sup_{x \in V} \left\|f(x)\right\|_{\mathbb{K}}$, where $\|\cdot\|_{\mathbb{K}}$ stands for the absolute value for $\mathbb{K} = \mathbb{R}$, the Euclidean norm for $\mathbb{K} = \mathbb{R}^2$, and the absolute value for $\mathbb{K} = \mathbb{C}$. In the following, we'll briefly write $\mathbb{C}(V)$ for $\mathbb{C}(V, \mathbb{K})$, when it is clear what the target space $\mathbb{K}$ is.

A path γ in V is a continuous mapping $\gamma : [0,1] \to V$, where $[0,1]$ denotes the closed real interval from 0 to 1. V s called connected, if for each $x, y \in V$ there is a path γ in V with $\gamma(0) = x$ and $\gamma(1) = y$. A compact set $V \subset \mathbb{K}$ is a closed and bounded subset of $\mathbb{K}$. V will be called closed region, if it is the closure of a non-void open and connected set. The path γ is called closed if $\gamma(0) = \gamma(1)$, and a connected V is called *simply connected*, if all closed paths in V are point homotopic in V, i.e.: if V has no holes. Let V be a simply connected, closed region and $f \in \mathcal{C}(V, \mathbb{K})$. Then for every piecewise continuously differentiable path $\gamma : [0,1] \to V$, the path integral $\int_\gamma f(s)ds := \int_0^1 f(\gamma(t))\frac{d\gamma(t)}{dt}dt$ is a well-defined, continuous linear functional on $\mathcal{C}(V, \mathbb{K})$. A function $f \in \mathcal{C}(V, \mathbb{K})$ is called *integrable*, if and only if $\int_\gamma f(s)ds = 0$ for every closed path γ in V. In all cases, if f is integrable, then the path integrals from a fixed startpoint in V to the variable endpoint in V define a function If, which is commonly called *primitive* of f. (Since two primitives of the same function f differ utmost by the choice of the startpoint, which adds an additive constant, the primitives are naturally defined as equivalence classes.) While this is trivial for one real dimension, i.e: for $V \subset \mathbb{R}$, and it is simple in the complex (also 1-dimensional) case, with two real dimensions $V \subset \mathbb{R}^2$, both $f \in \mathcal{C}(V, \mathbb{R})$ and $f \in \mathcal{C}(V, \mathbb{R}^2)$ there is a twist: primitives of integrable $f \in \mathcal{C}(V, \mathbb{R})$ are functions $If \in \mathcal{C}(V, \mathbb{R}^2)$, while the primitives of $f \in \mathcal{C}(V, \mathbb{R}^2)$ are functions $If \in \mathcal{C}(V, \mathbb{R})$. So, if If itself is integrable again to $I^2 f$, then $I^2 f$ will be in the same space of continuous functions on V as f, and the m^{th} order primitive $I^m f$ of f will be element of $\mathcal{C}(V, \mathbb{R})$ or $\mathcal{C}(V, \mathbb{R}^2)$, depending on whether m is even or odd.

For now, let us restrict to the unproblematic complex case $\mathcal{C}(V, \mathbb{C})$ with $V \subset \mathbb{C}$:
If $f \in \mathcal{C}(V, \mathbb{C})$ is integrable, then f can be uniquely path integrated from a fixed $z_0 = x_0 + iy_0$ in the interior of V to any other $z = x + iy \in V$, which – up to an additive constant of integration – defines a function $If \in \mathcal{C}(V, \mathbb{C})$, which is complex differentiable and for which $\frac{dIf(z)}{dz}$ holds. If is therefore called *anti-derivative* or *primitive* of f. Clearly, if f is integrable, then it is integrable to all orders, i.e. the n^{th} primitive $I^n f$ exists for all $n \in \mathbb{N}$. For the next, a definition of complex analyticity is needed: $If \in \mathcal{C}(V, \mathbb{C})$ is called *(complex) analytic*, if for all $z_0 \in V$ there is an environment $U_\epsilon(z_0)$, such that for all $z \in U_\epsilon(z_0)$: $f(z) = \sum_{k \geq 0} c_k (z - z_0)^k$ is on $U_\epsilon(z_0)$ the uniform limit of the power series $\sum_{k \geq 0} c_k (z - z_0)^k$, where $c_k \in \mathbb{C}$ for all k.

We'll refer to *Cauchy theory* as the contents of his original article [4]. Morera's theorem ([1][Ch. 4.2.3, p.122]) shows that analyticity and analytic continuation are rooted in the integrability only:

Proposition 1.1 (Morera's theorem: Corollary of Cauchy theory). *If $V \subset \mathbb{C}$ is a compact and simply connected region and $f : V \ni z \mapsto f(z)$ is continuous and integrable (w.r.t. dz), then f is analytic on V.*

Remark 1.2. Because of Cauchy's theorem every analytic function on V has a primitive (on V) so it is necessarily integrable. Morera's theorem hence implicitly states the equivalence of analyticity and integrability (on V).

The proof of this theorem uses the following

Lemma 1.3. *Let $V \subset \mathbb{C}$ be a compact and simply connected region.*

(i) If $f \in \mathcal{C}(V, \mathbb{C})$ is integrable and If is its primitive, then the square $If^2 := If \cdot If$ is integrable.

(ii) If $f, g \in \mathcal{C}(V, \mathbb{C})$ are integrable, the product $If \cdot Ig$ of their primitives If and Ig is integrable.

We'll skip the straightforward, well-known proofs (see: [1][Ch. 4.2.2]), as we'll extend Morera's theorem to $n \in \mathbb{N}$ dimensions in part 2.

The above offers numerous open topics to explore:

(1) The characteristic properties of integrable functions as a subspace of $\mathcal{C}(V, \mathbb{C})$ should be examined:
   - is it closed?
   - is it open?
   - does if have a topological compement, and if so: what is this complement?

(2) For $V \subset \mathbb{R}^2$ the *complex isomorphism* $\iota : V \ni (x, y) \mapsto x + iy \in \mathbb{C}$ isomorphically transforms $f \in \mathbb{C}(V, \mathbb{R}^2)$ to $\iota f \iota^{-1} \in \mathcal{C}(\iota V, \mathbb{C})$. So

$$T_\iota : \mathbb{C}(V, \mathbb{R}^2) \ni f \mapsto \iota f \iota^{-1} \in \mathcal{C}(\iota V, \mathbb{C})$$

is an isomorphism of Banach spaces, which will be called *complex isomorphism*, either. Then it is to expect that every relation for the complex functions can be mapped via T_ι^{-1} from $\mathcal{C}(\iota V, \mathbb{C})$ to $\mathbb{C}(V, \mathbb{R}^2)$, and this includes integrability and analyticity along with it. By the Weierstraß convergence theorem ([1][Ch. 8 1.1]) this pulled-back space of analytic functions should be closed in $\mathbb{C}(V, \mathbb{R}^2)$, and therefore the complex analytic functions would be closed in $\mathcal{C}(\iota V, \mathbb{C})$.

2. Integrability decomposition

Let V be a simply connected, closed and compact region of $\mathbb{R}^2$ or $\mathbb{C}$ and $f \in \mathbb{C}(V, \mathbb{K})$, where $\mathbb{K}$ stands for either $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{C}$. f will be called *integrable at* the point $z \in V$, if and only if there are some $h_0 > 0$ such that the path integrals $\int_{\gamma_h} f(s)ds$ of positively (i.e. counter-clockwise) orientated, closed paths once around the boundaries of circles of radius $h < h_0$ around z are defined, and such that $\int_{\gamma_h} f(s)ds = o(h^m)$ holds for any $m \in \mathbb{N} \cup \{0\}$ as $h \to 0$, which means that $\frac{1}{h^m}\left|\int_{\gamma_h} f(s)ds\right| \to 0$ as $h \to 0$. Because the value of the path integration gets inverted in sign, $\frac{1}{h^m}\left|\int_{\gamma_h} f(s)ds\right| \to 0$ for $h \to 0$ likewise holds if the paths γ_h go the opposite way with negative orientation. A function $f \in \mathbb{C}(V, \mathbb{K})$, which is not integrable at $z \in V$ will be called *unintegrable at* z. As such f is unintegrable at $z \in V$, if and only if there is some $h_0 > 0$, some $C_0 > 0$, and $m \in \mathbb{N}$, such that for any $\delta > 0$ with $\delta < h_0$ there is a positive $h < \delta$: $\left|\int_{\gamma_h} f(s)ds\right| \geq C_0 h^m$, where again $(\gamma_h)_{h>0}$ is the family of positively orientated paths with (winding) index 1 along circles of radius h around z.

Proposition 2.1 (Integrability decomposition). *Let $V \subset \mathbb{R}^2$ be be a simply connected compact region and $\mathbb{K}$ stand for either $\mathbb{R}$ or $\mathbb{R}^2$.*

(i) $\mathbb{C}(V, \mathbb{K})$ *is the topological direct sum of two subspaces: the space of integrable functions $\mathcal{Y}_+(V, \mathbb{K})$ and a complementary space $\mathcal{Y}_-(V, \mathbb{K})$ of unintegrable functions.*
(ii) $\mathbb{C}(\iota V, \mathbb{C})$ *is the topological direct sum of two subspaces: the space of integrable functions $\mathcal{Y}_+(\iota V, \mathbb{C})$ and a space $\mathcal{Y}_-(\iota V, \mathbb{C})$ of (strictly) unintegrable functions.*

Proof. The asserted decomposition of $\mathbb{C}(\iota V, \mathbb{C})$ follows from the decomposition of $\mathbb{C}(V, \mathbb{K})$ through the complex isomorphism T_ι. So it suffices to prove the first statement.

So, let $f \in \mathbb{C}(V, \mathbb{K})$. Then f is to be continuous on an open superset U of V, and we define $\mathcal{Q}$ as set of all squares $Q(d, x, y) = \{(x', y') \in \mathbb{R}^2 \mid |x' - x|, |y' - y| \leq d/2\}$ for $(x, y) \in V$ and some $d > 0$. Let $\Gamma(\mathcal{Q})$ be the set of all positively (i.e.: anti-clockwise) orientated paths $\gamma(d, x, y)$ around the boundaries of the $Q(h, x, y)$ with $d > 0$ and $(x, y) \in V$. Then $p_\gamma : f \mapsto p_\gamma(f) := \left\|\int_\gamma f(s)ds\right\| \geq 0$, $(\gamma \in \Gamma(\mathcal{Q}))$, defines a family of continuous seminorms on $\mathbb{C}(V, \mathbb{K})$. The set of all $f \in \mathbb{C}(V, \mathbb{K})$, for which $p_\gamma(f) = 0$ for all $\gamma \in \Gamma(\mathcal{Q})$ then is closed in $\mathbb{C}(V, \mathbb{K})$, since it is the intersection of the closed sets. It contains all integrable(, continuous) functions on V.

Let $\mathcal{Y}_+(V, \mathbb{K})$ denote this closed space of $\mathbb{C}(V, \mathbb{K})$. Then its complement is an algebraic subspace, which is open in $\mathcal{C}(V, \mathbb{K})$. We call it space of *non-integrable* functions and denote it by $\mathcal{Y}_-(V, \mathbb{K})$.

To finish up, it remains to be shown that $\mathcal{Y}_+(V, \mathbb{K})$ is also open, or equivalently to prove that $\mathcal{Y}_-(V, \mathbb{K})$ is closed. We need to refine this family of seminorms, in order to make further progress:
For each $f \in \mathcal{C}(V, \mathbb{K})$ the function

$$F : [0, d] \times V \ni (h, x, y) \mapsto \int_{\gamma(h,x,y)} f(s)ds \in \mathbb{K}$$

is uniformly continuous on $[0, d] \times V$, but also: $|F(h, x, y) - F(h', x, y)| = o(h - h')$ (for $h, h' < d$). So, F is (right) differentiable (at $h = 0$) in its first argument for $h \to 0$, and F is continuously differentiable in h for each $(x, y) \in V$ for $0 < h < d$. And because every $f \in \mathcal{C}(V, \mathbb{K})$ can be isometrically extended as a continuous function onto the closed d-environment of V, the mapping

$$p : \mathcal{C}(V, \mathbb{K}) \ni f \mapsto sup_{h\in[0,d],(x-y)\in V} \frac{1}{4h} |F(h, x, y)| \geq 0$$

is a well-defined semi-norm on $\mathcal{C}(V, \mathbb{K})$, and it is a norm on its (open) subspace $\mathcal{Y}_-(V, \mathbb{K})$ of unintegrable functions. Let's inspect the last statement in detail: For $f \in \mathcal{Y}_-(V, \mathbb{K})$, there is some $(x, y) \in V$, such that for any $\delta > 0$ there is an $h > 0$ with $h < \delta$ and $\left|\int_{\gamma(h,x,y)} f(s)ds\right| > 0$, where $\gamma_{(h,x,y)}$ is the path once around the boundary of the h-square centered at (x, y). So, $\gamma_{(h,x,y)}$ is the sum of two paths, $\gamma_{(h,x,y)} = \gamma_{(h,x,y),R} - \gamma_{(h,x,y),L}$, where $\gamma_{(h,x,y),L}$ starts from the lower left corner along the y-axis to the upper left corner, then along the upper upper side along the x-axis from top left to upper right corner, and $\gamma_{(h,x,y),R}$ is the path from the lower left corner to upper right corner across the lower right corner. Unintegrability of f at (x, y) then mandates $\int_{\gamma(h,x,y)} f(s)ds = 2\int_{\gamma(h,x,y),R} f(s)ds$. So, by continuity of f:

$$lim_{h\to 0} \sup_{(x,y)\in V} \left|\frac{1}{4h}\int_{\gamma(h,x,y)} f(s)ds\right| \geq \left|f(x, y)\int_{\gamma(h,x,y),R} \frac{1}{4h}ds\right|,$$

and therefore $p(f) \geq \frac{1}{2}\sup_{(x,y)\in V}\left|f(x, y)\right|$. So, p is stronger than the supremum norm, so p itself is a norm on $\mathcal{Y}_-(V, \mathbb{K})$. On the other hand, clearly: $p(f) \leq \sup_{(x,y)\in V}\left|f(x, y)\right|$, so on $\mathcal{Y}_-(V, \mathbb{K})$, p is equivalent to the supremum norm. Hence $\mathcal{Y}_-(V, \mathbb{K})$ is closed, its algebraic complement $\mathcal{Y}_+(V, \mathbb{K})$ is open, the canonical projections to the quotient spaces $\pi_\pm : \mathcal{C}(V, \mathbb{K}) \ni f \mapsto [f]_\pm \in \mathcal{C}(V, \mathbb{K})/\mathcal{Y}_\pm(V, \mathbb{K})$ are (bi-)continuous, and $\mathcal{C}(V, \mathbb{K})$ is the topological direct sum of its closed and open subspaces $\mathcal{Y}_\pm(V, \mathbb{K})$ – as was asserted. □

The decomposition into the spaces $\mathcal{Y}_\pm(V, \mathbb{K})$ and $\mathcal{Y}_\pm(\iota V, \mathbb{C})$ resp. is a provisional result and not the final decomposition: One would want the integrable and unintegrable subspaces to be isomorphic. We'll see next, that there are conjugations on $\mathcal{C}(V, \mathbb{R}^2)$ and $\mathcal{C}(\iota V, \mathbb{C})$, which map the $\mathcal{Y}_-$-spaces into their complementary $\mathcal{Y}_+$-spaces, but leave a subspace of the $\mathcal{Y}_+$-spaces invariant. The goal then will be to extract that subspace and to decompose $\mathcal{Y}_+$ further.

3. Conjugation, Jacobians, and $\mathcal{C}_0$-spaces

Again, let $V \subset \mathbb{R}^2$ be a simply connected compact region.
For $f = (f_1, f_2) \in \mathcal{C}(V, \mathbb{R}^2)$ and $f = Re(f) + iIm(f) \in \mathcal{C}(\iota V, \mathbb{C})$ the functions

$$f^c := (f_1 - f_2) \text{ and } f^c := \bar{f} := Re(f) - iIm(f)$$

will be called *conjugates* of f, where in particular $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. So, in the complex case, $f^c(z) := \overline{f(z)}$.

Then the *conjugation* is a an isometric isomorphism on $\mathcal{C}(V, \mathbb{R}^2)$ and an isometric antilinear bijection on $\mathcal{C}(\iota V, \mathbb{C})$, such that $(f^c)^c = f$ for all f, i.e.: the conjugation is an idempotent mapping in all cases.

We now examine the spaces of integrable and unintegrable functions, in order to identify the conjugation-invariant subspaces. We shall restrict (mainly) to $\mathcal{C}(V, \mathbb{R}^2)$. For the complex case, $\mathcal{C}(V, \mathbb{C})$, results will be constructed from this via the complex isomorphism.

Both, $\mathcal{C}(V,\mathbb{R}^2)$ and $\mathcal{C}(\iota V,\mathbb{C})$, have the infinitely differentiable functions $\mathcal{C}^\infty(V,\mathbb{R}^2)$ and $\mathcal{C}^\infty(\iota V,\mathbb{C})$ as dense subspaces (see: [6]). Restricting to these has the advantage that the structure of the subspaces can be classified by the types of the Jacobi matrices (i.e.: the derivatives) of its elements. With this we have: The derivative of every continuously differentiable $f \in \mathcal{C}(V,\mathbb{R}^2)$ can be represented by matrix-valued function Df, called the *Jacobian*, given by

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}, \quad \text{with } a,b,c,d \in \mathcal{C}(V,\mathbb{R})$$

By Green's theorem (see e.g.: [1][Ch. 5 5.2]), a continuously differentiable function $f \in \mathcal{C}(V,\mathbb{R}^2)$ is integrable if and only if its Jacobian Df is a symmetric matrix

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}, \quad \text{where } a,b,c \in \mathcal{C}(V,\mathbb{R}).$$

These then comprise all continuously differentiable elements from $\mathcal{Y}_+(V,\mathbb{R}^2)$. And the unintegrable, continuously differentiable $f\mathcal{Y}_-(V,\mathbb{R}^2)$ then have the Jacobian Df

$$Df(x,y) = \begin{pmatrix} 0 & -b(x,y) \\ b(x,y) & 0 \end{pmatrix}, \quad \text{where } b \in \mathcal{C}(V,\mathbb{R}) \setminus \{0\}.$$

The conjugation on $\mathcal{C}(V,\mathbb{R}^2)$ now maps the Jacobian

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}$$

for an abitrary continuously differentiable $f \in \mathcal{C}(V,\mathbb{R}^2)$ to:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} = \begin{pmatrix} a(x,y) & b(x,y) \\ -c(x,y) & -d(x,y) \end{pmatrix},$$

hence it inverts the integrability: It maps $\mathcal{Y}_-(V,\mathbb{R}^2)$ into $\mathcal{Y}_+(V,\mathbb{R}^2)$, but it is not onto, because its image does not contain the diagonal elements

$$\begin{pmatrix} a(x,y) & 0 \\ 0 & d(x,y) \end{pmatrix}.$$

This determines its invariant subspace w.r.t. integrability inversion, which will be denoted by $\mathcal{C}_0(V,\mathbb{R}^2)$. Because these diagonal matrix functions are globally diagonal on V, a must not change in the y-direction, and d must be constant in the x-direction. So,

$$Df(x,y) = \begin{pmatrix} a(x,y) & 0 \\ 0 & d(x,y) \end{pmatrix}$$

mandate $a(x,y) = a(x)$ and $d(x,y) = d(y)$ for $(x,y) \in V$, so a and d are functions on the x- and y-coordinate projections on V, namely $V_x := \{x \in \mathbb{R} \mid (x,y) \in V\}$ and $V_y := \{y \in \mathbb{R} \mid (x,y) \in V\}$, and both are bounded, closed intervals, since V is to be a simply connected compact region. And a and b have primitives given by $Ia(x) := \int_{-\infty}^{x} a(t)dt$ and $Ib(y) := \int_{-\infty}^{y} b(t)dt$, so that the primitive of Df is given by the pair of functions $f : V \ni (x,y) \mapsto (Ia(x), Ib(y))$. And because the set of continuously differentiable functions is dense in $\mathcal{C}(V,\mathbb{R}^2)$, it follows that $\mathcal{C}_0(V,\mathbb{R}^2)$ is the closed subspace of all $f : V \ni (x,y) \mapsto (f_1(x), f_2(y)) \in \mathbb{R}^2$ with $f_1 \in \mathcal{C}(V_x,\mathbb{R})$ and $f_2 \in \mathcal{C}(V_y,\mathbb{R})$. Next, $\mathcal{C}_0(V,\mathbb{R}^2)$ is open too, because for every $f = (f_1, f_2) \neq 0$, either $f_1 \neq 0$ or $f_2 \neq 0$, where both are continuous, real-valued functions on intervals. If $f_1 \neq 0$, then $\left|f_1(x)\right| > \epsilon$ for some $(x,y) \in V$ an some $\epsilon > 0$. Then there is a function $g \in \mathcal{C}(V_x,\mathbb{R})$ contained in the ϵ-environment of f_1, and likewise there is for f_2, in case $f_2 \neq 0$. That proves the openedness of $\mathcal{C}_0(V,\mathbb{R}^2)$.

As announced above, we then define $\mathcal{C}_+(V,\mathbb{R}^2) := \mathcal{Y}_+(V,\mathbb{R}^2)/\mathcal{C}_0(V,\mathbb{R}^2)$, rename $\mathcal{C}_-(V,\mathbb{R}^2) := \mathcal{Y}_-(V,\mathbb{R}^2)$, and get the desired decomposition

$$\mathcal{C}(V,\mathbb{R}^2) = \mathcal{C}_+(V,\mathbb{R}^2) \oplus \mathcal{C}_0(V,\mathbb{R}^2) \oplus \mathcal{C}_-(V,\mathbb{R}^2)$$

into the the topological direct sum of its constituents $\mathcal{C}_\pm(V,\mathbb{R}^2)$ and $\mathcal{C}_0(V,\mathbb{R}^2)$.

We consider $\mathcal{C}(V,\mathbb{R})$: There is no conjugation defined on it, yet the $\mathcal{Y}_\pm(V,\mathbb{R})$ are both non-trvial, and they have an integrability inversion with a nontrivial $\mathcal{C}_0(V,\mathbb{R})$ as invariant subspace of $\mathcal{C}(V,\mathbb{R})$:
If $f \in \mathcal{C}(V,\mathbb{R})$ is continuously differentiable, then its derivative is given by its gradient $\nabla f := (\partial_x f, \partial_y f)$. It exists irrespective of whether ∇f is integrable again to its primitive, or not. Suppose, that ∇f was not integrable. What we know from the above is that ∇f is the sum of three components $\nabla f = g_+ + g_- + g_0$ with $g_\pm \in \mathcal{C}_\pm(V,\mathbb{R}^2)$ and $g_0 \in \mathcal{C}_0(V,\mathbb{R}^2)$, where $g_- \neq 0$. To enforce the integration of ∇f back to f, g_- must be transformed to its integrable counterpart via conjugation: $(Ig_0^c)^c$; this would allow to retain f from ∇f, even when non-integrable.

It is well-known that $\mathcal{C}_\pm(V,\mathbb{R})$ are both non-trivial:
$f(x = r\cos(t), y = r\sin(t)) := r^2 \sin(t/r)$ for $(x,y) \neq 0$ and $f(0,0) := 0$ with (x,y) in $V := \{(x,y) \mid -1 \leq x,y \leq 1\}$, is an example of an unintegrable function at the origin, so represents a non-zero element $f \in \mathcal{C}_-(V,\mathbb{R})$, and hence its conjugate represents a member of $\mathcal{C}_+(V,\mathbb{R})$.
To show that $\mathcal{C}_0(V,\mathbb{R})$ is non-trivial either, it suffices to integrate $f \in \mathcal{C}_0(V,\mathbb{R}^2)$: $f(x,y) = (f_1(x), f_2(y))$ is integrable and has $If(x,y) = If_1(x) + If_2(y)$ as primitives, where again If_1, If_2 are the primitives $If_1(x) := \int_{-\infty}^x f_1(t)dt$ and $If_2(y) := \int_{-\infty}^y f_2(t)dt$. By differentiating the continuously differentiable $f \in \mathcal{C}_0(V,\mathbb{R}^2)$, we even get the general result directly for all $g \in \mathcal{C}_0(V,\mathbb{R})$: it consists of all functions $g = g_1 + g_2$ with $g_1 \in \mathcal{C}_0(V_x,\mathbb{R})$ and $g_2 \in \mathcal{C}_0(V_y,\mathbb{R})$. And again, this is an open and closed subspace of $\mathcal{C}(V,\mathbb{R})$.

An immediate consequence of the above is that primitives of (integrable) functions of $\mathcal{C}(V,\mathbb{K})$ are integrable again to any order.

(The special case $\mathbb{K} = \mathbb{C}$ is analogous to $\mathbb{K} = \mathbb{R}$.)
As to the differentiation, the situation then is similar: if $f \in \mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K})$ is n times continuously differentiables, then all its n derivatives are in $\mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K})$ for some $\mathbb{K} = \mathbb{R}, \mathbb{R}^2, \mathbb{C}, \mathbb{C}^2$. However: if $f \in \mathcal{C}_-(V,\mathbb{K})$, then latest at the 2^{nd} derivative, the anti-symmetry of the Jacobian

$$Dg(x,y) = \begin{pmatrix} 0 & -b(x,y) \\ b(x,y) & 0 \end{pmatrix}, \quad \text{where } b \neq 0$$

impedes further differentiability, because of $\partial_y\partial_x g(x,y) = \partial_x b(x,y) = -\partial_x\partial_y g(x,y)$. That said, $f \in \mathcal{C}(V,\mathbb{K})$ is continuously differentiable to an order of 2 or more, only if $f \in \mathcal{C}(V,\mathbb{K}) \in \mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K})$.

Summarizing, it was shown:

Proposition 3.1. *1. The subspaces $\mathcal{Y}_+(V,\mathbb{K})$ and $\mathcal{Y}_+(\iota V,\mathbb{C})$ contain open and closed invariant subspaces $\mathcal{C}_0(V,\mathbb{K})$ and $\mathcal{C}_0(\iota V,\mathbb{C})$ consisting of continuous functions f, for which $\partial_x\partial_y f = \partial_y\partial_x f \equiv 0$ holds.*
2. $\mathcal{Y}_-(V,\mathbb{K})$ and $\mathcal{Y}_-(\iota V,\mathbb{C})$ are isomorphic to the quotient spaces $\mathcal{Y}_+(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ and $\mathcal{Y}_+(\iota V,\mathbb{C})/\mathcal{C}_0(\iota V,\mathbb{C})$, resp.
3. For $\mathcal{C}(V,\mathbb{R}^2)$ and $\mathcal{C}(\iota V,\mathbb{C})$, the conjugation $f \to f^c$ maps $\mathcal{Y}_-(V,\mathbb{K})$ onto $\mathcal{Y}_+(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$, and $\mathcal{Y}_-(\iota V,\mathbb{C})$ onto $\mathcal{Y}_+(\iota V,\mathbb{C})/\mathcal{C}_0(\iota V,\mathbb{C})$.
4. Primitives of integrable functions are integrable.

We define $\mathcal{C}_+(\iota V,\mathbb{C}) := \mathcal{Y}_+(\iota V,\mathbb{C})/\mathcal{C}_0(\iota V,\mathbb{C})$ and $\mathcal{C}_-(\iota V,\mathbb{C}) := \mathcal{Y}_-(\iota V,\mathbb{C})$ in line with $\mathcal{C}_+(V,\mathbb{K}) := \mathcal{Y}_+(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ and $\mathcal{C}_-(V,\mathbb{K}) := \mathcal{Y}_-(V,\mathbb{K})$, the corresponding canonical projections will be denoted by

$\Pi_0 : \mathcal{C}(V,\mathbb{K}) \to \mathcal{C}_0(V,\mathbb{K})$,
$\Pi_0 : \mathcal{C}(\iota V,\mathbb{C}) \to \mathcal{C}_0(\iota V,\mathbb{C})$,
$\Pi_\pm : \mathcal{C}(V,\mathbb{K}) \to \mathcal{C}_\pm(V,\mathbb{K})$ as well as
$\Pi_\pm : \mathcal{C}(\iota V,\mathbb{C}) \to \mathcal{C}_\pm(\iota V,\mathbb{C})$.

Since integrable functions have been defined as elements from the $\mathcal{Y}_+$-spaces, which include the $\mathcal{C}_0$-spaces as a subspace, the functions from $\mathcal{C}_+(V,\mathbb{K})$ and $\mathcal{C}_+(\iota V,\mathbb{C})$ will be called *strictly integrable.*

Then we can state:

Corollary 3.2. *The following holds as a topological direct sum:*
1. $\mathcal{C}(V,\mathbb{K}) = \mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_0(V,\mathbb{K}) \oplus \mathcal{C}_-(V,\mathbb{K})$
2. $\mathcal{C}(\iota V,\mathbb{C}) = \mathcal{C}_+(\iota V,\mathbb{C}) \oplus \mathcal{C}_0(\iota V,\mathbb{C}) \oplus \mathcal{C}_-(\iota V,\mathbb{C})$

From inspection of the Jacobians, note that the product of two integrable functions from $\mathcal{C}(V,\mathbb{R})$ or $\mathcal{C}_+(V,\mathbb{R}^2)$ is integrable again (where for $\mathcal{C}_+(V,\mathbb{R}^2)$ the product is a function in $\mathcal{C}(V,\mathbb{R})$).

4. Conformality, holomorphic and anti-holomorphic functions

As was shown above, $\mathcal{C}(\iota V,\mathbb{C})$ splits into the topological sum of a strictly integrable, a strictly unintegrable, and an invariant subspace. From Proposition 1.1 we know that all integrable functions are analytic, and then it will be straightforward to derive the analyticity of the unintegrable ones (see below).

The a-priori concern however is, how the vector space of holomorphic functions will fit into this, especially regarding the closedness of the space $\mathcal{C}_+(\iota V,\mathbb{C})$ in $\mathcal{C}(\iota V,\mathbb{C})$. So, let's look into this:

The Jacobian for a continuously differentiable $f \in \mathcal{C}_+(V,\mathbb{R}^2) \oplus \mathcal{C}_0(V,\mathbb{R}^2)$ is given by

$$Df(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}, \quad \text{where } a,b,c \in \mathcal{C}(V,\mathbb{R}).$$

Under the complex isomorphism T_ι it transforms to

$$D(T_\iota f)(x,y) = D(\iota f \iota^{-1})(x,y) = \begin{pmatrix} a(x,y) & -ib(x,y) \\ ib(x,y) & c(x,y) \end{pmatrix}, \quad \text{where } a,b,c \in \mathcal{C}(\iota V, \mathbb{R}).$$

But: The definition of an holomorphic function demands $c \equiv a$ (see: e.g. [1]). This is solved by splitting the diagonal matrix up into the sum of a symmetric and an anti-symmetric part:

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+c & 0 \\ 0 & a+c \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a-c & 0 \\ 0 & -(a-c) \end{pmatrix},$$

which defines continuous projections on $\mathcal{C}_0(V,\mathbb{R}^2)$ and $\mathcal{C}_0(\iota V,\mathbb{C})$, respectively. The spaces $\mathcal{C}_0(V,\mathbb{R}^2)$ and $\mathcal{C}_0(\iota V,\mathbb{C})$ therefore decompose into toplogical direct sums of symmetric subspaces $\mathcal{C}_{0,sym}(V,\mathbb{R}^2)$ and $\mathcal{C}_{0,sym}(\iota V,\mathbb{C})$, as well as anti-symmetric subspaces $\mathcal{C}_{0,asym}(V,\mathbb{R}^2)$ and $\mathcal{C}_{0,asym}(\iota V,\mathbb{C})$. So,

$$\mathcal{C}(V,\mathbb{R}^2) = \mathcal{C}_{conf}(V,\mathbb{R}^2) \oplus \mathcal{C}_{aconf}(V,\mathbb{R}^2), \text{ where}$$

$$\mathcal{C}_{conf}(V,\mathbb{R}^2) := \mathcal{C}_{+}(V,\mathbb{R}^2) \oplus \mathcal{C}_{0,sym}(V,\mathbb{R}^2),$$

$$\mathcal{C}_{aconf}(V,\mathbb{R}^2) := \mathcal{C}_{-}(V,\mathbb{R}^2) \oplus \mathcal{C}_{0,asym}(V,\mathbb{R}^2) \text{ and likewise}$$

$$\mathcal{C}(\iota V,\mathbb{C}) = \mathcal{C}_{conf}(\iota V,\mathbb{C}) \oplus \mathcal{C}_{aconf}(\iota V,\mathbb{C}), \text{ where}$$

$$\mathcal{C}_{conf}(\iota V,\mathbb{C}) := \mathcal{C}_{+}(\iota V,\mathbb{C}) \oplus \mathcal{C}_{0,sym}(\iota V,\mathbb{C}), \text{ and}$$

$$\mathcal{C}_{aconf}(\iota V,\mathbb{C}) := \mathcal{C}_{-}(\iota V,\mathbb{C}) \oplus \mathcal{C}_{0,asym}(\iota V,\mathbb{C}).$$

The functions of $\mathcal{C}_{conf}(V,\mathbb{R}^2)$ and $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ are called *conformal*, and the functions of $\mathcal{C}_{aconf}(V,\mathbb{R}^2)$ and $\mathcal{C}_{aconf}(\iota V,\mathbb{C})$ are defined as *anti-conformal* functions. With this, a real-valued function $f \in \mathcal{C}(V,\mathbb{R})$ will be called conformal, if and only if it is integrable and its primitive (which then is an $\mathbb{R}^2$-valued function) is conformal.

Remark 4.1. $\mathcal{C}_{conf}(V,\mathbb{R}^2)$ is the closure of the subspace of all differentiable $f = (f_1,f_2)$ of $\mathcal{C}(V,\mathbb{R}^2)$, for which $\partial_x f_1 = \partial_y f_2$ holds, $\mathcal{C}_{aconf}(V,\mathbb{R}^2)$ the closure of differentiable $f \in \mathcal{C}(V,\mathbb{R}^2)$, for which $\partial_x f_1 = -\partial_y f_2$. Analogously, $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ is the closure of all $f \in \mathcal{C}(\iota V,\mathbb{C})$, for which the partial derivatives exist and $\partial_x Re(f) = \frac{\partial Im(f)(x+iy)}{i\partial_y}$ holds, and $\mathcal{C}_{aconf}(\iota V,\mathbb{C})$ the closure of f with existing partial derivatives, such that $\partial_x Re(f) = -\frac{\partial Im(f)(x+iy)}{i\partial_y}$.

The decomposition of $\mathcal{C}(V,\mathbb{R}^2)$ and $\mathcal{C}(V,\mathbb{C})$ into the topological direct sum of their conformal and anti-conformal subspaces will be called *conformal split*.
Then we get:

Proposition 4.2. *Let $V \subset \mathbb{R}^2$ be a simply connected compact region. The functions of $\mathcal{C}_{conf}(\iota V,\mathbb{C})$ are exactly those, which obey the Cauchy-Riemann equations (see:* [1]*), which – by the definition – are holomorphic functions on V. Its (complex) conjugated space $\mathcal{C}_{aconf}(\iota V,\mathbb{C})$ therefore consists of all* anti-holomorphic functions *on V.*

Proof. The functions in $\mathcal{C}_{conf}(\iota V, \mathbb{C})$ are integrable. By Proposition 1.1 these functions then are analytic on V, so continuously differentiable on V in its x- and y-coordinates. Because all elements of $\mathcal{C}_{conf}(\iota V, \mathbb{C})$ are conformal, they are holomorphic (which by definition means that they satisfy the functions are continuously differentiable in x- and y-coordinate and satisfy the Cauchy-Riemann equations). All non-zero elements in its topological complement are either not integrable or anti-conformal, conflicting the Cauchy-Riemann equations. So, no other holomorphic functions exist on V. □

Remark 4.3. The conformal split allows a pragmatic access to integrability: $f_{conf} = (f_1, f_2) \in \mathcal{C}_{conf}(V, \mathbb{R}^2)$ if and only if $f_1 \equiv f_2$. Likewise, $f_{aconf} = (f_1, f_2)$ is in $\mathcal{C}_{aconf}(V, \mathbb{R}^2)$ if and only if $f_1 \equiv -f_2$. So, $f_{conf} = (g, g)$ and $f_{aconf} = (h^c, -h^c)$ for some conformal functions $g, h \in \mathcal{C}(V, \mathbb{R})$. As a conformal function, g is integrable to a function (Ig, Ig), so the primitive If_{conf} of f_{conf} is Ig, which we can write as $If_{conf} = Ig(1,1)$; the second order primitive of f_{conf} then writes to $I^2 f_{conf} = (I^2 g, I^2 g)$, and so forth. Analogously, we can assign $If_{aconf} := (Ih)^c(1, -1)$ as the primitive of f_{aconf}, $I^2 f_{aconf} := ((I^2 h)^c, (I^2 h)^c)$ as 2^{nd}-order primitive, and so forth.

Lemma 4.4. *Let $V \subset \mathbb{R}^2$ be a simply connected compact region. If $f \in \mathcal{C}(\iota V, \mathbb{C})$ is analytic on ιV, then its conjugate f^c is analytic on $(-i)\overline{\iota V}$.*

Proof. If $f(z) = \sum_k c_k (z - z_0)^k$ is analytic (on V), then $\bar{f}(z) := \sum_k \bar{c}_k (z - z_0)^k$ is analytic (on V). The conjugate f^c is defined by $f^c : z \mapsto \overline{f(z)}$, so we have $f^c(z) = \bar{f}(\bar{z})$. Now, $g : (-i)\overline{\iota V} \ni (ix + y) \mapsto \sum_k \bar{c}_k (-i)^k ((ix + y) - (ix_0 + y_0))^k$ is analytic on $(-i)\overline{\iota V}$, and $f^c = g$, since $(-i)((ix + y) - (ix_0 + y_0)) = (x - iy) - (x_0 - iy_0)$. □

Because every $f \in \mathcal{C}(\iota V, \mathbb{C})$ can be extended to a continuous function $\tilde{f}$ on a square area $Q(h) \supset \iota V$ with the origin as center and of sufficiently large side length $h > 0$, such that $\sup_{z \in Q(h)} \left|\tilde{f}(z)\right| \leq 2 \sup_{z \in \iota V} \left|f(z)\right|$, $\mathcal{C}(\iota V, \mathbb{C})$ is continuously embedded into $\mathcal{C}(Q(h), \mathbb{C})$, and we can ensure ιV to contain $-iz$ and $\bar{z}$ with every $z \in \iota V$. So, there appears to be no substantial reason, to exclude conjugates of analytic functions from being analytic functions.

The results can be summarized for $\mathcal{C}(\iota V, \mathbb{C})$ as:

Corollary 4.5. *Let $\iota V \subset \mathbb{C}$ be a simply connected compact region. $\mathcal{C}(\iota V, \mathbb{C})$ is the topological direct sum of the subspace $\mathcal{C}_{conf}(\iota V, \mathbb{C})$ of analytic and holomorphic functions $f(x + iy) = g(x) + ih(iy)$, and its conjugated subspace $\mathcal{C}_{aconf}(\iota V, \mathbb{C})$ of anti-holomorphic functions.*

That solves the integrability and analyticity posed as to the complex space, but still we have no analogous results for the spaces $\mathcal{C}(V, \mathbb{R}^2)$ (and $\mathcal{C}(V, \mathbb{R})$. This asks for some explanation:
Complex analysis is essentially built upon the 2-dimensional Laplace equation

$$\Delta f(x, y) := (\partial_x^2 + \partial_y^2) f(x, y) \equiv 0.$$

Within $\mathbb{C}$, Δ factors into the commuting product $\Delta = (\partial_x - i\partial_y)(\partial_x + i\partial_y)$. Hence, in there, $\Delta f \equiv 0$ reduces to first order differential equations, and the solutions are the sums of functions that solve $(\partial_x - i\partial_y)f = 0$ or $(\partial_x + i\partial_y)f = 0$. So, the idea was to pick any differentiable function $f(x+iy)$, for which then $(\partial_x - i\partial_y)f(x+iy) \equiv 0$, so $\Delta f \equiv 0$. The hindsight: these functions are analytic (by Cauchy theory). The problem: By the Weierstraß convergence theorem, these functions proved not to be dense in the space of continuous functions $f : V \ni z \mapsto f(z) \in \mathbb{C}$, where $V \neq \emptyset$ is a simply connected open region in $\mathbb{C}$. What was proved in here was, that the conjugated differentiable functions $f^c : z \mapsto f(\bar{z})$ are needed either, in order to get $\Delta f \equiv 0$ fulfilled for a dense set of continuous functions $f : U \to \mathbb{C}$.

What one would then obviously would want to do, is to pull the results in the complex via the complex isomorphism T_ι^{-1} to the $\mathcal{C}(V, \mathbb{R}^2)$. The concern is, that for a well-behaved, integrable complex function $f(re^{i\phi})$, the preimage $T_\iota^{-1} f$ is a function $g(r,\phi)$ with a polar symmetry, which generally will be strictly unintegrable at the origin: For example, if $g(r,\phi) = r^2 \sin(4\phi)$, the path integral along ϕ from 0 to 2π will not vanish. And as discussed above, this means that $\partial_x \partial_y g = -\partial_y \partial_x g$ (at the origin), which in turn suggests to look for a (possibly compact) Lie group to apply. However, there is apparently no suitable one. To get at results for $\mathcal{C}(V, \mathbb{R}^2)$ at all, it will be necessary to build from ground up.

5. Algebraic extension of $\mathbb{R}^2$ and $\mathcal{C}(V, \mathbb{R}^2)$

An *orientation* on the vector field $\mathbb{R}^n$ is an embedding

$$\varphi : \mathbb{K}^n \ni (x_1, \ldots, x_n) \mapsto \sum_{1 \le k \le n} a_k x_k \in \mathcal{A}$$

into an associative algebra $\mathcal{A}$ over the field $\mathbb{R}$ with unit element 1, such that $a_k a_j = -a_j a_k$ for all $1 \le k < j \le n$ and $a_k^2 = 1$ for all $k = 1, \ldots, n$. (For $\mathbb{K} = \mathbb{C}$ and $n = 1$ the orientation is implicitly interpreted to be "in line with" or as "given by" the direction of the real part.)

In the 2-dimensional case, $n = 2$, we define two numbers $\mathfrak{e}_1$ and $\mathfrak{e}_2$ (not contained in $\mathbb{C}$), for which

(i) $\mathfrak{e}_1\mathfrak{e}_1 = \mathfrak{e}_2\mathfrak{e}_2 \equiv 1$,
(ii) $\mathfrak{e}_1\mathfrak{e}_2 = -\mathfrak{e}_2\mathfrak{e}_1$, and
(iii) $\mathfrak{e}_1\mathfrak{e}_2 \equiv +i$.

(From conditions (i) and (ii) follows that $\mathfrak{e}_1\mathfrak{e}_2 = \pm i$, and in order to determine the sign of that value, (iii) is needed.)

Then $\varphi_+ : \mathbb{R}^2 \ni (x,y) \mapsto \zeta := \mathfrak{e}_1 x + \mathfrak{e}_2 y$ and $\varphi_- : \mathbb{R}^2 \ni (x,y) \mapsto \tilde{\zeta} := \mathfrak{e}_1 x - \mathfrak{e}_2 y$ are a vector space isomorphisms of $\mathbb{R}^2$ onto the target spaces $\varphi_\pm \mathbb{R}^2$, which we denote by $\mathbb{R}^2_\pm$.

By defining on $\mathbb{R}^2_\pm$ the metrics, induced by the quadratic form

$$Q : \varphi_\pm \mathbb{R}^2 \ni \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \mapsto (\mathfrak{e}_1 x \pm \mathfrak{e}_2 y)^2 = x^2 + y^2 = \left\| \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \|^2 \right\|,$$

$\varphi_\pm$ become isometries.

Along with $\zeta = \mathfrak{e}_1 x + \mathfrak{e}_2 y$ also $\zeta' = \mathfrak{e}_2 x + \mathfrak{e}_1 y$ solves the algebraic equation $(a+b)^2 = a^2 + b^2$. Because of $\mathfrak{e}_1\mathfrak{e}_2 = i$, $i(\mathfrak{e}_2 x + \mathfrak{e}_1 y) = \mathfrak{e}_1 x - \mathfrak{e}_2 y$, and $i\zeta' = (\mathfrak{e}_1 x - \mathfrak{e}_2 y)$ follows. To be in line with the complex functions, $i\zeta'$ will be called *conjugate* of ζ and denoted with either ζ^c or $\tilde{\zeta}$.

$$\varphi_\pm : \mathbb{R}^2 \ni (x,y) \mapsto \zeta = \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \in \mathbb{R}^2_\pm$$

then define two global coordinate charts over the manifold $(\mathbb{R}^2, \varphi_\pm)$ of positive and negative orientation.

Remark 5.1.
1. $\mathfrak{e}_1$ and $\mathfrak{e}_1$ are numbers, not just symbols: they are defined solely based on the imaginary i, which is not a symbol, but a number.
2. Sofar, $\mathbb{R}^2_\pm$ are vector spaces, which are equivalent to $\mathbb{R}^2$, but they readily extend to a non-commutative, associative algebra, which will be denoted by $\mathbb{A}$, in which the product is defined as algebra extension of:

$$\cdot : \mathbb{R}^2_\pm \times \mathbb{R}^2_\pm \ni (\mathfrak{e}_1 x \pm \mathfrak{e}_2 y, \mathfrak{e}_1 x' \pm \mathfrak{e}_2 y') \mapsto xx' + yy' \pm (ixy' - iyx') \in \mathbb{A}.$$

3. Due to $\mathfrak{e}_1\mathfrak{e}_2 = i$, the algebra $\mathbb{A}$ is inevitably complex. However it is not an algebra over the field $\mathbb{C}$: As an algebra over $\mathbb{C}$, i would commute with all elements, which is not the case for $\mathbb{A}$.
4. Anti-commutativity of $\mathfrak{e}_1$ and $\mathfrak{e}_2$ with $\mathfrak{e}_1\mathfrak{e}_2 \equiv +i$ implies: $i\mathfrak{e}_k = -\mathfrak{e}_k i$, $(k = 1,2)$, so $(\mathfrak{e}_k x + iy)^2 = x^2 - y^2$ follows for $k = 1,2$.

Next, $\zeta^2 > 0$ for all non-zero $\zeta \in \mathbb{R}^2_\pm$. Therefore the the Euclidean topology of $\mathbb{R}^2$ (and its isometric space $\mathbb{R}^2_\pm$) extends onto $\mathbb{A}$, so $\mathbb{R}^2$ and $\mathbb{R}^2_\pm$ are isometrically embedded into $\mathbb{A}$.

With this we define $\mathcal{C}_+(\varphi_+ V, \mathbb{A})$ as vector space of all functions $T_{\varphi_+} := \varphi_+ f_{conf} \varphi_+^{-1}$, where $f_{conf} \in \mathcal{C}_{conf}(V, \mathbb{R}^2)$, and likewise $\mathcal{C}_-(\varphi_- V, \mathbb{A})$ is defined as vector space of all $T_{\varphi_-} := \varphi_- f_{aconf} \varphi_+^{-1}$ with $f_{aconf} \in \mathcal{C}_{aconf}(V, \mathbb{R}^2)$. For $\zeta = \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \in \mathbb{R}^2_\pm \setminus \{0\}$ the multiplicative inverse $\frac{1}{\zeta} = \frac{\zeta}{x^2+y^2}$ is well-defined, and likewise $\zeta^m = (\mathfrak{e}_1 x \pm \mathfrak{e}_2 y)^m$ exists for $m \in \mathbb{N}$.

Since $\mathbb{R}^2_\pm$ and $\mathbb{A}$ are finite dimensional normed spaces, the vector spaces $\mathcal{C}(\varphi_\pm V, \mathbb{A})$ of $\mathbb{A}$-valued continuous functions on $\varphi_\pm$ are well-defined, and are Banach spaces with the supremum norm, which isometrically embed $\mathcal{C}_\pm(\varphi_\pm V, \mathbb{A})$ as closed subspaces.

For $\zeta_0 \in \varphi_\pm V$ a function $f \in \mathcal{C}_\pm(\varphi_\pm V, \mathbb{A})$ will be called *differentiable* in ζ_0 if and only if $\frac{df(\zeta=\zeta_0)}{d\zeta} := \lim_{\zeta\to\zeta_0}(f(\zeta) - f(\zeta_0))\frac{1}{\zeta-\zeta_0}$ exists (as an $\mathbb{A}$-valued function). $\frac{df(\zeta)}{d\zeta}$ will be called *derivative* of f.

Remark 5.2.
(i) Note that the divisional term $\frac{1}{\zeta-\zeta_0}$ is factored to the right side of f: This is to ensure uniqueness of the limit in the case that the target values $f(\zeta)$ do not commute with the variable ζ. As long as $f(\zeta)$ is real-valued, however, the ordering of the product is irrelevant: "left" and "right" derivative coincide.
(ii) In particular, we then have: $(\frac{df(\zeta)}{d\zeta})^c = \frac{df^c(\tilde{\zeta})}{d\tilde{\zeta}}$, where $f^c : \tilde{\zeta} \mapsto (f(\tilde{\zeta}))^c$.

Since $\mathbb{A}$ is a finite-dimensional algebra, the Euclidean metrics defines a natural topology on $\mathcal{A}$, through which differentiability of functions $f : U \to \mathbb{A}$ for open $U \subset \mathbb{A}$ get well-defined.
The chain rule also holds for differentiable functions $g : \varphi_\pm V \to \mathbb{A}$ and $f : g(\varphi_\pm V) \to \mathbb{A}$, where $\frac{df(u=g(\zeta))}{du}$ now denotes the derivative $Df(u)$ of f at $u \in g(\varphi_\pm V) \subset \mathbb{A}$.
Also, the product rule holds for two commuting, differentiable functions $f, g : \varphi_\pm V \to \mathbb{A}$: if $f(\zeta)g(\zeta) = g(\zeta)f(\zeta)$ for all $\zeta \in \varphi_\pm V$, then $\frac{d(f(\zeta)g(\zeta))}{d\zeta} = \frac{df(\zeta)}{d\zeta} g(\zeta) + f(\zeta)\frac{dg(\zeta)}{d\zeta}$.

In view of the isometry of $\varphi_\pm : \mathbb{R}^2 \to \mathbb{R}^2_\pm$, a real-valued function $f \in \mathcal{C}(V, \mathbb{R})$ is differentiable in some point (x_0, y_0), if and only if $\varphi_\pm V \ni \zeta \mapsto f(\zeta = \varphi_\pm(x, y))$ is differentiable in $\zeta_0 = \mathfrak{e}_1 x_0 \pm \mathfrak{e}_2 y_0$.

Because $\mathcal{C}_\pm(\varphi_\pm V, \mathbb{A})$ are the images of conformal and anti-conformal subspaces of $\mathcal{C}(V, \mathbb{R}^2)$, every $f \in \mathcal{C}_+(\varphi_+ V, \mathbb{A})$ writes as

$$f(\zeta) = \mathfrak{e}_1 g(\zeta) + \mathfrak{e}_2 g(\zeta), \text{ where } g : V \ni (x, y) \mapsto g(\zeta := \mathfrak{e}_1 x + \mathfrak{e}_1 x) \in \mathbb{R}$$

is conformal. Then path integration of f along a path $\gamma \subset \varphi_+ V$ in $\mathfrak{e}_1 x$- and $\mathfrak{e}_2 y$-coordinates from $\zeta_0 = \mathfrak{e}_1 x_0 + \mathfrak{e}_2 y_0$ to $\zeta = \mathfrak{e}_1 x + \mathfrak{e}_2 y$ equals the path-invariant integral of $G : (x, y) \mapsto (g(x, y), g(x, y))$ from (x_0, y_0) to (x, y), so f is integrable, and $If = (Ig)_1 \equiv (Ig)_2$, where $(Ig)_1$ and $(Ig)_2$ denote the projections of $Ig = ((Ig_1, (Ig)_2)$ onto its x- and y-coordinates. The 2^{nd} primitive $I^2 f_{conf}$ of f_{conf} then results into

$$I^2 f : \zeta \mapsto \mathfrak{e}_1 I^2 g(x, y) + \mathfrak{e}_2 I^2 g(x, y).$$

By induction, f_{conf} is integrable to all orders, If is differentiable, and $\frac{dIf_{conf}}{d\zeta} = f_{conf}$.

Likewise, for $f \in \mathcal{C}(\varphi_- V, \mathbb{A})$, $f(\tilde{\zeta}) = \mathfrak{e}_1 g^c(x, y) - \mathfrak{e}_2 g^c(x, y)$, where $g \in \mathcal{C}(V, \mathbb{R})$ is conformal again, and $If(\tilde{\zeta}) = (Ig^c(x, y))^c$ defines the primitive of f. Then $I^2 f = (I^2 g^c(x, y))^c$ is its second primitive, f has primitives of all orders, If is differentiable on $(\varphi_- V)$ w.r.t. $\tilde{\zeta}$, and $\frac{dIf(\tilde{\zeta})}{d\tilde{\zeta}} = f(\tilde{\zeta})$.

Next, we define analyticity:
A function $f \in \mathcal{C}(\varphi_+ V, \mathbb{A})$ is called *analytic* on $\varphi_+ V$, if for each ζ_0 in $\varphi_+ V$ there is an open neighbourhood $U \subset \mathbb{R}^2_+$, of ζ_0, such that

$$f(\zeta) = \sum_{k \geq 0} c_k (\zeta - \zeta_0)^k, \quad (c_k \in \mathbb{A},\ k \in \mathbb{N}),$$

where the power series is to converge uniformly on U. Analogously, $f \in \mathcal{C}(\varphi_- V, \mathbb{A})$ is called analytic, if every $\tilde{\zeta}_0 \in \varphi_- V$ has an open neighbourhood $U \subset \mathbb{R}^2_-$ of $\tilde{\zeta}_0$, on which f the uniformly converging limit $f(\tilde{\zeta}) = \sum_{k \geq 0} c_k (\tilde{\zeta} - \tilde{\zeta}_0)^k$.

On the positive/negative orientated $\varphi_\pm \mathbb{R}^2$ let

$$\Psi_\pm : \zeta = \mathfrak{e}_1 x \pm \mathfrak{e}_2 y \mapsto \frac{1}{\zeta}$$

be the *Cauchy function*. Then $\Psi_\pm(\zeta_0 - \zeta) = \frac{1}{\zeta_0}\sum_{k\geq 0}\left(\zeta_0^{-1}\zeta\right)^k$ exists for $\left|\zeta_0^{-1}\zeta\right| < 1$, and the series uniformly converges in ζ on all compact simply connected regions not containing the pole ζ_0. So, it is *analytic* on these regions.

Remark 5.3. Ψ_+ is conformal on simply connected regions not containing the origin, because

(1) the constant function and the identity $id : \mathbb{R}^2_+ \ni \zeta \mapsto \zeta \in \mathbb{A}$ are conformal,
(2) the addition $f + g$ of two conformal functions f and g is conformal,
(3) if f is conformal, then $\frac{1}{f}$ is conformal on all simply connected regions, on which f has no zeros.

Since also the product of two conformal functions is conformal again, the path integral $\int_\gamma f(\zeta)\Psi_+(\zeta_0 - \zeta)d\zeta$ for $f \in \mathcal{C}_+(\varphi_+ V, \mathbb{A})$ along a (piecewise smooth) path $\gamma \subset \varphi_+ V \setminus \{\zeta_0\}$ is a conformal function of ζ_0.

6. Analyticity of $\mathcal{C}(V, \mathbb{R}^2)$

For $r > 0$ the paths $\gamma_\pm : [0, 2\pi] \ni t \mapsto r(\mathfrak{e}_1 \cos(t) \pm \mathfrak{e}_2 \sin(t)) \in \mathbb{R}^2_\pm$ are circular paths around the origin with positive and negative orientation from and to $\mathfrak{e}_1 r$. The path integrals $\int_{\gamma_+} \Psi_+(\zeta)d\zeta$ and $\int_{\gamma_-} \Psi_-(\tilde{\zeta})d\tilde{\zeta}$ along these paths then calculate to

$$\int_{\gamma_+} \Psi_+(\zeta)d\zeta = \int_0^{2\pi} (\mathfrak{e}_1 \cos(t) + \mathfrak{e}_2 \sin(t))(-\mathfrak{e}_1 \sin(t) + \mathfrak{e}_2 \cos(t))dt \tag{6.1}$$

$$= \int_0^{2\pi} (\mathfrak{e}_1\mathfrak{e}_2(\cos^2(t) + \sin^2(t)))dt = \int_0^{2\pi} idt = 2\pi i, \text{ and}$$

$$\int_{\gamma_-} \Psi_-(\tilde{\zeta})d\tilde{\zeta} = \int_0^{2\pi} (\mathfrak{e}_1 \cos(t) - \mathfrak{e}_2 \sin(t))(-\mathfrak{e}_1 \sin(t) - \mathfrak{e}_2 \cos(t))dt \tag{6.2}$$

$$= \int_0^{2\pi} -(\mathfrak{e}_1\mathfrak{e}_2(\cos^2(t) + \sin^2(t)))dt = \int_0^{2\pi} -idt = -2\pi i.$$

This gives

Proposition 6.1. *1. Every conformal $f_{conf} \in \mathcal{C}(V, \mathbb{R}^2)$ extends as an analytic function $f_+ : \varphi_+ V \to \mathbb{A}$, where $\varphi_+ : \mathbb{R}^2 \to \mathbb{R}^2_+$ is the chart with positive orientation. The Cauchy-formula holds for f_+: $\int_\gamma f_+(\zeta)\frac{1}{\zeta - \zeta_0}d\zeta = 2\pi i f_+(\zeta_0)$, where $\gamma \subset \varphi_+ V$ is a positively orientated Jordan curve around ζ_0 (i.e: a piecewise continuously differentiable closed curve looping once around ζ_0 at some distance $\epsilon > 0$ from ζ_0 with positive orientation).*

2. Every anti-conformal $f_{aconf} \in \mathcal{C}(V, \mathbb{R}^2)$ extends as analytic function $f_- : \varphi_- V \to \mathbb{A}$, where $\varphi_- : \mathbb{R}^2 \to \mathbb{R}^2_-$ is the chart with negative orientation.

The Cauchy-formula holds for f_-*:* $\int_\gamma f_-(\tilde{\zeta})\frac{1}{\tilde{\zeta}-\tilde{\zeta}_0}d\tilde{\zeta} = -2\pi i f_-(\tilde{\zeta}_0)$*, where* $\gamma \subset \varphi_- V$ *is a negatively orientated Jordan curve around* $\tilde{\zeta}_0$*.*

Proof. Since $f \in \mathcal{C}_+(\varphi_+ V, \mathbb{R}^2)$ is integrable on V, the path integrals (within $\varphi_+ V$) from startpoint $a \in \varphi_+ V$ to endpoint $b \in \varphi_+ V$ are path independent. By the above, the Cauchy function $\Psi_+(\zeta) = \frac{1}{\zeta}$ is analytic on convex sets not containg the origin, hence integrable on there. The Cauchy-formula $f(\zeta_0) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\frac{1}{\zeta-\zeta_0}dz$ then follows from equation 6.1 togther with the continuity of f for all closed, positively orientated Jordan curves $\gamma \subset \iota V$ around ζ_0. Then, as in Proposition 1.1, $f(\zeta_0)$ is within the encircled open region the uniform limit of a power series on ϵ-neighbourhoods of ζ_0, so analytic in there.
For $f \in \mathcal{C}_-(\varphi_- V, \mathbb{R}^2)$ the the proof is analogous with equation 6.2. ☐

Corollary 6.2. *The complex isomorphism* T_ι *maps* $\mathcal{C}_+(\varphi_+ V, \mathbb{R}^2)$ *onto the complex subspace* $\mathcal{C}_{conf}(\iota V, \mathbb{C})$ *of holomorphic functions, and is given by:* $T_\iota : f \mapsto \mathfrak{e}_1 f \mathfrak{e}_1$*. Hence, the power series expansion* $f(z) = \sum_k c_k (z-z_0)^k$ *of any holomorphic function* $f \in \mathcal{C}(\iota V, \mathbb{C})$*, determines the power series expansion for* $T_\iota^{-1} f \in \mathcal{C}_+(\varphi_+ V, \mathbb{R}^2)$ *to be* $(T_\iota^{-1} f)(\zeta) = \sum_k (\mathfrak{e}_1 c_k)(\zeta - \zeta_0)^k$*.*

Part 2. Integrability and orientation of continuous functions in $n \in \mathbb{N}$ dimensions

7. Preliminaries

Let $\mathbb{K}$ denote either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. For $n \in \mathbb{N}$ let $V \subset \mathbb{R}^n$ be bounded and be the closure of a non-trivial open subset of $\mathbb{R}^n$. A path γ in V is a continuous, piecewise smooth mapping $\gamma : [0,1] \to V$, where $[0,1]$ denotes the closed real interval from 0 to 1. As γ is piecewise smooth, it is the sum $\gamma = \gamma_1 + \cdots + \gamma_k$ of $k \in \mathbb{N}$ continuously differentiable paths γ_k. $|\gamma| := \sum_{1 \le l \le k} \int \|\frac{d\gamma_l(t)}{dt}\| dt$ is called *arc length* of γ, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. V is called *connected*, if for each $x, y \in V$ there is a path γ in V with $\gamma(0) = x$ and $\gamma(1) = y$. A *compact set* $V \subset \mathbb{R}^n$ is a closed and bounded subset of $\mathbb{R}^n$. The path γ is called closed if $\gamma(0) = \gamma(1)$, and a connected V is called *simply connected*, if all closed paths in V are point homotopic in V, i.e.: if V has no holes.

If V is also simply connected (i.e.: if all closed paths in V are 0-homotopic in V), then V will be called a *simply connected compact region.* A function f from V to $\mathbb{K}^m$ for some $m \in \mathbb{N}$ is called *continuous on* V, if is well-defined and continuous in an open environment $U \subset \mathbb{R}^n$ of V. The set of continuous $\mathbb{K}^m$-valued functions on V then is a Banach space $\mathbb{C}(V, \mathbb{K}^m)$ with the supremum norm $\|\cdot\| : f \mapsto sup_{x \in V} \big\|f(x)\big\|_{\mathbb{R}^m}$, where $\|\cdot\|_{\mathbb{R}^m}$ denotes the Euclidean norm of $\mathbb{K}^m$. In the following, we'll briefly write $\mathbb{C}(V)$ for $\mathbb{C}(V, \mathbb{K}^m)$, when it is clear what the target space $\mathbb{K}^m$ is.

Let V be a simply connected, closed region and $f \in \mathcal{C}(V, \mathbb{R}^m)$. Then for every piecewise continuously differentiable path $\gamma : [0,1] \to V$, the path integral $\int_\gamma f(s)ds := \int_0^1 f(\gamma(t))\frac{d\gamma(t)}{dt}dt$ is a well-defined, continuous linear functional on $\mathcal{C}(V, \mathbb{R}^m)$. A function $f \in \mathcal{C}(V, \mathbb{R}^m)$ is called *integrable*, if and only if $\int_\gamma f(s)ds = 0$ for every closed path γ in V. In all cases, if f is integrable, then the path integrals from a fixed startpoint in V to the variable endpoint in V define a function If, which is commonly called *primitive* of f. (Since two primitives of the same function f differ utmost by the choice of the startpoint, which adds an additive constant, the primitives are naturally defined as equivalence classes.)

In this paper, we restrict integrability considerations on two complementary cases, $f \in \mathcal{C}(V, \mathbb{R}^n)$ and $f \in \mathcal{C}(V, \mathbb{R})$, that is: we'll assume either $m = n$ or $m = 1$, where $V \subset \mathbb{R}^n$. The situation then is in line with the 2-dimensional case: Given that $V \subset \mathbb{R}^n$ is a simply connected compact region, the primitive If of an integrable $f \in \mathcal{C}(V, \mathbb{R}^n)$ is a (continuously differentiable) real-valued function $If : V \to \mathbb{R}$, while for integrable $f \in \mathcal{C}(V, \mathbb{R})$ the primitive is $\mathbb{R}^n$-valued.
It will be seen below, that these functions are twice integrable, if they are once integrable. Hence, the integrable real-valued or $\mathbb{R}^n$-valued functions f have primitives $I^m f$ of all orders $m \in \mathbb{N}$, which toggle in the dimensionality of their range at each order of integration (from $\mathbb{R}$ to $\mathbb{R}^n$).

Proposition 7.1 (Integrability). *Let $V \subset \mathbb{R}^n$ be a simply connected compact region with $n > 2$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{R}^n$. Then the following statements are equivalent for $f \in (V, \mathbb{K})$:*

(1) f is integrable on V.

(2) There exists an $\epsilon > 0$, such that for all $x_0 \in V$ and all spheres $S(r)$ of radius $r < \epsilon$ around x_0: $\int_{S(r)} f(x)da \equiv 0$, where $da := \sum_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \cdots \wedge dx_n$ denotes the differential of the $(n-1)$-dimensional surface element of $S(r)$.

(3) There exists an $\epsilon > 0$, such that for all $x_0 \in V$ and all r-balls $B(r)$ of radius $r < \epsilon$ around x_0: $\int_{B(r)} f(x)d^n x = \int_{B(r)} f(x)dx_1 \wedge \cdots \wedge dx_n \equiv 0$ for all permutations of the integrals along the coordinates.

(Note that the differential $d^n x := dx_1 \wedge \cdots \wedge dx_n$ is the alternating n-form, and it is not the Lebesgue measure of the (then always non-negative) volume.)

Proof. (1)⇒(2) follows indirectly: If for a given ordering of the coordinates and some r-sphere $S(r) \subset V$: $\int_{S(r)} f(x)da \neq 0$, then locally and in spherical coordinates, the surface integral is the (ordered) product of one integration along an azymuthal angle ϕ_1 from 0 to 2π and $n-2$ integrations along the polar angles $\vartheta_1, \ldots, \vartheta_{n-2}$ from 0 to π, each. Applying the mean value theorem to the polar angles $\vartheta_1, \ldots, \vartheta_{n-2}$ iteratively one by one, there must exist $\vartheta_1, \ldots, \vartheta_{n-2}$ such that $\frac{\partial}{\partial\vartheta_{n-2}} \cdots \frac{\partial}{\partial\vartheta_1} \int_{S(r)} f(x)da \neq 0$, so $\int_\gamma f(s)ds \neq 0$ follows for some closed, circular curve γ of radius $r' \leq \epsilon$ in the azymuthal plane around some $x' \in V$.

Conversely, if f is integrable on V, the path integrals of f along all circles on the r-spheres $S(r)$ vanish, so starting with the r-circular path γ_{12}

around fixed $x \in V$ in the azymuthal x_1x_2-plane, the integrals of $\int_{\gamma_{12}} f(s)ds$ from 0 to $t \leq \pi$ along any of the $n-2$ polar coordinates must be zero either, and iterative integration w.r.t. the orthogonal polar coordinates from to $t \leq \pi$, again gives zero. So, $\int_{S(r)} f(x)da \equiv 0$, regardless of the order of the coordinate integration.

Finally, (1) and (2) are equivalent, because in local spherical coordinates $\int_{B(r)} f(x)d^nx = \int_0^r \int_{S(r')} f(x)dadr'$. □

Remark 7.2. Note that (3) of Proposition 7.1 can be understood as Poincaré's lemma in terms of differential n-forms:
The volume integral of f over $B(r)$ can be written as integral of the differential form $d\alpha := f(x)dx_1 \wedge \cdots \wedge dx_n$, which possesses an exact primitive α on $B(r)$, i.e.: a differential form ω exists on $B(r)$ such that $\alpha = d\omega$, then by Poincaré's lemma: $\int_{B(r)} d\alpha = \int_{B(r)} f(x)dx_1 \wedge \cdots \wedge dx_n \equiv 0$ (see: [3][Theorem 2.12.1]), and we then have (by Stoke's theorem): $\int_{\partial B(r)} \alpha = \int_{B(r)} d\alpha \equiv 0$, where $\partial B(r)$ is the boundary of $B(r)$, which is $S(r)$ (see: [3][Theorem 4.4.1]).

So far, we are in line with the 2-dimensional case: The concept of integrability does not depend on the orientation of the vector space. To proceed, we now need the concept of orientation on the $\mathbb{R}^n$ for $n > 2$. While in the 2-dimensional case, i.e. for $n = 2$, the orientated $\mathbb{R}^2$ is isomorphic to the field $\mathbb{C}$ of complex numbers (see: Part 1), for $n > 2$ there is no field structure any more, and the strong concept of complex differentiability is not applicable. This is a general problem for multi-dimensional complex analysis, and it is dealt with in there by projection to affine 2-dimensional sections, which then allow a pairwise treatment. This we can also do in here:

For $1 \leq k < l \leq n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $\pi_{kl}(x) : \mathbb{R}^2 \ni (y_k, y_n) \mapsto (x_1, \ldots, x_k + y_k, \ldots, x_l + y_l, \ldots, x_n) \in \mathbb{R}^n$ be the affine plane sections through x in the kl-plane, and let $\pi_{kl} : \mathbb{R}^2 \ni x \mapsto (x_k, x_l) \in \mathbb{R}^2$ be the canonical coordinate projection.

For a simply connected compact region $V \subset \mathbb{R}^n$ and $f \in \mathcal{C}(V, \mathbb{R}^n)$ then f is integrable on V if and only if for all $x \in V$ and all $1 \leq k < l \leq n$ there is an ϵ-environment $U_\epsilon \subset \mathbb{R}^2$ of the origin, such that $U_\epsilon \ni (y_k, y_l) \mapsto \pi_{kl} f(\pi_{kl}(x)(y_k, y_l))$ is integrable. And for real-valued $f \in \mathcal{C}(V, \mathbb{R})$, the integrability condition is simpler: $f \in \mathcal{C}(V, \mathbb{R})$ is integrable on V if and only if for each $x \in V$ and all $1 \leq k < l \leq n$ there is an ϵ-environment U_ϵ of the origin in 2 dimensions, such that $U_\epsilon \ni (y_k, y_l) \mapsto f(\pi_{kl}(x)(y_k, y_l))$ is integrable (in U_ϵ).
So, this allows us to fall back to the 2-dimensional functions, for which it was shown in Part 1, that an integrable function (on a compact simply connected region) is twice integrable. Therefore, we have

Proposition 7.3. *As above, let $V \subset \mathbb{R}^n$ be a simply connected compact region with $n > 2$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{R}^n$. If $f \in \mathcal{C}(V, \mathbb{K})$ is integrable, then it is twice integrable. So, if at all a primitive If of f exists on V, then it has primitives $I^m f$ of all orders $m \in \mathbb{N}$.*

Let's me restate the definition of orientation on $\mathbb{R}^n$ from Section 5:

Definition 7.4 (Orientation). An *orientation* on the vector field $\mathbb{R}^n$ is an embedding

$$\varphi : \mathbb{K}^n \ni (x_1, \ldots, x_n) \mapsto \sum_{1 \leq k \leq n} a_k x_k \in \mathcal{A}$$

into an associative and non-commutative algebra $\mathcal{A}$ over the field $\mathbb{R}$ with unit element 1, such that $a_k a_j = -a_j a_k$ for all $1 \leq k < j \leq n$ and $a_k^2 = 1$ for all $k = 1, \ldots, n$.

Remark 7.5. (i) $\mathcal{A}$ is the real-valued algebra generated by the elements $a_1, \ldots, a_n$, defined above.

(ii) Other than in the case of $n = 2$ dimensions, for all higher dimensions, we can always find $n \times n$ (Hermititian) matrices as representations for $a_1, \ldots, a_n$.

In particular, the product $a_1 \cdots a_n$ satisfies: $(a_1 \cdots a_n)^2 = -1$ for $n = 2, 3, 6, 7, 10, 11, \ldots$ and $(a_1 \cdots a_n)^2 = +1$ for $n = 4, 5, 8, 9, \ldots$. In order to reduce ambiguity of the a_k, one might additionally demand $a_1 \cdots a_n := i$ for $n = 2, 3, 6, 7, \ldots$ and $a_1 \cdots a_n := 1$ for $n = 4, 5, 8, 9, \ldots$. Though, it is not necessary, so it is left out in here.
Monomials $a_{k_1} a_{k_2} \ldots a_{k_l}$ with $1 \leq k_1 < \cdots < n$ and their cyclic permutations are said to be *positively orientated.* Otherwise, $a_{k_1} a_{k_2} \cdots a_{k_l}$ is called *negatively orientated.*
This defines the (l-volume) differentials $a_{k_1} a_{k_2} \cdots a_{k_l} dx_{k_1} \cdots dx_{k_l}$ the same way as the $dx_{k_1} \wedge \cdots \wedge dx_{k_l}$ for differential forms, the only difference being that we have here the a_k, replacing the wedges.

Remark 7.6 (Graßmann algebra and differential forms). I am well-aware that an orientation also can be defined on the Graßmann algebra of alternating mappings, on which then differential forms are constructed. I refrain from doing this, since it will not suffice our purpose, for one major reason: the Graßmann algebra does not extend the vector space $\mathbb{R}^n$. Because the alternating product $a \wedge a \equiv 0$ for any $a \in \mathbb{R}^n$, differential forms are limited in order to the number of dimensions n, and alternating products of order $m > n$ are all identical zero. So, we have to do it differently, in order to catch up with power series expansions etc..

For $1 \leq k_1 \neq k_2 \cdots \leq n$, the monomials $a_{k_1} a_{k_2} \cdots$ of the $a_k \in \mathbb{A}$ will in the following take on the role of the alternating monomials $e_{k_1} \wedge e_{k_2} \wedge \cdots$ of the orthogonal base vectors e_k, while keeping the squares $a_k^2 \equiv 1$ in line with the expected Euclidean product $e_k^2 \equiv 1$.

This sets up the toolset to generalize the 2-dimensional results from Part 1 to n dimensions.

8. Integrability decomposition

Proposition 8.1 (Integrability decomposition). *Let $V \subset \mathbb{R}^2$ be be a simply connected compact region and $\mathbb{K}$ stand for either $\mathbb{R}^n$ or $\mathbb{R}$.*

$\mathbb{C}(V, \mathbb{K})$ *is the topological direct sum of two subspaces: the space of integrable functions* $\mathcal{Y}_+(V, \mathbb{K})$ *and a complementary space* $\mathcal{Y}_-(V, \mathbb{K})$ *of unintegrable functions.*

Proof. So, let $f \in \mathbb{C}(V, \mathbb{K})$. Then f is to be continuous on an ϵ-environment $U \supset V$ of V, and we define $\mathcal{B}$ as set of all balls $B(r, x) = \{(x') \in \mathbb{K} \mid |x' - x| \leq r\}$ for $(x, y) \in V$, where $r > 0$ and $r < \epsilon$. Let $\Gamma(\mathcal{B})$ be the set of all positively orientated boundaries $S(r, x)$ of the balls $B(r, x)$ with $d > 0$ and $(x) \in V$. Then $p_{S(r,x)} : f \mapsto p_{S(r,x)}(f) := \left|\int_{S(r,x)} f(a)da\right| \geq 0$, $(S(r, x) \in \Gamma(\mathcal{B}))$, defines a family of continuous seminorms on $\mathbb{C}(V, \mathbb{K})$. The set of all $f \in \mathbb{C}(V, \mathbb{K})$, for which $p_{S(r,x)}(f) = 0$ for all $\gamma \in S(r, x) \in \Gamma(\mathcal{B}))$ then is closed in $\mathbb{C}(V, \mathbb{K})$, since it is the intersection of the closed sets. It contains all integrable(, continuous) functions on V.

Let $\mathcal{Y}_+(V, \mathbb{K})$ denote this closed space of $\mathbb{C}(V, \mathbb{K})$. Then its complement is an algebraic subspace, which is open in $\mathcal{C}(V, \mathbb{K})$. We call it space of *non-integrable* functions and denote it by $\mathcal{Y}_-(V, \mathbb{K})$.

To finish up, it remains to be shown that $\mathcal{Y}_+(V, \mathbb{K})$ is also open, or equivalently to prove that $\mathcal{Y}_-(V, \mathbb{K})$ is closed. We need to refine this family of seminorms, in order to make further progress:
For each $f \in \mathcal{C}(V, \mathbb{K})$ the function

$$F : [0, \epsilon] \times V \ni (r, x) \mapsto \int_{S(r,x)} f(a)da \in \mathbb{K}$$

is uniformly continuous on $[0, d] \times V$, but also: $|F(h, x) - F(h', x)| = o(h - h')$ (for $h, h' < d$). So, F is (right) differentiable (at $h = 0$) in its first argument for $h \to 0$, and F is continuously differentiable in h for each $(x, y) \in V$ for $0 < h < d$. Let $|S_n|$ be the area of the n-dimensional unit sphere. Because every $f \in \mathcal{C}(V, \mathbb{K})$ can be isometrically extended as a continuous function onto the closed ϵ-environment of V, the mapping

$$p : \mathcal{C}(V, \mathbb{K}) \ni f \mapsto sup_{h \in [0,\epsilon], x \in V} \frac{1}{r^{(n-1)}|S_n|} |F(h, x)| \geq 0$$

is a well-defined semi-norm on $\mathcal{C}(V, \mathbb{K})$, and it is a norm on its (open) subspace $\mathcal{Y}_-(V, \mathbb{K})$ of unintegrable functions. And, because $p(f) \leq \frac{1}{|S_n|} sup_{x \in V} |f(x)|$, p is a continuous seminorm on $\mathcal{C}(V, \mathbb{K})$. But for $f \in \mathcal{Y}_-(V, \mathbb{K})$ we also have $p(f) \geq \sup_{x \in V, r < \epsilon} \inf_{y \in S(r,x)} |f(y)|$, where $\inf_{y \in S(r,x)} |f(y)| \to |f(y)|$ uniformly for $x \in V$ as $r \to 0$, because of uniform continuity of f on V. So, on $\mathcal{Y}_-(V, \mathbb{K})$, the norm p is equivalent to the supremum norm, which is the norm of the Banach space $\mathcal{C}(V, \mathbb{K})$. This proves that $\mathcal{Y}_-(V, \mathbb{K})$ is a closed subspace of $\mathcal{C}(V, \mathbb{K})$, and as it was already shown to be open, both complementary subspaces $\mathcal{Y}_\pm(V, \mathbb{K})$ are both closed and open.

Therefore, the canonical projections to the quotient spaces $\pi_\pm : \mathcal{C}(V, \mathbb{K}) \ni f \mapsto [f]_\pm \in \mathcal{C}(V, \mathbb{K})/\mathcal{Y}_\pm(V, \mathbb{K})$ are (bi-)continuous, and $\mathcal{C}(V, \mathbb{K})$ is the topological direct sum of its closed and open subspaces $\mathcal{Y}_\pm(V, \mathbb{K})$ – as was asserted. □

Remark 8.2. Note that the restriction to positive orientated boundaries of the balls $B(r, x)$ is unnecessary for the above proof: With this, the number of elements in $\Gamma(\mathcal{B})$ is just halved. The values of integration in opposite orientation is only an additional factor -1.

The decomposition into the spaces $\mathcal{Y}_\pm(V,\mathbb{K})$ is a provisional result and not the final decomposition: One would want the integrable and unintegrable subspaces to be isomorphic. We'll see next, that there are conjugations on $\mathcal{C}(V,\mathbb{K})$, which map the $\mathcal{Y}_-$-spaces into their complementary $\mathcal{Y}_+$-spaces, but leave a subspace of the $\mathcal{Y}_+$-spaces invariant. The goal then will be to extract that subspace and to decompose $\mathcal{Y}_+$ further.

9. Jacobians, and $\mathcal{C}_0$-spaces

Again, let $V \subset \mathbb{R}^n$ be a simply connected compact region. We now examine the spaces of integrable and unintegrable functions:

$\mathcal{C}(V,\mathbb{R}^n)$ contains the space of infinitely differentiable functions $\mathcal{C}^\infty(V,\mathbb{R}^n)$ as a dense subspace (see: [6]). Restricting to these has the advantage that the structure of the subspaces can be classified by the types of the Jacobi matrices (i.e.: the derivatives) of its elements. With this we have: The derivative of every continuously differentiable $f \in \mathcal{C}(V,\mathbb{R}^n)$ can be represented by matrix-valued function Df, called the *Jacobian*, given by

$$Df(x,y) = \begin{pmatrix} g_{11}(x,y) & g_{12}(x) & \cdots & g_{1n}(x) \\ g_{21}(x) & g_{23}(x) & \cdots & g_{2n}(x) \\ & \cdots & & \\ g_{n1}(x) & g_{n2}(x) & \cdots & g_{nn}(x) \end{pmatrix}, \text{ with } g_{kl} \in \mathcal{C}(V,\mathbb{R})$$

By Poincaré's lemma (see e.g.: [3][Ch. 1 2.12.1]), a continuously differentiable function $f \in \mathcal{C}(V,\mathbb{R}^n)$ is integrable if and only if its Jacobian Df is a symmetric matrix, i.e: $g_{kl} = g_{lk}$ holds for all $1 \le k,l \le n$. These then comprise all continuously differentiable elements from $\mathcal{Y}_+(V,\mathbb{R}^n)$. And the unintegrable, continuously differentiable $f\mathcal{Y}_-(V,\mathbb{R}^n)$ then have the anti-symmetric Jacobian Df

$$Df(x,y) = \begin{pmatrix} 0 & -g_{21}(x) & \cdots & -g_{n1}(x) \\ g_{21}(x) & 0 & \cdots & -g_{n2}(x) \\ & \cdots & & \\ g_{n1}(x) & g_{n2}(x) & \cdots & 0 \end{pmatrix}, (g_{kl} = -g_{lk} \in \mathcal{C}(V,\mathbb{R}))$$

For any $f \in \mathcal{C}^\infty(V,\mathbb{R}^n)$ the Jacobian Df can be broken into the sum $Df = Df_{sym} + Df_{asym}$ of a symmetric matrix $Df_{sym} = \frac{1}{2}(Df + Df^t)$ and an anti-symmetric matrix $Df_{asym} := \frac{1}{2}(Df - Df^t)$, where Df^t is defined as the transpose of Df. That decomposes $\mathcal{C}^\infty(V,\mathbb{R}^n)$ into the algebraic direct sum of a symmetric and an anti-symmetric subspace. Since $\mathcal{C}^\infty(V,\mathbb{R}^n)$ is dense in $\mathcal{C}(V,\mathbb{R}^n)$, if we see that the closures of these subspaces w.r.t. the supremum norm are just the spaces $\mathcal{Y}_\pm(V,\mathbb{K})$. There is just one problem: $\mathcal{Y}_+(V,\mathbb{R}^n)$ does contain elements f, for which Df has non-zero diagonal elements, so $\mathcal{Y}_-(V,\mathbb{R}^n)$ is not isomorphic (or even isometric) with $\mathcal{Y}_+(V,\mathbb{K})$.
So, we want to extract these diagonal matrices, integrate these, prove that

their primitives constitute their own subspace, and show that this is an open and closed subspace of $\mathcal{C}(V,\mathbb{R}^n)$:

Let Df be a diagonal $n\times n$-matrix, where its diagonal elements are continous functions $g_{11},\dots,g_{nn}\in\mathcal{C}(V,\mathbb{R})$. Then Df is integrable, because for any piecewise continuously differentiable path $\gamma:[0,1]\to V$ the integral $f(x)=\int_\gamma Df(s)ds=\int_0^1 Df(\gamma(t))\cdot\frac{d\gamma(t)}{dt}dt$ does only depend on the endpoints $\gamma(0)$ and $\gamma(1)$: on convex subsets, $f=(f_k)_{1\le k\le n}$ is given by $f_k(x)=\int_{\gamma(0)_k}^{\gamma(1)_k} g_{kk}(x_1,\dots,s,\dots,x_n)$, $k=1,\dots,n$. The set of all continuously differentiable vector functions with diagonal matrix Df is a vector space, $\mathcal{Z}$, say. For all $f=(f_k)_k\in\mathcal{Z}$ and all $1\le k\ne l\le n$ then: $\frac{\partial f_k}{\partial x_l}\equiv 0$. We equip $\mathcal{Z}$ with the supremum norm and denote its closure by $\mathcal{C}_0(V,\mathbb{R}^n)$. (Then again all $f=(f_k)_k$ $f=(f_k)k\in\mathcal{C}_0(V,\mathbb{R}^n)$ satisfy: $\frac{\partial f_k}{\partial x_l}\equiv 0$ for $1\le k\ne l\le n$.)

Further, $\mathcal{C}_0(V,\mathbb{R}^n)$ also is open, because for any $f\ne 0$ in $\mathcal{C}_0(V,\mathbb{R}^n)$, there must exist some $x\in V$, for which $\left|f(x)\right|>\epsilon>0$. Let $\Omega(\epsilon)$ be the open set of all $g\in\mathcal{C}_0(V,\mathbb{R}^n)$, for which $\sup_{x\in V}\left|g(x)\right|<\epsilon/2$. Then $f+\Omega(\epsilon/2)$ is an open subset of $\mathcal{C}_0(V,\mathbb{R}^n)$ not containing 0, so $\mathcal{C}_0(V,\mathbb{R}^n)$ is also open. Then again, the canonical projection $\pi_0:\mathcal{Y}_+(V,\mathbb{R}^n)\to\mathcal{C}_0(V,\mathbb{R}^n)$ is continuous, so $\mathcal{C}_0(V,\mathbb{R}^n)$ possesses a topological complement

$$\mathcal{C}_+(V,\mathbb{R}^n):=\mathcal{Y}_+(V,\mathbb{R}^n)/\mathcal{C}_0(V,\mathbb{R}^n).$$

The functions in $\mathcal{C}_+(V,\mathbb{R}^n)$ will be called *purely integrable.*

We gain more insight into the space $\mathcal{C}_0(V,\mathbb{R}^n)$ considering the space of real-valued functions, $\mathcal{C}(V,\mathbb{R})$:
It has a dense subspace $\mathcal{C}^1(V,\mathbb{R})$ of continuously differentiable functions, and the space of its derivatives just equals $\mathcal{C}(V,\mathbb{R}^n)$. And because the functions $f\in\mathcal{C}_0(V,\mathbb{R}^n)$ are integrable to continuously differentiable functions $If\in\mathcal{C}(V,\mathbb{R})$, the primitives If of $\mathcal{C}_0(V,\mathbb{R}^n)$ satisfy $\partial_k\partial_l If\equiv 0$ for all $1\le k\ne l\le n$. Their closure w.r.t the supremum norm, denoted by $\mathcal{C}_0(V,\mathbb{R})$, then consists of all functions, which locally are the sums of 1-dimensional functions: $f:x=(x_1,\dots,x_n)\mapsto f_1(x_1)+\dots+f_n(x_n)$. And this, of course, is a closed and open subspace of $\mathcal{C}(V,\mathbb{R})$.

For convenience, the 1-dimensional space $Const$ of additive constants will be included in $\mathcal{C}_0(V,\mathbb{K})$.

In all, we proved:

Proposition 9.1. *Let V be a compact simply connected region in $\mathbb{R}^n$, and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{R}^n$. Then $\mathcal{C}(V,\mathbb{K})$ is – modulo additive constants – the topological direct sum of three subspaces $\mathcal{C}(V,\mathbb{K})=\mathcal{C}_+(V,\mathbb{K})\oplus\mathcal{C}_0(V,\mathbb{K})\oplus\mathcal{C}_-(V,\mathbb{K})$ of the space of purely integrable functions $\mathcal{C}_+(V,\mathbb{K}):=\mathcal{Y}_+(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$, an isomorphic space of unintegrable functions $\mathcal{C}_-(V,\mathbb{K}):=\mathcal{Y}_-(V,\mathbb{K})$, and a complementary $\mathcal{C}_0(V,\mathbb{K})$.*

10. Partially integrable and unintegrable functions

In this section the subspace $\mathcal{C}(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K}) = \mathcal{C}_+(V,\mathbb{K}) \oplus \mathcal{C}_-(V,\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{R}^n$ is to be decomposed further:

As immediate consequence of Proposition 7.1, a function $f \in \mathcal{C}(V,\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{R}^n$ (on a simply connected compact region $V \subset \mathbb{R}^n$) is integrable, if and only if it is locally integrable on all convex, compact, closed regions $U \subset V$. So, without loss of generality, we may assume V to be convex. Then, by Proposition 7.1 f is integrable on that convex V, if and only if f is integrable in all its 2-dimensional orthogonal affine sections, of which there are $\binom{n}{2} = \frac{n(n-1)}{2}$, one for each pair of coordinate indices (μ,ν), $(1 \le \mu < \nu \le n)$.

Let $\Gamma(V)$ be the set of all piecewise smooth paths $\gamma : [0,1] \ni s \mapsto \gamma(s) \in V$, and let $|\gamma|$ denote the arc length of γ, $(\gamma \in \Gamma(V))$. Then the supremum norm on $\mathcal{C}(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ is equivalent to the supremum norm

$$f \mapsto \sup_{\gamma \in \Gamma(V)} \frac{1}{|\gamma|} \left| \int_\gamma f(s)ds \right| .$$

Assuming convexity of V as above, each γ is the sum

$$\gamma = \sum_{1 \le \mu < \nu \le n} \gamma_{\mu\nu}$$

of $\binom{n}{2}$ paths $\gamma_{\mu\nu} = \pi_{\mu\nu}\gamma$, where $\pi_{\mu\nu} : V \ni (x_1,\dots,x_n) \mapsto (x_\mu x_\nu) \in \mathbb{R}^2$ are the orthogonal projections to the $x_\mu x_\nu$-coordinate planes. Let $\Gamma_{\mu\nu}(V)$ be the set of all paths $\pi_{\mu\nu}\gamma$, $(\gamma \in \Gamma(V))$. The supremum norm on $\mathcal{C}(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ then is also equivalent to the sum of seminorms

$$q_{\mu\nu} : f \mapsto \sup_{\gamma \in \Gamma_{\mu\nu}(V)} \frac{1}{|\gamma|} \left| \int_\gamma f(s)ds \right| .$$

The vector space $\mathcal{C}^{\mu\nu}(V,\mathbb{K})$ of all $f \in \mathcal{C}(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$, for which $q_{kl}(f) = 0$ holds for all $(k,l) \neq (\mu,\nu)$ and $(1 \le k < l \le n)$, therefore is an open and closed Banach subspace of $\mathcal{C}(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$.

And because $\mathcal{C}(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ does not contain the invariant subspace $\mathcal{C}_0(V,\mathbb{K})$, the subspaces $\mathcal{C}^{\mu\nu}(V,\mathbb{K})$ are mutually disjoint, i.e:

$$\mathcal{C}^{\mu\nu}(V,\mathbb{K}) \cap \mathcal{C}^{kl}(V,\mathbb{K}) = \{0\}, \quad (\mu,\nu) \neq (k,l), \quad (1 \le \mu, k < \nu, l \le n).$$

So, $\mathcal{C}(V,\mathbb{K})/\mathcal{C}_0(V,\mathbb{K})$ is the toplogical direct sum of its subspaces $\mathcal{C}^{\mu\nu}(V,\mathbb{K})$, $(1 \le \mu < \nu \le n)$.

Either by applying [5][Proposition 2.1] or by following the proof of Proposition 8.1 above, it follows, that each subspace $\mathcal{C}^{\mu\nu}(V,\mathbb{K})$ decomposes further into the topological direct sum of two subspaces $\mathcal{C}^{\mu\nu}_\pm(V,\mathbb{K})$:

Let $\Gamma^c_{\mu\nu} \subset \Gamma_{\mu\nu}$ be the set of all closed curves, which are contained in some convex subset $U \subset V$. We define $\mathcal{C}^{\mu\nu}_+(V,\mathbb{K})$ as subspace of all $f \in \mathcal{C}^{\mu\nu}(V,\mathbb{K})$, for which

$$p_{\mu\nu} : f \mapsto \sup_{\gamma \in \Gamma^c_{\mu\nu}(V)} \frac{1}{|\gamma|} \left| \int_\gamma f(s)ds \right| = 0$$

and $\mathcal{C}_-^{\mu\nu}(V,\mathbb{K})$ as its algebraic complement in $\mathcal{C}^{\mu\nu}(V,\mathbb{K})$.

Both spaces then are closed and open subspaces of $\mathcal{C}^{\mu\nu}(V,\mathbb{K})$, and we obtain:

Proposition 10.1 (Partial integrability decomposition). *Let $V \subset \mathbb{R}^2$ a simply connected compact region and $\mathbb{K} = \mathbb{R}, \mathbb{R}^n$.*

1. *$\mathcal{C}_+(V,\mathbb{K})$ is the topological direct sum of the subspaces $\mathcal{C}_+^{\mu\nu}(V,\mathbb{K})$, i.e.:*

$$\mathcal{C}_+(V,\mathbb{K}) = \bigoplus_{1\leq\mu<\nu\leq n} \mathcal{C}_+^{\mu\nu}(V,\mathbb{K}).$$

2. *$\mathcal{C}_-(V,\mathbb{K})$ is the topological direct sum of the subspaces $\mathcal{C}_-^{\mu\nu}(V,\mathbb{K})$, i.e.:*

$$\mathcal{C}_-(V,\mathbb{K}) = \bigoplus_{1\leq\mu<\nu\leq n} \mathcal{C}_-^{\mu\nu}(V,\mathbb{K}).$$

While the spaces $\mathcal{C}_+^{\mu\nu}(V,\mathbb{K})$ will be referred to as *partially integrable subspaces*, the spaces $\mathcal{C}_-^{\mu\nu}(V,\mathbb{K})$ will be called *partially unintegrable.*

11. Conjugation, orientation, and parity

In Part 1, conjugation, anti/conformality and parity (i.e. orientation inversion) were defined, and it was shown that these played together, mapping unintegrable functions isomorphically onto strictly integrable functions, while the supplementary $\mathcal{C}_0$-space is invariant w.r.t. conjugation and parity, and can be broken up into the toplogical direct sum of a conformal and an anticonformal subspace. Because conjugation, conformality, and parity are defined as inversions between coordinate pairs, these are uniquely defined for 2 dimensions, but for $n > 2$, because of the lack of uniqueness for $n > 2$.

In order to achieve a holistic view and treatment for $n > 2$, it is therefore necessary to extend the definitions for conjugation, conformality, and parity:

Remark 11.1 (Recapture from Part 1: $\mathbb{R}^2$ is unorientated). As to Euclidean geometry, $\mathbb{R}^2$ represents the coordinates of points w.r.t. a right or positively orientated orthonormal pair of base vectors. If coordinate values (x, y) were all non-negative, everything would be fine: because then $(x, -y)$ and $(-x, y)$ with $x, y \geq 0$ will be identified as the coordinate (x, y) of the left or negatively orientated pair of base vectors. But with $x, y \in \mathbb{R}$ we loose that information: $\mathbb{R}^2$ is unorientated.

The complexification $(x, y) \mapsto x + iy := (x, iy)$, however, introduces an orientation: irrespective of the sign of x and y, (x, iy) refers to the unique point in the right/positively orientated system. An orientated coordinate system therefore differentiates between parity, which is the inversion $(x + iy) \mapsto x - iy$ and the inversion of handedness $x + iy \mapsto ix + y$. Both inversions anti-commute, and their product is $\pm i$, not 1. In Part 1, the orientation was based on that product of these inversions, and it was shown that orientation (as defined in 7.4) in 2 dimensions allows a pullback of complex calculus from $\mathbb{C}$ to $\mathbb{R}^2$.

While complex analysis – because of its field structure – is confined to 2 dimensions, the concept of orientation similarly allows the extension to any dimensions $n > 2$ (incl. odd numbers of dimension), but comes with a defect:

As in definition 7.4, let

$$\varphi : \mathbb{R}^n \ni x = (x_1, \ldots, x_n) \mapsto \zeta := \sum_{1 \leq k \leq n} a_k x_k := (a_1 x_1, \ldots, a_n x_n) \in \mathbb{A}$$

be an orientation of $\mathbb{R}^n$. We define φ to be of *(totally) positive/right handed orientation* and set $\varphi_+ := \varphi$ as well as $\mathbb{R}^n_+ := \varphi_+ \mathbb{R}^n$. $\mathbb{R}^n_+$ is called *positive orientated* $\mathbb{R}^n$.

Remark 11.2. $\varphi \mathbb{R}^n$ is (of course) given the norm

$$|\cdot| : \zeta \mapsto (\zeta^2)^{1/2} = (\sum_k x_k^2)^{1/2} \geq 0,$$

which makes φ an isometry of Banach spaces.

For each pair (k, l) with $1 \leq k < l \leq n$, then the mapping

$$\vartheta^{kl} : \mathbb{R}^n_+ \ni \zeta = \sum_k a_k x_k \mapsto a_1 x_1 + \cdots + a_{k-1} x_{k-1} + a_l x_k + \cdots + a_k x_k + \cdots + a_n x_n,$$

which interchanges the algebra element a_k with the algebra element a_l, defines an *orientation inversion*, such that for any two pairs ϑ^{kl} and $\vartheta^{k'l'}$ the orientations of $\vartheta^{kl} \mathbb{R}^n_+$ and $\vartheta^{kl} \mathbb{R}^n_+$ are equivalent and will be defined to be *negatively orientated* with respect to $\mathbb{R}^n_+$. And consequently, any odd number of transpositions compose to a orientation inversion, while any even number of these are orientation preserving. In other words, the orientation is an equivalence relation on the group of permutations of $a_1, \ldots, a_n$, by which all ϑ^{kl} are equivalent orientation inversions, so that we may pick ϑ^{12} as representative of the equivalence class $\vartheta := [\vartheta^{12}]$ and set $\mathbb{R}^n_- := \vartheta \mathbb{R}^n_+$(, which is represented by $\vartheta^{12} \mathbb{R}^n_+$). $\mathbb{R}^n_-$ will be defined as *negative orientated.*

The problem with that definition of orientation inversion ϑ is:
While φ strictly determines the positive orientation from dimension n down to dimension 2 in all its othogonal projections, ϑ only partially inverts the orientation and hides the rest behind an equivalence class:

Example: For $n = 3$ the inversion $x_1 \mapsto -x_1$ inverts the orientation of 12- and 13-projections, while it leaves the orientation of 23-projections invariant – up to equivalence. But if instead the orientation is inverted by $x_2 \mapsto -x_2$, then 13-projections keep their orientation – again up to equivalence.

That calls for an improved redefinition of ϑ:

Definition 11.3 (Orientation inversion). For $1 \leq \mu < \nu \leq n$ let $\pi_{\mu\nu} : \mathbb{R}^n_+ \ni \zeta = \sum_k a_k x_k \mapsto a_\mu x_\mu + a_\nu x_\nu \in \mathbb{R}^2_+$ be the orthogonal, 2-dimensional projections.

We represent each $\zeta \in \mathbb{R}^n_+$ as the tuple $(\pi_{\mu\nu}\zeta)_{1\leq\mu<\nu\leq n}$. The *orientation inversion* is then defined as the mapping

$$\vartheta : (\pi_{\mu\nu}\mathbb{R}^n_+)_{1\leq\mu<\nu\leq n} \ni \zeta \mapsto (\vartheta^{\mu\nu}\pi_{\mu\nu}\zeta)_{1\leq\mu<\nu\leq n}.$$

While $\mathbb{R}^n_+$ will be called to have *positive orientation*, $\mathbb{R}^n_- = \vartheta\mathbb{R}^n_+$ will be said to have *negative orientation*. Further, for a compact region $V \subset \mathbb{R}^n$ we define $V_+ := \varphi V$ and set $V_- := \vartheta V_+$.

The vector space $\mathbb{R}^n_-$ and the 2^n-dimensional algebra $\mathbb{A}$ are equipped with the Euclidean norm (as with $\mathbb{R}^n_+$), by which they become Banach spaces. So, $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ are isometric (and isomorphic), and the orientation inversion ϑ is a conjugation on these spaces, i.e.: its square $\vartheta^2 = \mathbb{1}$ is the identity $\mathbb{1}$ on these conjugated spaces. Because the spaces $\mathbb{R}^n_\pm$ are identical up to their opposite orientation, it makes no sense to distinguish between these as target space, and we may replace both by their superspace $\mathbb{A}$.
In line with the above, we set $\varphi_+ := \varphi$ and $\varphi_- := \vartheta\varphi_+$, and define:

$$\Phi_\pm : \mathcal{C}(V, \mathbb{R}) \ni f \mapsto f\varphi_\pm^{-1} \in \mathcal{C}(V_\pm, \mathbb{R}) \quad \text{and}$$

$$\Phi_\pm : \mathcal{C}(V, \mathbb{R}^n) \ni f \mapsto \varphi_\pm f\varphi_\pm^{-1} \in \mathcal{C}(V_\pm, \mathbb{R}^n_\pm).$$

From Proposition 10.1 we know that $\mathcal{C}(V, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{R}^n$ is the topological direct sum $const \oplus \bigoplus_{\mu\nu}\big(\mathcal{Y}_+(V,\mathbb{K}) \oplus \mathcal{Y}_-(V,\mathbb{K})\big)$. With this, the *orientation inversion* Θ is defined as:

$$\Theta : \mathcal{C}(V_+, \mathbb{K}) \ni f = c + \sum_{\mu\nu} f_+^{\mu\nu} + f_-^{\mu\nu} \mapsto c + \sum_{\mu\nu}\big(f_+^{\mu\nu}\vartheta^{\mu\nu} + f_+^{\mu\nu}\vartheta^{\mu\nu}\big) \in \mathcal{C}(V_-, \mathbb{K}).$$

Under the condition that path integration factors are taken in fixed order to the right of the integrand, Proposition 10.1 extends straightforward from $\mathcal{C}(V, \mathbb{K})$ to $\mathcal{C}(V, \mathbb{K})$, which allows the extension of Θ to

$$\Theta : \mathcal{C}(V_+, \mathbb{A}) \to \mathcal{C}(V_-, \mathbb{A}).$$

As a convenient shorthand, we'll write $f\vartheta := \Theta f$. Along with ϑ, Θ is a conjugation on both $\mathcal{C}(V_\mp, \mathbb{A})$.

Proposition 11.4. *Let $V \subset \mathbb{R}^n$ be a compact simply connected region and $V_{pm} := \varphi_\pm V$. Then $\Theta : \mathcal{C}(V_\pm, \mathbb{A}) \to \mathcal{C}((\vartheta V)_\mp, \mathbb{A})$ is an inversion, i.e. an homeomorphism of Banach spaces, such that Θ^2 is the identity, and the following holds: Θ maps*

1. *$\mathcal{C}_\pm(V_\pm, \mathbb{A})$ onto $\mathcal{C}_\mp(V_\mp, \mathbb{A})$, and*
2. *$\mathcal{C}_0(V_\pm, \mathbb{A})$ onto $\mathcal{C}_0(V_\mp, \mathbb{A})$.*

In other words: Θ maps unintegrable functions onto strictly integrable functions, and the "straight" $\mathcal{C}_0$-spaces onto $\mathcal{C}_0$-spaces.

We can therefore use φ_+ and Φ_+ to integrate $\mathcal{C}_+(V, \mathbb{A})$ as well as φ_- and Φ_- to integrate $\mathcal{C}_-(V, \mathbb{A})$.

For convenience and in analogy to Part 1 we write: $\bar{f} := \Theta f$ for $f \in \mathcal{C}_\pm(V, \mathbb{A})$ and call $\bar{f}$ *the conjugate* of f.

12. Conformality

We begin with conformality, which – as shown in Part 1 – is an optional feature for analyticity, which was discussed in order to get analyticity in line with holomprphy in complex analysis.

Again, let $V \subset \mathbb{R}^n$ be a compact simply connected region.

Proposition 12.1. *$\mathcal{C}_0(V, \mathbb{A})$ is the topological direct sums of pairwise conformal and anti-conformal subspaces $\mathcal{C}_{0,conf}(V, \mathbb{A})$ and $\mathcal{C}_{0,aconf}(V, \mathbb{A})$.*

Proof. We prove the two cases for $\mathbb{K} = \mathbb{R}, \mathbb{A}/\mathbb{R}$ separately. For $\mathbb{K} = \mathbb{R}$, we know from 9 that $f \in \mathcal{C}_0(V, \mathbb{R})$ (locally) is the sum of 1-dimensional, real-valued functions $f(x = (x_1, \ldots, x_n)) = \sum_k f_k(x_k)$. This sum can be split into a sum of $\binom{n}{2}$ pairs: $f = \frac{1}{n-1} \sum_{1 \leq k < l \leq n} (f_k + f_l)$, which allows the pairwise conformality decomposition as in 1[Sec. 4]:

$$f_k(x_k) + f_l(x_l) =$$
$$(\frac{f_k(x_k) + f_l(x_l)}{2} + \frac{f_k(x_k) + f_l(x_l)}{2}) + (\frac{f_k(x_k) - f_l(x_l)}{2} - \frac{f_k(x_k) - f_l(x_l)}{2})$$

For $\mathbb{K} = \mathbb{A}/\mathbb{R}$, we have $f(x) = (f_1(x_1), \ldots, f_n(x_n))$, which splits accordingly into the sum of pairwise vectors $\frac{1}{n-1}(f_k, f_l)$ for $1 \leq k < l \leq n$:

$$\begin{pmatrix} f_k(x_k) \\ f_l(x_l) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f_k(x_k) + f_l(x_l) \\ f_k(x_k) + f_l(x_l) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} f_k(x_k) - f_l(x_l) \\ -(f_k(x_k) - f_l(x_l)) \end{pmatrix}.$$

Because in both cases the decompositions define continuous projections onto complementary subspaces, they give the desired decomposition for $\mathcal{C}_0(V, \mathbb{R})$ and $\mathcal{C}_0(V, \mathbb{A} \setminus \mathbb{R})$. The other remaining 2 cases the follow from the topological isomorpisms Φ_+ (defined in the previos section). □

We may now add the conformal part of $\mathcal{C}_0(V, \mathbb{A})$ to $\mathcal{C}_+(V, \mathbb{A})$ and the anti-conformal part of $\mathcal{C}_0(V, \mathbb{A})$ to $\mathcal{C}_-(V, \mathbb{A})$, which gives us, what we'll call *conformality decomposition* or *conformality split*:

$$\mathcal{C}(V, \mathbb{A}) = \mathcal{C}_{conf}(V, \mathbb{A}) \oplus \mathcal{C}_{aconf}(V, \mathbb{A}),$$

where $\mathcal{C}_{conf}(V, \mathbb{A}) = \mathcal{C}_+(V, \mathbb{A}) \oplus \mathcal{C}_{0,conf}(V, \mathbb{A})$ is defined to be *conformal*, and $\mathcal{C}_{aconf}(V, \mathbb{A}) = \mathcal{C}_-(V, \mathbb{A}) \oplus \mathcal{C}_{0,aconf}(V, \mathbb{A})$ is called *anti-conformal.*
For $\mathcal{C}(V_\pm, \mathbb{A})$, the conformal and anti-conformal subspaces $\mathcal{C}_{conf}(V_\pm, \mathbb{A})$ and $\mathcal{C}_{aconf}(V_\pm, \mathbb{A})$ are defined analogously.

13. Differential calculus on $\mathcal{C}(V_\pm, \mathbb{A})$ (and $\mathcal{C}(V_\pm, \mathbb{R})$)

As discussed in Part 1 for two dimensions, the algebra $\mathbb{A}$ with n anti-commuting "generators" $a_1, \ldots, a_n$ of unit length is a finite dimensional vector space over the field of real numbers, while φ is an orthogonal injection into $\mathbb{A}$, so continuity and partial differentiability of functions $f : V_\pm \to \mathbb{A}$ are well-defined (where of course $V \subset \mathbb{R}^n$ is assumed to be a simply connected compact region throughout this section).

And, because $\mathbb{R}$ is a subspace of $\mathbb{A}$, we may – for the largest part – restrict to consider just $\mathcal{C}(\varphi_\pm V, \mathbb{A})$.

Let $f \in \mathcal{C}(V_\pm, \mathbb{A})$. Then f will be called *differentiable* at $\zeta_0 \in V_\pm$, if $\frac{df(\zeta=\zeta_0)}{d\zeta} := \lim_{\zeta\to\zeta_0}(f(\zeta) - f(\zeta_0))\frac{1}{\zeta-\zeta_0}$ exists as element $f'(\zeta_0) \in \mathbb{A}$. f is called *differentiable on* $V_\pm$, if f is differentiable in all points $\zeta_0 \in V_\pm$.
$f'(\zeta) := \frac{df(\zeta)}{d\zeta}$ will be called *derivative* of f.

Remark 13.1. (i) Because $\frac{1}{\zeta-\zeta_0} = \frac{\zeta-\zeta_0}{|\zeta-\zeta_0|^2}$, where $|\zeta - \zeta_0| = \sum_k (x_k - x_{0k})^{1/2}$, $\frac{1}{\zeta-\zeta_0}$ exists.

(ii) Note that the divisional term $\frac{1}{\zeta-\zeta_0}$ is factored to the right side of f: This is to ensure uniqueness of the limit in the case that the target values $f(\zeta)$ do not commute with the variable ζ. As long as $f(\zeta)$ is real-valued, however, the ordering of the product is irrelevant: "left" and "right" derivative coincide.

(iii) For $f \in \mathcal{C}(V_-, \mathbb{A})/\mathcal{C}_0(V_-, \mathbb{A})$ differentiability means, that for its tuple-representation $f = (f_{\mu\nu})_{\mu\nu}$ (as of Proposition 10.1), each $f_{\mu\nu}$ is differentiable w.r.t. $\bar{\zeta}$, where $\bar{\zeta} := \vartheta\zeta$ and $\zeta \in V_+$.

It follows:

Proposition 13.2. *1. (Anti-integral): The derivative is the inverse of the primitive of an integrable $f \in \mathcal{C}(V_\pm, \mathbb{A})$.*

2. (Anti-derivative): Conversely, if $f \in \mathcal{C}(V_\pm, \mathbb{A})$ is continuously differentiable on (a simply connected compact region) $V \subset \mathbb{R}^n$, then its derivative f' is integrable, and If' equals f modulo constant of integration.

3. Let $f \in \mathcal{C}(V_\pm, \mathbb{A})$. Then $f : \zeta \mapsto f(\zeta)$ is differentiable in $\zeta_0 \in V_\pm$ if and only if $f^c : V_\mp \ni \bar{\zeta} \mapsto f(\zeta)$ is differentiable in $\bar{\zeta}_0$, where $\bar{\zeta} := \vartheta\zeta$.

4. (Chain rule): Let $f \in \mathcal{C}(V_\pm, \mathbb{A})$ and $g \in \mathcal{C}(W_\pm, \mathbb{R}^n_\pm)$ be differentiable on $V_\pm$ and $W_\pm$, where V and W are assumed to be compact simply connected regions of $\mathbb{R}^n$, such that $ran(g) := g(W_\pm) \subset V_\pm$. Then the composed function $f \circ g : W_\pm \to \mathbb{A}$ is differentiable, and the chain rule holds: $\frac{df\circ g(\zeta)}{d\zeta} = \frac{df(\eta)}{d\eta}\frac{d\eta}{d\zeta}$, where $\eta := g(\zeta)$.

But:

Proposition 13.3. *If $f \in \mathcal{C}(V_\pm, \mathbb{A})$ is differentiable and non-constant, then its conjugate Θf is undifferentiable. In other words: The conjugation is not differentiable(, although it is a topological homeomorphism).*

Proof. The conjugation Θ is a linear (and continuous) mapping on $\mathcal{C}(V_\pm, \mathbb{A})$ onto $\mathcal{C}(V_\mp, \mathbb{A})$, so the limit $\lim_{\zeta\to\zeta_0}(\bar{\zeta} - \bar{\zeta}_0)\frac{1}{\zeta-\zeta_0}$ exists and is Θ. For $f \in \mathcal{C}_+(V_\pm, \mathbb{A}) \oplus \mathcal{C}_-(V_\pm, \mathbb{A})$ however, Θ is itself not an element of $\mathbb{A}$, but only a linear mapping of $\varphi_\pm\mathbb{R}^n$ to $\mathbb{A}$. This makes Θ a globally non-differentiable transformation. □

So, the conjugation Θ sets the boundary for both, integrability and differentiability. (Which is to be expected, since derivatives proved to be anti-primitives and vice versa.) However, by Section 11, Θ also inverts integrability (and therefore differentiability as the inverse of the primitives)!

The mappings

$$\varphi_\pm : \mathbb{R}^n \ni x \mapsto \zeta \in \mathbb{R}^n_\pm$$

hence define two *global coordinate charts* over the manifold $(\mathbb{R}^n, \varphi_\pm)$ of positive and negative orientation, which in turn lets us define $\mathcal{C}_+(V_+, \mathbb{A})$ to be *positive orientated*, $\mathcal{C}_-(V_-, \mathbb{A})$ to be *negative orientated*, and $\mathcal{C}_0(V_\pm, \mathbb{A})$ will be called *straight*.
With this, integrability on simply connected compact regions is a matter of orientation: integrable functions can be integrated w.r.t. positive orientation, and the non-integrable functions are integrable w.r.t. the negative orientation.

A function $f \in \mathcal{C}(\varphi_\pm V, \mathbb{A})$ is called *analytic* on $\varphi_\pm V$, if for each ζ_0 in $\varphi_\pm V$ there is an open neighbourhood $U \subset \mathbb{R}^n_+$, of ζ_0, such that

$$f(\zeta) = \sum_{k \geq 0} c_k(\zeta - \zeta_0)^k, \quad (c_k \in \mathbb{A},\ k \in \mathbb{N}),$$

where the power series is to converge uniformly on U.

Remark 13.4. It immediately follows that analytic functions on $\varphi_\pm V$ are both integrable and differentiable on $\varphi_\pm V$ in all orders. Hence, the product $f \cdot g : \zeta \mapsto f(\zeta)g(\zeta)$ of an integrable continuous function f and an analytic function g on $\varphi_\pm V$ is integrable on $\varphi_\pm V$.

On the positive/negative orientated $\mathbb{R}^n_\pm$ let

$$\Psi_\pm : \mathbb{R}^n_\pm \setminus \{0\} \ni \zeta \mapsto \frac{\sum_k a_k}{\zeta^{n-1}} \in \mathbb{A}$$

be the *Cauchy function*.
Then

$$\Psi_\pm(\zeta_0 - \zeta) = \frac{\sum_k a_k}{\zeta_0^{n-1}} \sum_{k \geq 0} \frac{(k+n-2)!}{(n-2)!k!} \left(\zeta_0^{-1}\zeta\right)^k$$

exists for $\left|\zeta_0^{-1}\zeta\right| < 1$, and the series uniformly converges in ζ on all compact simply connected regions not containing the pole ζ_0. So, it is analytic on these regions.

Let us first consider the case $f \in \mathcal{C}_+(\varphi V, \mathbb{R})$. Because $f = \sum_k a_k f_k$ is integrable on (the simply connected compact region) V, it follows by 7.1, that

$$\int_{S(r,\zeta_0)} f(\zeta)\Psi_+(\zeta - \zeta_0) d^{n-1}\zeta = \pm(a_1 \cdots a_n)|S_n| \sum_k f_k(\zeta_0),$$

where $S(r, \zeta_0)$ is the sphere of radius $r > 0$ around ζ_0, $|S_n|$ is the area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, and where the integral is the surface integral over $S(r, \zeta_0)$, and where the sign ± 1 is determines orientation the integration over the r-sphere: it is $+1$ for the positive orientation and -1 for the opposite.
The complementary case is that $f \in \mathcal{C}_+(\varphi V, \mathbb{A}/\mathbb{R})$: then its primitive If is in $\mathcal{C}_+(\varphi V, \mathbb{R})$, so the above delivers the same result for If. Cauchy theory applies (as shown in Part 1[Proposition 1.1]), by which f (and If) prove

to be analytic on φV. (And for $\mathcal{C}_-(\varphi V, \mathbb{A})$ the equivalent results follow with Ψ_-.) Therefore:

Proposition 13.5. *For a simply connected compact region $V \subset \mathbb{R}^n$ and $\mathbb{K}_\pm = \mathbb{R}, \mathbb{A}$ all functions of $\mathcal{C}_+(\varphi_+ V, \mathbb{K})_+$ are analytic w.r.t. the positive orientation, while all functions of $\mathcal{C}_-(\varphi_- V, \mathbb{K}_-)$ are analytic w.r.t. the negative orientation, and the straight subspace $\mathcal{C}_0(\varphi_\pm V, \mathbb{K}_\pm)$ is analytic w.r.t both, positive and negative orientation.*

Remark 13.6 (Laplace equation). The above proposition is not really a surprise: it can be predicted from ground: the Laplace equation $\Delta := \sum_k \partial_k^2 \equiv 0$ tells it all:
We know that its solutions are the harmonic functions, of which we know that, restricting to pairs, these are all (pairwise) real analytic. So, the harmonic functions all are globally real analytic. Hence, any irregularity of the continuous functions must come with the sources of the Laplace operator. Now, where are the sources? If the sources are contained within V, then they are not contained in the outside of V, which itself is within the complement of V, and vice versa. This is the essence of the preceeding proposition.

Part 3. Mechanical Dynamical Systems

14. Lagrangian dynamical systems

Wrapped in the terminology of differential topology, a *Lagragian system* is defined in [7] as a pair (Y, L) consisting of a smooth n-dimensional fiber bundle $Y \to X$ over the time axis $\mathbb{R}$ and a differential n-form on an r-order jet-manifold $J^r Y$ of Y.

V. I. Arnold's original definition (see: [2][Ch. 1, §2.2]) brings it to a clearer point:
A *Lagrangian system* is a pair (M, L) of an n-dimensional (smooth) manifold M of the generalized location variables $q_1, \dots, q_n$ and a function $L : I \times TM \to \mathbb{R}$, where TM is the tangent space of M and I is the time interval.
Its meaning: The Euclidean space of location coordinates is (contravariantly) curved in order that the curved velocity variables become "flat" tangent vectors: that is one half of Einstein's equivalence principle.

From what was dealt with in this paper was about till here, it should be clear that we will refine the charts (U_α, ϕ_α) of the manifold's atlas to positive and negative orientated ones. For now, we simply leave the space of generalized coordinates "unbent", which reduces the Lagrangian system to its original model in classical mechanics, where the *Lagrange function*

$$L : I \times V \times \mathbb{R}^n \ni (t, q, \dot{q}) \to \mathbb{R}$$

is a function on the (cartesian) product of a (non-trivial) closed and bounded interval $I = [t_0, t_1]$ of time, a simply connected compact region $V \subset \mathbb{R}^n$ of location coordinates, and the space of velocity components $\mathbb{R}^n$. L then defines the equation of motion as solution of all continuously differentiable paths $\gamma : [t_0, t_1] \ni t \mapsto \gamma(t) \in U$, for which $S : \gamma \mapsto \int_{t_0}^{t_1} L(t, q(t), \frac{dq}{dt})dt$ is extremal, i.e.: $\delta S(\gamma) \equiv 0$.

We define the Lagrangian system to be *mechanical*, if the Lagrange function is strictly convex in all velocity coordinates $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)$, and restrict to such systems in the sequel. Then the Legendre transformation $\dot{q}_i \to p_i := \frac{\partial L}{\partial q_i}$ can be applied, which transforms $L(t, q(t), \dot{q}(t))$ to $p \cdot \dot{q} - H(t, q(t), p(t))$, where $H := \dot{q} \cdot \frac{\partial L}{\partial \dot{q}}$ is the energy function, known as Hamilton function. Because the domain of the velocity coordinates was supposed to be $\mathbb{R}^n$, the momentum coordinates either have $\mathbb{R}^n$ as domain(, although it would suffice to be an open, convex subset).

Remark 14.1. Note that the existence of the Legendre transformation demands that all particle masses are to be unequal zero.

Then $dS = Ldt = p(t) \cdot dq - H(t, q(t), p(t))dt$ is a differentiable 1-form, and

$$f : [0, 1] \times V \times \mathbb{R}^n \ni (t, q, p) \mapsto (p(t), -H(t, q(t), p(t))) \in \mathbb{R}^{n+1}$$

a vector-valued function, which we may restict in the momentum coordinate space to a sufficiently large, compact simply connected region $W \subset \mathbb{R}^n$, say, such that f is continous on $I \times V \times W$. We can now (uniquely) decompose f either into the sum $f = (f_+ + f_0) + f_-$ of an integrable function $f_+ + f_0$ and an unintegrable complement f_-, or into a conformal and anti-conformal part, introduce the orientation $\varphi : \mathbb{R}^{n+1} \to \mathbb{A}$ with its positive and negative orientation, then integrate the integable or conformal function w.r.t. φ_+ and the the unintegrable or anti-conformal part w.r.t. φ_-. In particular, $\Phi_\pm f_\pm$ and their primitives $S_\pm : \varphi_\pm I \times V \to \mathbb{R}$ then are analytic. We showed

Proposition 14.2. *Let $L : I \times V \times W \ni (t, q, p) \mapsto p \cdot \dot{q} - H(t, q, p) \in \mathbb{R}$ be the Legendre transformed Lagrange function of a mechanical Lagrangian system, where $I \times V \times W$ is a simply connected compact region, such that L is continuous on $I \times V \times W$. Then there exists a decomposition of the system into the sum of two subsystems $dL = dL_+ + dL_- = p \cdot dq - H(t, q, p)dt$ of positive and negative orientation, such that $L_\pm$ can be integrated to analytic action functions $S_\pm : \varphi_\pm I \times V \to \mathbb{R}$.*

Indeed, S_+ is nothing but the classical well-known *Hamilton-Jacobi function* S, which is calculated from the Legendre transformed (hence: mechanical) Lagrangian dynamical system, using the following sufficient condition: There exists a diffeomorphism of the variables $\Omega : (q, p) \to (Q, P)$, which leaves the Hamilton equations invariant (called *canonic transformation*), such that $H(t, Q, P) \equiv 0$. That is: A canonic transformation Ω to a free mechanical system must exist.
In this case, $P \cdot dQ - H(t, Q, P) = P \cdot dQ$ evidently is an exact differential 1-form, because $\frac{\partial P_k}{\partial Q_l} \equiv 0$ for $(1 \leq k, l \leq n)$, so the system is globally integrable w.r.t. the transformed canonical coordinates, and the inverse of the canonical transformation delivers a global action integral for the original, untransformed system. This is the Hamilton-Jacobi function.

There are several important remarks to be made on Hamilton-Jacobi functions:

Remark 14.3 (Hamilton-Jacobi functions). (1) The domain of definition of the Hamilton-Jacobi function is space time, and it is not the direct product of time cross phase space, where the phase space is defined as the direct product of location and momentum coordinates.

(2) The essence of Hamilton-Jacobi theory is the (global) integrability of the 1-form $p \cdot dq - H$, or equivalently, the integrability of the vector-valued function $f : (t, q, p) \mapsto (p(t), -H(t, q(t), p(t)))$. By that, all path integrals of f from a start point $(t_0, q(t_0), p(t_0))$ to the end point $(t_1, q(t_1), p(t_1))$ result in the vary same action value. The name "principle of stationary action" for the Lagrangian mechanical system seems misleading: all paths then are extremal.

(3) The Hamilton Jabobi function completely solves the mechanical system: given a particle or system constallation at $q(t_0)$ for time t_0 and initial momentum $p(t_0)$, then by S, we know the action value $S(t_0, q(t_0))$. And $\frac{\partial S(t,q(t))}{\partial q_k} = p_k(t)$, $(1 \leq k \leq n)$, give us the direction and momentum for each $t \in [t_0, t_1]$.

What the Hamilton-Jacobi function then tells us, is that S is to be grasped as $(n+1)$-dimensional manifold of space time, on which the system's possible motions are the exactly along geodesics of that manifold. This is the complementary part of Arnold's definition of the Lagrangian system in this sections's beginning: The Lagrangian mechanics contains in itself Einstein's principle of equivalence: Either the action function S is curved and the motion of the particles occurs on the geodesics of that manifold, or – equivalently – space time is bent along S, and the particles then move freely along geodesics of that curved space time. In both cases, we could and should define an oriented manifold on the space time domain along S. The transformation of the coordinates $(q, p) \mapsto (Q, P)$ then becomes the canonical Hamitom-Jacobi transformation above.

(4) Once S determined for one particle, then – as long as the particles don't interact with eachother – we can place any amount of particles altogether on the surface of S with any initial momentum. The result will be a stream of particles along the geodesics: it is the idea behind D. Bernoulli's hydromechanics.

Because the functions $S_\pm : \varphi_\pm(I \times V) \to \mathbb{R}$ are analytic, they have a radius of convergence for each $(t, q) \in I \times V$, and, since $I \times V$ is compact, a minimal radius $r > 0$ exists, within which all perturbations still lead to a convergent behavior of the $S_\pm$. In this sense, a mechanical Lagrangian system should sustain a limited amount of "chaotic backgound noise".

15. Outlook

Restricting to conservative mechanical systems, time is extractable from the space time coordinate system as an external parameter. The above can then be applied to KAM-theory.

Also, note that the part S_-, which is the action function for the negative orientated system, can be viewed as S_+-function for the outside of the bounded internal region. And this S_- is (completely) anti-symmetric. The particle system or system of states, which it describes, behave as fermions. So, it is possible to base the quantum mechanical notion of spin on geometric grounds, where fermions would be orientation inverted bosons.

16. Information disclosure and interest statement

The author declares that all data supporting the findings of this study are available within the article. The author states that there is no conflict of interest.

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