Geometric Facts of a Circle Through its Equation

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Abstract

We derive equation of a circle passing through three distinct points in a different way and demonstrate that several basic geometric properties of a circle can be derived easily by means of this equation alone.

1 Introduction

In the complex plane, the equation $|z-c|=r$ represents a circle with centre c and radius r. This is also known as centre-radius form of a circle. The equation of circle passing through three distinct and non-collinear points z_1, z_2, z_3 is generally written as $\arg\left(\frac{z-z_1}{z-z_2}\frac{z_3-z_2}{z_3-z_1}\right)=0$ or π, see [1]. We here consider centre-radius form of a circle and obtain c and r in terms of the expressions involving z_1, z_2, z_3. Then, we demonstrate that some geometric properties of circles and triangles can be derived by means of this equation alone. The underlying thought is that a unique circle and a unique triangle pass through three distinct points and therefore its equation must yield some geometrical properties of circles and the triangles.

2 Equation of Circle

We begin with writing the equation of circle in a form which immediately implies several geometric properties of a circle

Theorem. *Let $|z-c|=r$ be the equation of circle passing through z_1, z_2, z_3. Let $\alpha=(z_2-z_3)\left(\overline{z_1}-\overline{z_3}\right)$. Then,*

$$c=\left(1-i\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}\right)\frac{z_1}{2}+\left(1+i\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}\right)\frac{z_2}{2}, \tag{1}$$

$$r=\sqrt{1+\left(\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}\right)^2}\frac{|z_2-z_1|}{2}. \tag{2}$$

Proof. We have $|z_j - c| = r$, therefore, $f(z_j) = |z_j|^2 + |c|^2 - 2\operatorname{Re} c\overline{z_j} = r^2$ for, $j = 1, 2, 3$. Then, $f(z_3) - f(z_1) = 0$ and $f(z_2) - f(z_3) = 0$ give two linear equations in c and $\overline{c}$ the solution of which gives (1). Further, for $|z_1 - c| = |z_2 - c| = r$, we have $2r^2 = \sum_{j=1}^{2} |z_i - c|^2$. Also, $\sum_{i=1}^{2} \left(z_i - \frac{z_1+z_2}{2}\right) = 0$. Therefore

$$r^2 = \tfrac{1}{2} \sum_{i=1}^{2} \left| z_i - \frac{z_1 + z_2}{2} + \frac{z_1 + z_2}{2} - c \right|^2 = \left| \frac{z_2 - z_1}{2} \right|^2 + \left| \frac{z_1 + z_2}{2} - c \right|^2. \quad (3)$$

Inserting (1) in (3) and simplify, we immediately get (2). ■

3 Geometric Facts of a Circle

We now demonstrate that the above theorem yields the geometric properties of circles.

(i) *The diameter is the largest chord in a circle.* From (2), $|z_2 - z_1| \leq 2r =$ diameter.

(ii) *The line segment joining the mid point of a chord and centre of a circle is perpendicular to the chord.* From (1), $c - \frac{z_1+z_2}{2} = ik(z_1 - z_2)$ where $k = -\frac{1}{2}\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}$ is a real number. It follows that $c - \frac{z_1+z_2}{2}$ is perpendicular to $z_1 - z_2$.

(iii) *Equal chords of a circle are equidistant from the centre of the circle.* For any four distinct points z_1, z_2, z_3, z_4 on $|z - c| = r$, we have from (3),

$$\left| c - \frac{z_1 + z_2}{2} \right|^2 = r^2 - \left| \frac{z_1 - z_2}{2} \right|^2 \text{ and } \left| c - \frac{z_3 + z_4}{2} \right|^2 = r^2 - \left| \frac{z_3 - z_4}{2} \right|^2. \quad (4)$$

It follows from (4) that $\left|c - \frac{z_1+z_2}{2}\right| = \left|c - \frac{z_3+z_4}{2}\right|$ if and only if $|z_1 - z_2| = |z_3 - z_4|$.

(iv) *Equal chords of a circle subtend equal angles at the centre of the circle.* From (4), $|z_1 - z_2| = |z_3 - z_4|$ if and only if $\left|c - \frac{z_1+z_2}{2}\right| = \left|c - \frac{z_3+z_4}{2}\right|$if and only if $\operatorname{Re}(c - z_1)(\overline{c} - \overline{z_2}) = \operatorname{Re}(c - z_3)(\overline{c} - \overline{z_4})$ if and only if chords subtend equal angles at the centre.

(v) *A triangle inscribed in a circle is right triangle if and only if one side of the triangle is the diameter of the circumcircle (Thales's theorem).* From (2), $|z_2 - z_1| = 2r$ if and only if $\operatorname{Re}\alpha = 0$ if and only if triangle is right triangle at z_3.

(vi) *The circumcenter lies on triangle if and only if triangle is right triangle and circumcenter lies on hypotenuse (Thales's theorem).* From (1), we see that $c = \frac{z_1+z_2}{2}$ if and only if $\operatorname{Re}\alpha = 0$. This implies $c = \frac{z_1+z_2}{2}$ if and only if triangle with the vertices z_1, z_2, z_3 is right triangle at z_3. Further, $c = \frac{z_1+z_2}{2}$ if and only if $z_1$and z_2 are the end points of the diameter.

(vii) *A chord subtends equal angle at all points on a circle on one side of the chord and supplementary angle on other side of the chord* (cf. [2, p. 13-14]). Let θ be the angle between $z_2 - z_3$ and $z_1 - z_3$. Then, $\operatorname{Re}\alpha = |z_2 - z_3|\,|z_1 - z_3|\cos\theta$ and $\operatorname{Im}\alpha = |z_2 - z_3|\,|z_1 - z_3|\sin\theta$. We therefore have $\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha} = \cot\theta$. Then, the circumcenter (1) can equivalently be written as

$$c = (1 - i\cot\theta)\,\frac{z_1}{2} + (1 + i\cot\theta)\,\frac{z_2}{2}. \tag{5}$$

It is clear from (2) that the value of $\left|\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}\right|$ remain same for any two fixed points z_1, z_2 on the circle. Hence, $|\cot\theta|$ is constant for any point z_3 on the circle.

Let $z_j = c + re^{i\theta_j}$, $j = 1, 2, 3$. Then $\alpha = 4r^2 \sin\frac{\theta_3-\theta_2}{2}\sin\frac{\theta_3-\theta_1}{2} e^{i\left(\frac{\theta_2-\theta_1}{2}\right)}$. For $\theta_1 < \theta_2 < \theta_3 < 2\pi$, $\operatorname{Im}\alpha > 0$ but $\operatorname{Re}\alpha > 0$ when $\theta_2 - \theta_1 \le \pi$, otherwise $\operatorname{Re}\alpha < 0$. For $\theta_1 < \theta_3 < \theta_2 < 2\pi$, $\operatorname{Im}\alpha < 0$ but $\operatorname{Re}\alpha < 0$ when $\theta_2 - \theta_1 \le \pi$ otherwise $\operatorname{Re}\alpha > 0$. Thus, if $\theta_2 - \theta_1 \le \pi$ and z_j's are arranged in counterclockwise direction as z_1, z_2, z_3, we have $0 < \theta \le \frac{\pi}{2}$ ($\theta = \phi_1$ say) and if z_j;s are arranged as z_1, z_3, z_2, we have $\frac{\pi}{2} \le \theta \le \pi$ $(\theta = \phi_2)$. But $|\cot\theta|$ is constant, and it follows that $\phi_1 + \phi_2 = \pi$. The arguments for the case $\theta_2 - \theta_1 \ge \pi$ are same.

(viii) *Sine formula of triangle.* We have $\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha} = \cot\theta$, therefore from (2) we find that

$$r = \sqrt{1 + \cot^2\theta}\,\frac{|z_2 - z_1|}{2} = \frac{|z_2 - z_1|}{2\sin\theta}. \tag{6}$$

Further, if z_1, z_2, z_3 are vertices of a triangle and θ, ϕ, ψ be the angles opposite to sides $z_1 - z_2$, $z_2 - z_3$, $z_3 - z_1$ respectively, then (6) yields sine rule of triangles, $\frac{|z_2 - z_1|}{\sin\theta} = \frac{|z_2 - z_3|}{\sin\phi} = \frac{|z_3 - z_1|}{\sin\psi} = 2r$.

(ix) *Euler's line of an isosceles triangle is perpendicular to the third side.* The centre c as in (1) can also be expressed in terms of chords $z_2 - z_3$ and $z_1 - z_3$. Then, with notation $\beta = (z_3 - z_1)\left(\overline{z_2} - \overline{z_1}\right)$ and $\gamma = (z_1 - z_2)\left(\overline{z_3} - \overline{z_2}\right)$, we find that

$$c = \frac{z_1 + z_2 + z_3}{3} + \frac{1}{6i}\left(\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}(z_1 - z_2) + \frac{\operatorname{Re}\beta}{\operatorname{Im}\beta}(z_2 - z_3) + \frac{\operatorname{Re}\gamma}{\operatorname{Im}\gamma}(z_3 - z_1)\right) \tag{7}$$

For $|z_3 - z_1| = |z_3 - z_2|$, $\beta = \gamma$. So, from (7), $c - \frac{z_1 + z_2 + z_3}{3} = \frac{1}{6i}\left(\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha} - \frac{\operatorname{Re}\beta}{\operatorname{Im}\beta}\right)(z_1 - z_2)$. This shows that the Euler's line passing through circumcentre c and centroid $\frac{z_1+z_2+z_3}{3}$ is perpendicular to the third side $z_1 - z_2$.

(x) *The circumradius in terms of the lengths of the sides of triangle.* We have

$$|z_2 - z_1|^2 = |z_2 - z_3 + z_3 - z_1|^2 = |z_2 - z_3|^2 + |z_3 - z_1|^2 - 2\operatorname{Re}(z_2 - z_3)\left(\overline{z_1} - \overline{z_3}\right). \tag{8}$$

Denote $|z_2 - z_3|, |z_3 - z_1|$ and $|z_2 - z_1|$ by f_1, f_2 and f_3 respectively. Then, from (8),

$$\operatorname{Re}\alpha = \frac{f_1^2 + f_2^2 - f_3^2}{2} \text{ and } (\operatorname{Im}\alpha)^2 = |\alpha|^2 - (\operatorname{Re}\alpha)^2 = f_1^2 f_2^2 - \frac{\left(f_1^2 + f_2^2 - f_3^2\right)^2}{4}. \tag{9}$$

Therefore, from (2),

$$r^2 = \frac{f_1^2 f_2^2 f_3^2}{4 f_1^2 f_2^2 - \left(f_1^2 + f_2^2 - f_3^2\right)^2}. \tag{10}$$

(xi) *Distance between centre and chord.* From (4) and (10),

$$\left| c - \frac{z_1 + z_2}{2} \right|^2 = r^2 - \frac{f_3^2}{4} = \frac{1}{4} \frac{f_3^2 \left(f_1^2 + f_2^2 - f_3^2\right)^2}{4 f_1^2 f_2^2 - \left(f_1^2 + f_2^2 - f_3^2\right)^2}.$$

(xii) *The distance between the circumcentre and centroid of a triangle.* We note that $\operatorname{Im}\alpha = \operatorname{Im}\beta = \operatorname{Im}\gamma$ and therefore, from (7), with notation $\widetilde{z} = \frac{z_1 + z_2 + z_3}{3}$,

$$|c - \widetilde{z}|^2 = \frac{1}{36 \left(\operatorname{Im}\alpha\right)^2} \left((\operatorname{Re}\alpha)^2 f_3^2 + (\operatorname{Re}\beta)^2 f_1^2 + (\operatorname{Re}\gamma)^2 f_2^2 - 6 \operatorname{Re}\alpha \operatorname{Re}\beta \operatorname{Re}\gamma \right). \tag{11}$$

On using (9), we find from (11) that

$$|c - \widetilde{z}|^2 = \frac{f_1^2 f_2^2 f_3^2}{4 f_1^2 f_2^2 - \left(f_1^2 + f_2^2 - f_3^2\right)^2} - \frac{f_1^2 + f_2^2 + f_3^2}{9}.$$

4 Conclusion

The arithmetical and geometrical properties of complex numbers provide alternative ways to look at the related geometrical properties. We here demonstrated this in the special case when the geometrical properties of a circle follow from its equation in the complex form.

References

[1] Ahlforse, L.V. *Complex Analysis.* 3rd edition, New York, McGraw-Hill, Inc. (1979).

[2] Johnson, R. A. *Advanced Euclidean Geometry.* Dover Publication, Inc. New York. (1960).