Squaring the circle and approximations to Pi

Edgar Valdebenito

March 14, 2024

ABSTRACT

Since the 19th century when von Lindemann set out a proof of the transcendental properties of the mathematical constant Pi, mathematicians have taken the view that squaring the circle using a straight edge and compass is not possible.

I. Introduction

- The problem of the exact rectification of a circle cannot be solved by classical geometry.
- A German mathematician von Lindemann in 1882 has shown that the mathematical constant π is not an algebraic number.
- The rectification is an impossible construction to realize, as a consequence that the π is transcendental number.

II. Approximations to Pi

Some approximations to Pi (via squaring the circle):

- A. Kochański (1685):

$$\pi \approx \sqrt{\frac{40}{3} - 2\sqrt{3}} = 3.1415333\ldots$$

the exact value of pi is

$$\pi = \sqrt{\frac{40}{3} - 2\sqrt{3}} + 4\tan^{-1}\left(\sqrt{\frac{1-\sin\left(\frac{1}{2}\sqrt{\frac{40}{3}-2\sqrt{3}}\right)}{1+\sin\left(\frac{1}{2}\sqrt{\frac{40}{3}-2\sqrt{3}}\right)}}\right)$$

- J.Gelder (1849), S. Ramanujan (1913):

$$\pi \approx \frac{355}{113} = 3.141592920\ldots$$

the exact value of pi is

$$\pi = \frac{355}{113} - 4\tan^{-1}\left(\sqrt{\frac{1-\sin\left(\frac{355}{226}\right)}{1+\sin\left(\frac{355}{226}\right)}}\right)$$

- S. Ramanujan (1914):

$$\pi \approx \left(9^2 + \frac{19^2}{22}\right)^{1/4} = 3.14159265258 \ldots$$

the exact value of pi is

$$\pi = \left(9^2 + \frac{19^2}{22}\right)^{1/4} + 4\tan^{-1}\left(\sqrt{\frac{1 - \sin\left(\frac{1}{2}\left(\frac{2143}{22}\right)^{1/4}\right)}{1 + \sin\left(\frac{1}{2}\left(\frac{2143}{22}\right)^{1/4}\right)}}\right)$$

- E.W. Hobson (1913), R. Dixon (1991), F. Beatrix (2022):

$$\pi \approx \frac{6}{5}(1 + \phi) = 3.141640 \ldots$$

the exact value of pi is

$$\pi = \frac{6}{5}(1 + \phi) - 4\tan^{-1}\left(\sqrt{\frac{1 - \sin\left(\frac{3}{5}(1 + \phi)\right)}{1 + \sin\left(\frac{3}{5}(1 + \phi)\right)}}\right)$$

Remark: $\phi = \frac{1+\sqrt{5}}{2}$.

III. Miscellaneous approximations of Pi

$$\pi \approx 3 \implies \pi = 3 + 4\tan^{-1}\left(\sqrt{\frac{1 - \sin(3/2)}{1 + \sin(3/2)}}\right)$$

$$\pi \approx \sqrt{10} \implies \pi = \sqrt{10} - 4\tan^{-1}\left(\sqrt{\frac{1 - \sin\left(\sqrt{\frac{5}{2}}\right)}{1 + \sin\left(\sqrt{\frac{5}{2}}\right)}}\right)$$

$$\pi \approx \sqrt{2} + \sqrt{3} \implies \pi = \sqrt{2} + \sqrt{3} - 4\tan^{-1}\left(\sqrt{\frac{1 - \sin\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\sqrt{3}\right)}{1 + \sin\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\sqrt{3}\right)}}\right)$$

$$\pi \approx \frac{9}{5} + \sqrt{\frac{9}{5}} \implies \pi = \frac{9}{5} + \sqrt{\frac{9}{5}} - 4\tan^{-1}\left(\sqrt{\frac{1 - \sin\left(\frac{9}{10} + \frac{1}{2}\sqrt{\frac{9}{5}}\right)}{1 + \sin\left(\frac{9}{10} + \frac{1}{2}\sqrt{\frac{9}{5}}\right)}}\right)$$

IV. Pi formulas

Define c_n by

$$c_n = (-1)^n - \sum_{k=1}^{n} (-1)^k \binom{2n+1}{2k} c_{n-k} \quad , \; c_0 = 1 \, , \; n = 1, 2, 3, \ldots$$

$$c_n = \{1, 2, 16, 272, 7936, 353\,792, 22\,368\,256, \ldots\}$$

we have

Entry 1.

$$\pi = K + 4 \tan^{-1}\left(\frac{4 - K s}{4 + K s}\right)$$

where

$$K = \sqrt{\frac{40}{3} - 2\sqrt{3}}$$

$$s = \sum_{n=0}^{\infty} \frac{c_n}{(2n+1)!} \left(\frac{20 - 3\sqrt{3}}{24}\right)^n$$

Entry 2.

$$\pi = \sqrt{10} - 4 \tan^{-1}\left(\frac{\sqrt{10}\, s - 4}{\sqrt{10}\, s + 4}\right)$$

where

$$s = \sum_{n=0}^{\infty} \frac{c_n}{(2n+1)!} \left(\frac{5}{8}\right)^n$$

Entry 3.

$$\pi = \frac{22}{7} - 4 \tan^{-1}\left(\frac{11 s - 14}{11 s + 14}\right)$$

where

$$s = \sum_{n=0}^{\infty} \frac{c_n}{(2n+1)!} \left(\frac{11}{14}\right)^{2n}$$

Entry 4.

$$\pi = 2\sqrt{2} + 4 \tan^{-1}\left(\frac{\sqrt{2} - s}{\sqrt{2} + s}\right)$$

where

$$s = \sum_{n=0}^{\infty} \frac{c_n 2^{-n}}{(2n+1)!}$$

V. Endnote

Entry 5.

$$\pi = 3 + 4 \tan^{-1}\left(\frac{1}{28}\right) - 4 \tan^{-1}\left(\frac{29 \tan(3/4) - 27}{27 \tan(3/4) + 29}\right)$$

$$\pi = 3 + 4 \tan^{-1}\left(\frac{1}{29}\right) + 4 \tan^{-1}\left(\frac{14 - 15 \tan(3/4)}{15 + 14 \tan(3/4)}\right)$$

Entry 6.

$$\pi = \frac{22}{7} - 4 \tan^{-1}\left(\frac{1}{3163}\right) + 4 \tan^{-1}\left(\frac{1582 - 1581 \tan(11/14)}{1581 + 1582 \tan(11/14)}\right)$$

$$\pi = \frac{22}{7} - 4\tan^{-1}\left(\frac{1}{3164}\right) - 4\tan^{-1}\left(\frac{3163\tan(11/14) - 3165}{3165\tan(11/14) + 3163}\right)$$

Entry 7.

$$\pi = 3 + 4\tan^{-1}\left(1 - \frac{2}{1 + \cot(3/4)}\right)$$

$$\pi = \frac{22}{7} - 4\tan^{-1}\left(\frac{2}{1 + \cot(11/14)} - 1\right)$$

Entry 8.

$$\pi = 3 + 4\tan^{-1}\left(1 - \frac{4}{2 + \cot(3/8) - \tan(3/8)}\right)$$

$$\pi = \frac{22}{7} - 4\tan^{-1}\left(\frac{4}{2 + \cot(11/28) - \tan(11/28)} - 1\right)$$

Entry 9.

$$\pi = 3 + 4\tan^{-1}\left(-1 + \frac{2}{1+}\,\frac{3}{4\cdot 1-}\,\frac{9}{4\cdot 3-}\,\frac{9}{4\cdot 5-}\,\frac{9}{4\cdot 7-}\,\frac{9}{4\cdot 9-}\ldots\right)$$

VI. References

1. B. Lisiak SJ and L. Grzebien SJ, Correspondence of Adam Adamandy Kochański SJ (1657-1700). Wydawnictwo WAM, Kraków, 2005. In Latin with Polish commentary.
2. B. Lisiak SJ, Adam Adamandy Kochański (1631-1700). Studium z dziejów filozofii i nauki w Polsce XVII wieku. Wydawnictwo WAM, Kraków, 2005. In Polish.
3. A. A. Kochański, "Obsevationes cyclometricae ad facilitandam praxin accomodatae," Acta Eruditorum 4, 1685.
4. H. Fuks, Obsevationes cyclometricae by Adam Adamandy Kochański - Latin text with annotated English translation, 2011.
5. M. Szyszkowicz, Squaring the circle by using proportion. The Mathematical Gazette, 103(556), 2019.
6. F. Lindemann, Über die Zahl π. Mathematische Annalen, 20, 213-225, 1882.
7. S. Ramanujan, Squaring the Circle. Journal of the Indian Mathematical Society, 5, 1913.
8. S. Ramanujan, Modular equations and approximations to π. Quarterly Journal of Mathematics, 45, 1914.
9. E. W. Hobson, Squaring the Circle: A History of the Problem. Cambridge University Press. 1913.
10. R. Dixon, Squaring the circle. Mathographics. Blackwell, 1987. Reprinted by Dover Publications, 1991.
11. F. Beatrix, Squaring the circle like a medieval master mason. Parabola. UNSW School of Mathematics and Statistics. 58(2), 2022.
12. H. V. Chu, Square the circle in one minute. arXiv: 1908.01202v1, 2019.
13. WolframMathWorld, Pi Approximations. https://mathworld.wolfram.com