Geometric Interpretations of Riemann Hypothesis and the Proof

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Abstract: Riemann zeta function(RZF) $\zeta(s)$ is a function of a complex variable $s = x + iy$. Riemann hypothesis(RH) states that all the non-trivial zeros of RZF lie on the critical line, $x = \frac{1}{2}$. The symmetricity of RZF zeros implies that if $\zeta(\alpha + i\beta) = 0, 0 < \alpha < \frac{1}{2}$, then $\zeta(1 - \alpha + i\beta) = 0$, too. In geometric view, if RH is false, two trajectories $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$ must intersect at the origin when $y = \beta$. But, according to the functional equations of RZF, two trajectories $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$ can't intersect except when $\alpha = \frac{1}{2}$. So, they can't intersect at the origin, too, proving RH is true.

1. Introduction

RZF [1][2][3][4] $\zeta(s)$ is functions of a complex variable $s = x + iy$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{1.1}$$

The functional equations of RZF [4] relates $\zeta(s)$ and $\zeta(1 - s)$.

$$\zeta(s) = F(s)\zeta(1 - s). \tag{1.2}$$

$$F(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1 - s). \tag{1.3}$$

RH [5] states that all the non-trivial zeros of RZF are of the form $s = \frac{1}{2} + iy$. The symmetricity of RZF zeros implies that if $\zeta(\alpha + i\beta) = 0$, then $\zeta(1 - \alpha + i\beta) = 0$, too. So, the two zeros should be on the two edge lines of a strip $\alpha \le x \le 1 - \alpha, 0 < \alpha < \frac{1}{2}$. (From now on, suppose $0 < \alpha < 0.5$, otherwise specified.)

To satisfy $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta)$, two curves, $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$, must intersect at the origin when $y = \beta$.

We showed that, $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$ can't intersect at all, so, they can't intersect at the origin, too.

We define some terminologies.

Definition 1.1. Domain strip: A strip $\alpha \le x \le 1 - \alpha$, -∞ $\le y \le$ ∞.

Definition 1.2. Range strip: A strip generated by the mapping of a domain strip.

Definition 1.3. Edge lines: Two edge lines of a strip.

2. Symmetry Properties of RZF Zeros

The following three equations are well known. $\xi(s)$ is Riemann Xi function [6][7].

$$\xi(s) = \frac{1}{2} \pi^{\frac{-s}{2}} s(s - 1) \Gamma\left(\frac{s}{2}\right) \zeta(s). \tag{2.1}$$

$$\xi(s) = \xi(1 - s). \tag{2.2}$$

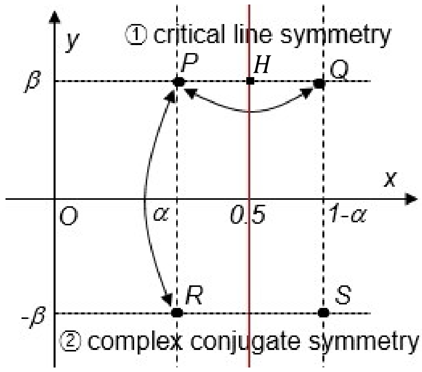
$$\zeta(\overline{s}) = \overline{\zeta(s)}. \tag{2.3}$$

The right side of (2.1) includes $\zeta(s)$, so, the zeros of $\zeta(s)$ are also the zeros $\xi(s)$. There are two types of symmetries of RZF zeros, as in Figure 1.

① **Critical line symmetry:** Symmetry of (2.2), which means that if $s = \alpha + i\beta$ is a zero, then $s = 1 - \alpha + i\beta$ is also a zero.

② **Complex conjugate symmetry:** Symmetry of (2.3), which means that if $s = \alpha + i\beta$ is a zero, then $s = \alpha - i\beta$ is also a zero.

Figure 1. Zero symmetries of RZF.



Lemma 2.1. To satisfy a critical line symmetry, $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$, a contour must be drawn by the movement of x in $\alpha \le x \le 1 - \alpha$.

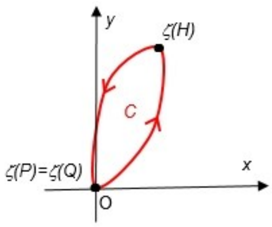
Proof. In Figure 1, *P(α, β)* and *Q(1 - α, β)* are critical line symmetry zeros, and *H(0.5, β)* lies on $x = \frac{1}{2}$. Then a contour must be drawn by the following 3 steps.

① **Initial step at $(\boldsymbol{\alpha}, \boldsymbol{\beta})$:** At *P(α, β)*, graph remains at the origin in Figure 2.

② **Movement to $\boldsymbol{H}(\mathbf{0.5}, \boldsymbol{\beta})$:** Graph leaves the origin and reaches $\zeta(H)$ in Figure 2.

③ **Movement to $\boldsymbol{Q}(\mathbf{1} - \boldsymbol{\alpha}, \boldsymbol{\beta})$:** Graph leaves $\zeta(H)$ and reaches back to the origin in Figure 2.

Figure 2. Contour *C* for $\alpha \le x \le 1 - \alpha$.



So, the RZF image of a line segment $\alpha \le x \le 1 - \alpha$ must be a closed curve. ∎

Note that the intersection $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$ is a special case of all the possible intersections $\zeta(\alpha + iy) = \zeta(1 - \alpha + iy)$. That is to say, if the intersections $\zeta(\alpha + iy) = \zeta(1 - \alpha + iy)$ is not possible, $\zeta(\alpha + iy) = \zeta(1 - \alpha + iy) = 0$ is not possible, too.

Also, $\zeta(\alpha + iy) = \zeta(1 - \alpha + iy)$ implies that the mapping of RZF is not planar, i.e., the range strip should be folded or overlapped. So, there should exist infinitely many duplicated values.

3. The Proof of RH

In the process of deriving the functional equation (1.2), we have [4]

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^{\infty}\left(x^{-\frac{s}{2}-\frac{1}{2}}+x^{\frac{s}{2}-1}\right)\psi(x)dx.$$

$$\psi(x)=\sum_{n=1}^{\infty}e^{n^2\pi x}.$$

$$R(s)=\frac{1}{s(s-1)}+\int_1^{\infty}\left(x^{-\frac{s}{2}-\frac{1}{2}}+x^{\frac{s}{2}-1}\right)\psi(x)dx.$$

$$R(s)=R(1-s). \tag{3.1}$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{3.2}$$

Equation (3.2) is equivalent to (1.2). What is important is that in (3.1), s and $1-s$ are interchangeable. That is to say $F(s)$ in (1.3) transforms $\zeta(1-s)$ to $\zeta(s)$. In view of variables, $F(s)$ transforms $1-s$ to s as follows.

$$1-s=1-\alpha-i\beta\rightarrow s=\alpha+i\beta.$$

$$1-\alpha\rightarrow\alpha. \tag{3.3}$$

$$-\beta\rightarrow\beta.$$

And when $\zeta(\alpha+i\beta)=\zeta(1-\alpha+i\beta)$

$$\zeta(s)=\zeta(\overline{1-s}),\overline{1-s}=1-\alpha+i\beta. \tag{3.4}$$

By the functional equation (1.1) and (3.4), we have

$$\zeta(s)=F(s)\zeta(1-s)=\zeta(\overline{1-s}). \tag{3.5}$$

So, $F(s)$ can simultaneously transform $\zeta(1-s)$ to $\zeta(s)$ and $\zeta(\overline{1-s})$. The transformation from $\zeta(1-s)$ to $\zeta(s)$ is guaranteed by (3.1). But, the transformation from $\zeta(1-s)$ to $\zeta(\overline{1-s})$ requires the following variable transformations.

$$1-s=1-\alpha-i\beta\rightarrow\overline{1-s}=1-\alpha+i\beta.$$

$$1-\alpha\rightarrow1-\alpha. \tag{3.6}$$

$$-\beta\rightarrow\beta.$$

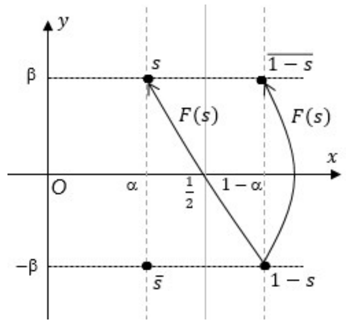
The only way to simultaneously satisfy (3.3) and (3.6) is when $s=\overline{1-s}$, as in Figure 3.

$$s=\alpha+i\beta=1-\alpha+i\beta=\overline{1-\mathrm{s}}.$$

$$2\alpha=1.$$

$$\alpha=\frac{1}{2}. \tag{3.7}$$

Figure 3. Simultaneous transformations by $F(s)$.



Consequently, we can assert that any two RZF trajectories of $x = \alpha$ and $x = 1 - \alpha$ can't intersect except when $\alpha = \frac{1}{2}$. So, RH is true.

5. Conclusion

The symmetricity of RZF zeros implies that if $\zeta(\alpha + i\beta) = 0$, then $\zeta(1 - \alpha + i\beta) = 0$, too. So, the two zeros should be on the two edge lines of the strip $\alpha \leq x \leq 1 - \alpha$. To satisfy this, in geometric view, two trajectories $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$ can intersect at some point when $y = \beta$. If they can't intersect, they can't intersect at the origin, too. The geometric intersection of the two trajectories $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$ also implies that the RZF mapping of the domain strip is not planar, that is to say, the range strip should be folded or overlapped, generating infinitely many duplicated values. But, the properties of the functional equations of RZF enforces $\alpha = \frac{1}{2}$, which means that two trajectories $\zeta(\alpha + iy)$ and $\zeta(1 - \alpha + iy)$ can't intersect when $y = \beta$.

References

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[7] https://en.wikipedia.org/wiki/Riemann_Xi_function

Figures