COLLATZ CONJECTURE IS TRUE-
A DEFINITE CONCLUSION IS DRAWN BY USING THE PRINCIPLE OF
NET INDUCTION RATE AND NET REDUCTION RATE

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ABSTRACT: This research work establishes a theory for concluding an affirmative answer to the famous, long-standing unresolved problem “The Collatz Conjecture”.

1 Introduction

The well-known Collatz problem [1] i.e., $3n + 1$ problem is the following :
If $f(n) = \frac{n}{2}$, when n is even and
.... $=3n + 1$, when n is odd,
do the sequential values of $f(n)$ eventually reach to 1, for every natural number n ?.
This problem was pursued by many researchers along different directions. Following are, among others[2, 7, 9], some of the important methods available in the literature:
1. Experimental evidence [8],
2. A probabilistic heuristic argument [3],
3. Use of encoding matrix [5],
4. Generalization of the problem [6, 4].

Obviously, experimental evidences cannot provide a satisfactory solution to the problem.

In probabilistic heuristic approach [3] the following idea is explained:

If we choose n at random in the sense that it is odd with probability $\frac{1}{2}$ and even with probability $\frac{1}{2}$ then the Collatz function $f_1 : N \to N$ increases n by a factor roughly $\frac{3}{2}$ half the time and decreases it by a factor of $\frac{1}{2}$ the time.

Furthermore, if n is uniformly distributed modulo 4, one easily verifies that f_1 is uniformly distributed modulo 2 and so f_1^2 should be roughly $\frac{3}{2}$ as

large as $f_1(n)$ half the time and roughly $\frac{1}{2}$ times as large as $f_1(n)$ the other half of the time. Continuing this at a heuristic level, we expect generally that
$f_1^{k+1}(n) \approx \frac{3}{2} f_1^k(n)$ half the time, and
$f_1^{k+1}(n) \approx \frac{1}{2} f_1^k(n)$ the other half of the time.
The logarithm $\log f_1^k(n)$ of this orbit can be modelled heuristically by a random walk with steps $\log\frac{3}{2}$ and $\log\frac{1}{2}$ occuring with equal probability. The expectation $\frac{1}{2}\log\frac{3}{2} + \frac{1}{2}\ \log\frac{1}{2} = \frac{1}{2}\log\frac{3}{4} < 0$ and so (by the classic gambler's ruin) we expect the orbit to decrease over the long term. This can be viewed as the heuristic justification of the Collatz Conjecture.
But this probabilistic approach cannot ensure a clear solution to the problem.

The generalized form [5] of the problem is

$T_n(x) = \frac{x}{p_{i_1} p_{i_2} \cdots\cdots p_{i_k}}$ where p_i's are primes less or equal to p_n dividing

the numerator and

$$= p_{n+1}x + 1, \text{ if no prime } p_i \leq p_n \text{ divides } x.$$

Using this generalized form, the author used the concept of encoding matrix and generalized some results in the paper of Terras [9]. But the author upheld a heuristic argument against the existence of divergent conjectures, which also cannot give a satisfactory answer to the main problem.

In 1985 J.C. Lagarias pursued the problem by using Weakly connected graph of the Collatz Graph. Lagarias had the following comment in his article 'The $3x + 1$ problem and its generalizations' :

"Of course there remains the possibility that someone will find some hidden regularity in the $3n + 1$ problem that allows some of the conjectures about it to be settled.
The existing general methods in number theory do not seem to touch the $3n + 1$ problem. In this sense it seems intractable at present.
Study of this problem has uncovered a number of interesting phenomena. It also serves as a benchmark to measure the progress of general mathematical theories. For example, future developments in solving exponential diphantine equations may lead to the resolution of the finite cycles conjecture."

A new approach is established in this research work which concludes an affirmative answer to the problem. This approach invents the 'Principle of Net Induction Rate and Net Reduction Rate'.

Five main sections are there in this article to achieve the required goal.
Second section contains preliminaries and some basic results.
Third section introduces concepts of Immediate odd predecessors and Immediate odd successor and studies some vital observations.
Fourth section deals with the study of dependence of C-convergence of naturals of the form $4n+3$ on C-convergence of the naturals of the form $3w-1$ and $3u-2$.
Fifth section introduces the concepts of Collatz Decisive Subset, Net Induction Rate, Net Reduction Rate and finally draws a definite conclusion to the problem.

2 Preliminaries

For any natural number n, if the Collatz sequence of n eventually reaches to 1, we write the fact as $n \to 1$. For example, $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. This will be written simply as $7 \to 11 \to 17 \to 13 \to 5 \to 1$, ignoring the even terms. This will be called Collatz sequence of 7, as we have started with 7 itself. Since 7 eventually reaches to 1, we write $7 \to 1$ and say that **'7 C-converges to** 1**'**.
Observation 1. **(i)** Any odd natural number $x(> 4)$ can be one of the following forms:
$x = 4n+1$ for some $n \in N$,
$x = 4n+3$ for some $n \in N$.
(ii) for any $n \in N$ we have the following observation on Collatz sequences :
$4n+1 \to 12n+4 \to 6n+2 \to 3n+1$.
This will be written in short as $4n+1 \to 3n+1$.
(iii) let n be odd. Then
$4n+1 \to 1$ iff $n \to 1$.
This follows immediately because
$4n+1 \to 3n+1$ and since n is odd, we have $n \to 3n+1$.

By a reduction of an odd natural x, we shall mean that in the Collatz sequence of x, x eventually reaches to some y, where $y < x$.

For example, $9 \to 28 \to 14 \to 7$. So 9 has a reduction.
Note that every even $2t$ has a reduction and for any $4t+1$, since $4t+1 \to 3t+1$, $4t+1$ has a reduction.

The following result holds :

Theorem 2. *The following statements are equivalent :*
(i) $n \to 1$ *for all* $n \in N$.
(ii) $4n+3$ *has a reduction for each* $n \in N$.

Clearly $(i) \Rightarrow (ii)$. Now let (ii) hold. Since any natural $n(> 4)$ is of the form $2t$ or $4t+1$ or $4t+3$ and in all cases it has a reduction, after a finite number of steps it follows that $n \to 1$. Thus (i) follows.

3 Immediate odd predecessors and immediate Odd Successor

For a natural x, we shall say that

(i) y is the immediate smaller odd predecessor(in short, i.s.o.p) if $3y = 2x - 1$.
For example, 7 is the i.s.o.p. of 11. Note that $7 \to 22 \to 11$, i.e., $7 \to 11$.

(ii) y is the immediate larger odd predecessor(in short, i.l.o.p) if $3y = 4x - 1$.
For example, 9 is the i.l.o.p. of 7. Note that $9 \to 28 \to 7$, i.e., $9 \to 7$.

(iii) y is the immediate odd successor of x (in short, i.o.s.of x) if $x \to y$ where y is the first odd obtained after Collatz operations on x and $x < y$.
For example, 11 is the i.o.s of 7. Note that $7 \to 22 \to 11$, i.e., $7 \to 11$.

The following observations are important for the subsequent sections:

Observations 3.
(a): There are three types of naturals:

(i) Type-A: $3u - 1$, for $u \in N$,
(ii) Type-B: $3u - 2$, for $u \in N$,
(iii) Type-C: $3u$, for $u \in N$.

(b): **(i)** Type-A naturals have the i.s.o.p. $2u - 1$, since $2u - 1 \to 6u - 2 \to 3u - 1$, i.e, $2u - 1 \to 3u - 1$.
Type-A naturals do not have the i.l.o.p. because, $4x - 1 = 4(3u - 1) - 1 = 12u - 5 \neq 3y$ for any $y \in N$.

(ii) Type-B naturals have the i.l.o.p. $4u - 3$, since $4u - 3 \to 12u - 8 \to 3u - 2$, i.e.,$4u - 3 \to 3u - 2$.
Type-B naturals do not have the i.s.o.p. because, $2x - 1 = 2(3u - 2) - 1 = 6u - 5 \neq 3y$ for any $y \in N$.

(iii) Type-C naturals neither have the i.s.o.p. nor have the i.l.o.p., since, for $x = 3u$, neither $2x - 1 = 3y$ nor $4x - 1 = 3y$ for any $y \in N$.

(c): In the Collatz sequence of any odd natural, at most one term may be of the Type-C and this term must be the initial term (if it is of that type). This is obvious from the observation (b)(iii) above.

In the next section we shall first establish one intimate connection among C-convergence of naturals of the type $4n + 3$, Type-A, and Type-B.

4 Dependence of C-convergence of the naturals $4n+3$ on C-convergence of the naturals of Type-A and Type-B, i.e. $3w-1$ and $3u-2$

Theorem 4. *For any n, $4n+3 = A.2^t - 1$, for some odd A and some natural $t \geq 2$.*

Proof. First, let n be even. Then $n = 2^j.H$ for some j and some odd H. So, $4n + 3 = 4H.2^j + 3 = 4(H.2^j + 1) - 1 = 2^2 A - 1$, say, where A is odd. Now let n be odd. Then $4n + 3 = 4(n + 1) - 1 = 4.2^j A - 1$, for some j and some odd A, because, $n + 1$ is even. Hence $4n + 3 = A.2^t - 1$, as required.

Theorem 5. *In the Collatz sequence of $4n+3$ for any n(however large n may be), $4n+3$ eventually reaches to naturals of Type-A and Type-B i.e., $3w-1$ and $3u-2$, for some w and u.*

Proof. Let $4n+3 = A.2^t - 1$, for some odd A and some natural $t \geq 2$.

$$\begin{aligned} \text{Then } 4n+3 &\to 3A.2^{t-1} - 1 \\ &\to 3^2 A.2^{t-2} - 1 \\ &\to 3^3 A.2^{t-3} - 1 \\ &\to \ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\to 3^{t-2} A.2^2 - 1 \\ &\to 2.A.3^{t-1} - 1 \text{ (only odd terms are written).} \end{aligned}$$

Clearly, $3A.2^{t-1} - 1 = 3w - 1$, for $w = A.2^{t-1}$.
We now claim that $2A.3^{t-1} - 1$ is of the form $4v+1$ for some v.
Note that $2A.3^{t-1} - 1$ is odd. Now if it is of the form $4w+3$ then $2A.3^{t-1} - 1 = 4w+3 \Rightarrow 2A.3^{t-1} = 4(w+1)$, a contradiction, since 4 is not a factor of $2A.3^{t-1}$. Hence $2A.3^{t-1} - 1 = 4v+1$ for some v. This justifies our claim.

Now note that $2A.3^{t-1} - 1 = 4v+1 \to 3v+1$ where $3v+1 = 3(v+1) - 2 = 3u - 2$, (here $u = v+1$).
This shows that $4n+3$ eventually reaches to $3w-1$ and $3u-2$, for some w and u.

Example 6. **(i)** Consider $4n+3 = 15$. Then $15 \to 23 \to 35 \to 53 \to 160 \to 80 \to 40$, where 23 is of the form $3w-1$ and 40 is of the form $3u-2$ (as argued in above theorem).

(ii) Consider $4n+3 = 63$. Then $63 \to 95 \to 143 \to 215 \to 323 \to 485 \to 364$ where 95 is of the Type-A, and 364 is of the Type-B.

Remark 7. From above theorem it follows that reduction of $4n+3$ completely depends upon reduction of naturals of Type-A and Type-B.

Theorem 8. *The following statements are equivalent :*

(i) $3u-1 \to 1$ *and* $3u-2 \to 1$ *for all* u.

(ii) $4n+3$ *has a reduction for all* n.

Proof. Let (i) hold. Consider any $4n+3$. By Theorem 4.2 above, there exist $3w-1$ and $3u-2$ for which $4n+3$ eventually reaches to $3w-1$ and $3u-2$. Hence by (i), $4n+3 \to 1$. So (ii) holds.

Conversely, let (ii) hold. Consider any $3u-1$. Then $3u-2$ is either even or of the form $4t+1$ or of the form $4w+3$. In all cases $3u-2$ has a reduction say V. Now if $V > 3$ then V is either even or $4t+1$ or $4w+3$. So, this process continues and finally after a finite number of steps we must have $3u-2 \to 1$. Similar argument can be given for $3u-1$. Hence (i) holds.

Theorem 9. *The following statements are equivalent :*

(i) $n \to 1$ *for all* n,

(ii) $3u-1 \to 1$ *and* $3u-2 \to 1$ *for all* $u \in N$.

Proof. Clearly, $(i) \Rightarrow (ii)$. Now let (ii) hold. Then by Theorem 4.5 and Theorem 2.2, (i) holds.

Theorem 10. *The following statements are equivalent :*

(i) $3u-1 \to 1$ *and* $3u-2 \to 1$ *for all* $u \in N$.

(ii) *Both* $3u-1$ *and* $3u-2$ *have reductions for all* $u \in N$.

Proof. Clearly $(i) \Rightarrow (ii)$. Now let (ii) hold. Consider any $3v-1$. Then by (ii), $3v-1$ has a reduction, say y_1. Then $y_1 < 3v-1$. Now by observation 3.1(c) above, y_1 is of the type $3t-1$ or $3t-2$ for some t. Then by (ii), y_1 has a reduction say y_2. Then $y_2 < y_1$. Again y_2 is of the type $3t-1$ or $3t-2$

for some t. Then by (ii), y_2 has a reduction say y_3. Continuing this process, after a finite number of steps we must have some y_n for which $y_n = 1$. This shows that $3v - 1 \to 1$.
Considering $3v - 2$, by the similar argument we can prove that $3v - 2 \to 1$. Thus if (ii) holds then (i) must hold.

Theorem 11. *The following statements are equivalent* :

(i) $n \to 1$ *for all* n.

(ii) *Both* $3u - 1$ *and* $3u - 2$ *have reductions for all* $u \in N$.

Proof is evident from Theorem 9 and Theorem 10.

5 Collatz Decisive Subset of $N \times N$, Concept of Net Induction Rate and Net Reduction Rate and Final Conclusion

It is clear that the sets of Type-A, Type-B and Type-C naturals have the same cardinality. Now a Type-A natural may be even **or** of the form $4t + 1$ **or** of the form $4t + 3$, and a Type-B natural may be even **or** of the form $4t + 1$ **or** of the form $4t + 3$.

Also note that

(i) if $3u - 2 = 2t$, where t is odd then $3u - 1 = 4v + 3$ for some v.

(ii) if $3u - 2 = 2t$, where t is even then $3u - 1 = 4v + 1$ for some v.

(iii) if $3u - 2 = 4t + 1$ then $3u - 1 = 2v$ for some odd v.

(iv) if $3u - 2 = 4t + 3$ then $3u - 1 = 2v$ for some even v.

The reverse cases of the above results (i), (ii), (iii), (iv) are also true.

Now let $C(S)$ denote the decisive subset of $N \times N$ where

$C(S) = \{(3u-2, 3u-1), u \in N\}$.
Let $\Omega_1 \subset C(S)$ where the components are of the form $2t$ and $4v+3$ in any order and $\Omega_2 \subset C(S)$ where the components are of the form $2t$ and $4v+1$ in any order.

Consequently, $C(S) = \Omega_1 \cup \Omega_2$ and cardinality of Ω_1 = cardinality of Ω_2.

Definition 12. **(i)** Net Induction Rate of elements of Ω_1.

Consider any ordered pair in Ω_1. Since $2t \to t$ is a reduction and $4v+3 \to 6v+5$ is an induction,

Net Induction Rate (N-I-R) of elements of $\Omega_1 = \frac{(6v+5)-(4v+3)}{4v+3} - \frac{2t-t}{2t} = \frac{1}{8v+6}$.

Note that as v becomes larger and larger, N-I-R approaches 0.

(ii) Net Reduction Rate of elements of Ω_2.

Consider any ordered pair in Ω_2. Since $2t \to t$ is a reduction and $4v+1 \to 3v+1$ is a reduction,

Net Reduction Rate (N-R-R) of elements of $\Omega_2 = \frac{(4v+1)-(3v+1)}{4v+1} + \frac{2t-t}{2t} = \frac{6v+1}{8v+2}$.

Note that as v becomes larger and larger, N-R-R approaches $\frac{3}{4}$.

Final Conclusion : Since N-R-R $>$ N-I-R, we conclude that Collatz Conjecture is true.

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