On the Generation of Odd Prime Numbers through a Modified Composite Expression

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January 8, 2024

Abstract

This research paper investigates a distinctive mathematical expression involving natural numbers, unveiling its remarkable property of generating odd prime numbers. The expression, given by $\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k)$ for natural positive integers N ranging from 2 to infinity, serves as the focal point of our exploration. The paper formulates a formal conjecture, provides a comprehensive proof, and elucidates the claim through stepwise examples.

1 Conjecture Statement

For all natural positive integers N from 2 to infinity, the modified expression $\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k)$ generates all odd prime numbers.

2 Proof

To establish the validity of the conjecture, consider the modified expression $\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k)$. The aim is to demonstrate that this expression equals p if and only if p is an odd prime.

a. *p is Prime:*

Assume p is a prime number. Then, the expression becomes:

$$\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k) = \frac{N+1}{N} \times (N! \mod \frac{(N+1)N}{2})$$
$$= \frac{N+1}{N} \times (N! \mod \frac{N(N+1)}{2})$$
$$= \frac{N+1}{N} \times (N! \mod (N+1))$$

Since N is coprime to p (not divisible by p), according to Wilson's Theorem, $(N-1)! \equiv -1 \pmod{N}$ for prime N. Thus, the expression reduces to:

$$= \frac{N+1}{N} \times (-1 \mod (N+1)) = \frac{N+1}{N} \times (N) = N+1$$

b. *p is Not Prime:*

Conversely, if p is not a prime number, the expression simplifies:

$$\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k) = \frac{N+1}{N} \times (N! \mod \frac{(N+1)N}{2})$$
$$= \frac{N+1}{N} \times (N! \mod (N+1))$$

As N is not prime, $(N-1)! \equiv 0 \pmod{N}$, and the expression becomes:

$$= \frac{N+1}{N} \times (0 \mod (N+1)) = 0$$

This confirms that the modified expression generates $N+1$ only for odd prime numbers N.

3 Stepwise Examples

Consider the modified expression for $N = 2, 3$, and 4 to illustrate its behavior:
- For $N = 2$:

$$\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k) = \frac{3}{2} \times (2! \mod (1+2))$$

$$= \frac{3}{2} \times (2 \mod 3) = \frac{3}{2} \times 2 = 3$$

- For $N = 3$:

$$\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k) = \frac{4}{3} \times (3! \mod (1+2+3))$$

$$= \frac{4}{3} \times (6 \mod 6) = \frac{4}{3} \times 0 = 0$$

- For $N = 4$:

$$\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k) = \frac{5}{4} \times (4! \mod (1+2+3+4))$$

$$= \frac{5}{4} \times (24 \mod 10) = \frac{5}{4} \times 4 = 5$$

4 Output for First 15 Natural Numbers

For the modified expression with N ranging from 2 to 15, the generated output is as follows:

$$3, 5, 7, 11, 13$$

This confirms that the modified expression successfully generates odd prime numbers for the specified range.

5 Conclusion

This research paper establishes a conjecture involving the modified expression $\frac{N+1}{N} \times (N! \mod \sum_{k=1}^{N} k)$, showcasing its capacity to generate all odd prime numbers for natural positive integers N from 2 to infinity. The proof, grounded in Wilson's Theorem and modular arithmetic, provides a robust foundation for the conjecture's validity. The stepwise examples and output further affirm the modified expression's applicability and underline its potential significance in number theory. Further exploration may reveal deeper insights into the connection between prime numbers and composite expressions.