A third order sequence

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ABSTRACT

We study the third order sequence:

$$u_n = 3\,u_{n-1} + 6\,u_{n-2} + 3\,u_{n-3}\;,\; u_0 = 1,\; u_1 = 3,\; u_2 = 15\;,\; n \geq 3$$

Keywords: third order sequences, binomial series, number Pi, combinatorial identities.

1. Introduction

Notations:

[x]: Floor function (Integer part)

Binomial coefficient: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$, $n \geq k > 0$

Entry 1:

$$u_n = 3\,u_{n-1} + 6\,u_{n-2} + 3\,u_{n-3}\;,\; u_0 = 1,\; u_1 = 3,\; u_2 = 15\;,\; n \geq 3$$

$$u_n = \{1, 3, 15, 66, 297, 1332, 5976, 26\,811, 120\,285, \ldots\}$$

Entry 2:

$$u_n = \sum_{k=0}^{[n/2]} \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \binom{n-k-m}{m} \binom{m}{k} 3^{n-k-m} \cdot 2^{m-k}\;,\; n \geq 0$$

Entry 3:

$$\alpha = \lim_{n\to\infty} \sqrt{\frac{u_n}{u_{n+1}}} = \sqrt[3]{\sqrt{\frac{13}{108}} + \frac{1}{2\sqrt{3}}} - \sqrt[3]{\sqrt{\frac{13}{108}} - \frac{1}{2\sqrt{3}}}$$

$$\alpha^{2k} = \lim_{n\to\infty} \left(\frac{u_n}{u_{n+k}}\right)\;,\; k = 1, 2, 3, \ldots$$

Entry 4:

$$v_n = \sqrt{3}\;v_{n-1} + \sqrt{3}\;v_{n-3}\;,\; v_0 = 1,\; v_1 = \sqrt{3}\,,\; v_2 = 3,\;\; n \geq 3$$

$$v_n = \left\{1, \sqrt{3}\,, 3, 4\sqrt{3}\,, 15, 18\sqrt{3}\,, 66, 81\sqrt{3}\,, 297, \ldots\right\}$$

Entry 5:

$$v_n = \sum_{k=0}^{[n/3]} \binom{n-2k}{k} \left(\sqrt{3}\right)^{n-2k}\;, n \geq 0$$

$$u_n = v_{2n} = \sum_{k=0}^{\left[\frac{2n}{3}\right]} \binom{2n-2k}{k} 3^{n-k}, \quad n \geq 0$$

Entry 6:

$$\alpha = \lim_{n \to \infty} \left(\frac{v_n}{v_{n+1}}\right) = \sqrt[3]{\sqrt{\frac{13}{108}} + \frac{1}{2\sqrt{3}}} - \sqrt[3]{\sqrt{\frac{13}{108}} - \frac{1}{2\sqrt{3}}}$$

$$\alpha^k = \lim_{n \to \infty} \left(\frac{v_n}{v_{n+k}}\right), \quad k = 1, 2, 3, \ldots$$

2. Pi formulas

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

Entry 7:

$$\pi = 6 \sum_{n=0}^{\infty} (-1)^n \alpha^{2n+1} \sum_{k=0}^{[(2n)/3]} (-1)^k \binom{2n-2k}{k} \left(\frac{1}{2n+1} + \frac{3\alpha^2}{2n+3}\right)$$

Entry 8:

$$\pi = 6\alpha + 6 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \alpha^{2n+1}}{2n+1} f(n)$$

$$f(n) = 3 \sum_{k=0}^{\left[\frac{2n-2}{3}\right]} (-1)^k \binom{2n-2k-2}{k} - \sum_{k=0}^{\left[\frac{2n}{3}\right]} (-1)^k \binom{2n-2k}{k}$$

Entry 9:

$$\pi = 6 \sum_{n=0}^{\infty} (-1)^n \alpha^{2n+1} \sum_{k=0}^{\left[\frac{2n+1}{3}\right]} \frac{(-1)^k}{2n-2k+1} \binom{2n-2k+1}{k}$$

Remark:

$$(2n+1) \sum_{k=0}^{\left[\frac{2n+1}{3}\right]} \frac{(-1)^k}{2n-2k+1} \binom{2n-2k+1}{k} = \sum_{k=0}^{\left[\frac{2n}{3}\right]} (-1)^k \binom{2n-2k}{k} - 3 \sum_{k=0}^{\left[\frac{2n-2}{3}\right]} (-1)^k \binom{2n-2k-2}{k}, \quad n \geq 0$$

Entry 10:

$$\pi = 6 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{3\alpha}\right)^{2n+1} {}_3F_2\left(\frac{2n+1}{3}, \frac{2n+2}{3}, \frac{2n+3}{3}; n+1, \frac{2n+3}{2}; \frac{3}{4}\right) -$$
$$2 \sum_{n=1}^{\infty} (-1)^{n-1} \alpha^{2n-1} {}_3F_2\left(\frac{n+1}{3}, \frac{n+2}{3}, \frac{n+3}{3}; n+1, \frac{3}{2}; \frac{3}{4}\right)$$

Remark: ${}_3F_2$ is the generalized hypergeometric function.

Entry 11:

$$\pi = 6 \sum_{n=0}^{\infty} \alpha^{n+3} \left(\sqrt{3}\right)^n \sum_{k=0}^{[n/6]} \frac{(-1)^k\, 3^{-3k}}{2k+1} \binom{n-4k}{n-6k}$$

Entry 12:

$$\pi = \frac{9}{2} \sum_{n=0}^{\infty} \alpha^{2n+1} \sum_{k=0}^{\left[\frac{2n+1}{3}\right]} \frac{(9/16)^{n-k}}{2n-2k+1} \binom{2n-2k+1}{k} \binom{2n-2k}{n-k}$$

Entry 13:

$$\pi = 6 \sum_{n=0}^{\infty} (-1)^n \alpha^{n+1} \sum_{k=\lceil n/3 \rceil}^{[n/2]} \frac{(-1)^k \left(\sqrt{3}\right)^{n-2k}}{6k-2n+1} \binom{4k-n}{n-2k}$$

Remark: $\lceil x \rceil$ is the ceiling function.

Entry 14:

$$\pi = 6 \tan^{-1}(\alpha) + 6 \sum_{n=0}^{\infty} (-1)^n \alpha^{n+3} \left(\frac{1}{\sqrt{3}}\right)^n \sum_{k=0}^{[n/6]} \frac{(-1)^k\, 3^{3k}}{2k+1} \binom{n-4k}{n-6k}$$

Entry 15:

$$\pi = 8\sqrt{3} \sum_{n=0}^{\infty} (-1)^n u_n \beta^{2n+1} \left(\frac{1}{2n+1} - \frac{3\beta^2}{2n+3}\right)$$

$$\pi = 8\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \beta^{2n+1} \sum_{k=0}^{\left[\frac{2n+1}{3}\right]} \frac{3^{n-k}}{2n-2k+1} \binom{2n-2k+1}{k}$$

$$\beta = \frac{\sqrt{2}-1}{\sqrt{3}} + \left(\frac{\sqrt{2}-1}{\sqrt{3}} + \left(\frac{\sqrt{2}-1}{\sqrt{3}} + \ldots\right)^3\right)^3$$

3. References

[1] K. Adegoke, Weighted Tribonacci sums, arXiv:1804.06449[math.CA], 2018.

[2] W. Gerdes, Generalized tribonacci numbers and their convergent sequences., The Fibonacci Quarterly 16:3 , 1978.

[3] S. Rabinowitz, Algorithmic manipulation of third order linear recurrences, The Fibonacci Quarterly, 34:5 , 1996.

[4] A. G. Shannon and A. F. Horadam, Some properties of third-order recurrence relations, The Fibonacci Quarterly 10:2, 1972.

[5] H. W. Gould, Combinatorial Identities, rev. ed., Morgantown, 1972.

[6] M. E. Larsen, Summa Summarum, Peters, 2007.