Solving the Monty Hall Problem Using Information Theory

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Abstract

The Monty Hall Problem, which involves probability, uncertainty and information is solved using information theory. When the host opens one or more doors that don't contain the prize, he adds information to the remaining unselected doors. Information theory shows that the added information increases the likelihood that the prize is behind an unselected door. The method was applied to a generalization of the Monty Hall Problem and showed that the probability or winning the prize is increased by switching doors.

Keywords: Monty Hall Problem, Information theory, entropy, probability theory

Nomenclature

I - information

M - number of doors Monty Hall opens

N - number of doors

n - number of events

p - probability

p_i - probability of i^{th} event

S - entropy

$\mathbf{S_F}$ - final entropy

$\mathbf{S_i}$ - initial entropy

V - number of events associated with information

W - number of possible answers associated with entropy

1. Introduction

The Monty Hall Problem is named after a TV game show where the contestant could win a car, if he chose one of three doors correctly. The correct door revealed the winning choice of a car and the other two doors each revealed a goat that signifies an incorrect choice. Without prior knowledge, it is assumed that each door has a probability of 1/3, therefore the contestant could choose any one of the three doors and be equally likely to win the car. However, prior to opening his chosen door, the game show moderator, Monty Hall, opens one of the two remaining doors that always reveals a goat. Monty Hall knows which door contains the car, so he always opens a door that reveals a goat. At this point the contestant is given a choice of switching his original door choice to the remaining

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unopened door. What should the contestant do to increase his odds of winning the car? Since there are only two unopened doors (his original choice and the remaining unopened door) most people believe the odds for each door are the same at 50 percent each. Therefore, there wouldn't be any advantage to switching doors. Nevertheless, by switching doors the contestant increases his odds of winning the car from 1/3 to 2/3, so the contestant should always switch doors. This problem caused much controversy and discussion among both laymen and scholars due to logical and probability considerations. Numerous explanations were given in papers and on internet platforms.

The current work provides a completely new method of resolving the Monty Hall Problem using information theory, which is appropriate for the problem and elucidates the issue that causes many to incorrectly answer the question. A short introduction to entropy and information theory is presented to familiarize the read with the concepts. Information theory is then applied to the Monty Hall Problem and its generalization.

2. Entropy and Information Theory Concepts

Entropy is a quantity that represents disorder or uncertainty in a system and has notable application in the area of thermodynamics. For example, when heat energy flows of a hot object to a cooler object entropy increases. Heat cannot flow from a cooler object to a hotter object, since the entropy would decrease, which is not permitted by the 2nd Law of Thermodynamics. Entropy has numerous applications in various areas, one of which involves probability and chance. Entropy is a quantitative measure of our lack of information. The logarithm function is used in quantifying entropy and information, since it insures the additivity of the respective quantities. If there are **W** possible answers of equal probability, then the associated entropy is given by eq. (1), where ln() represents the natural logarithm function.

$$\mathbf{S} = \ln(\mathbf{W}) \tag{1}$$

More generally, if we have one of n events that occur with probabilities p_1, p_2, p_3, p_4, $p_5 \ldots p_n$, subject to the normalization condition,

$$\sum_1^n p_i = 1 \qquad , \tag{2}$$

then the entropy is given by eq. (3).

$$\mathbf{S} = -\sum_1^n p_i \ln(p_i) \tag{3}$$

For example, if n = 5 and each p_i is equal to 1/5, then eq. (3) reduces to eq. (4), which agrees with eq. (1) with **W** = 5.

$$\mathbf{S} = \frac{\ln(5)}{5} + \frac{\ln(5)}{5} + \frac{\ln(5)}{5} + \frac{\ln(5)}{5} + \frac{\ln(5)}{5} = \ln(5) \tag{4}$$

Entropy can be reduced by adding Information, **I**, so that the final entropy is given by the initial entropy plus the added information, as shown in eq. (5).

$$\mathbf{S_F} = \mathbf{S_I} + \mathbf{I} \tag{5}$$

Information equivalent to one correct choice out of **V** choices of equal probability is quantified by eq. (6).

$$\mathbf{I} = -\ln(\mathbf{V}) = \ln\left(\frac{1}{\mathbf{V}}\right) \tag{6}$$

If the information in eq.(6) is added to the entropy in eq. (1), the final entropy is reduced, as shown in eq. (7).

$$\mathbf{S_F} = \ln(\mathbf{W}) - \ln(\mathbf{V}) = \ln\left(\frac{\mathbf{W}}{\mathbf{V}}\right) = \ln\left(\frac{1}{\mathbf{V}/\mathbf{W}}\right) \tag{7}$$

The probability associated with $\mathbf{S_F}$ in eq. (7) is given by **V/W**. If **V** equals **W**, then the information reduces the final entropy to zero, as given by eq. (8), since the natural logarithm of 1 is 0. This is equivalent to having complete knowledge of the correct choice, with a probability of 1.

$$\mathbf{S_F} = \ln\left(\frac{\mathbf{W}}{\mathbf{W}}\right) = \ln(1) = 0 \tag{8}$$

3. Application to the Monty Hall Problem

In the Monty Hall Problem, one of three closed doors conceals the prize, which is a car and each of the other two doors has a goat. Since the location of the car is assumed to be random, each door has a probability of 1/3 of concealing the car. The probability of the contestant's choice of one door is 1/3, which has an associated entropy given by eq. (9).

$$\mathbf{S} = \ln\left(\frac{1}{p}\right) = \ln\left(\frac{1}{1/3}\right) = \ln(3) \tag{9}$$

Since Monty Hall knows where the car is located, he opens one of the remaining two doors that always contains a goat. In opening the door Monty Hall provides information given by -ln(2), since he has a choice of two doors. Initially, the remaining unopened door has an entropy of ln(3), just as the contestant's original choice. However, the information added by Monty Hall decreases the entropy of the unopened door to ln(3/2), as shown in eq. (10).

$$\mathbf{S_F} = \mathbf{S_i} + \mathbf{I} = \ln(3) - \ln(2) = \ln\left(\frac{3}{2}\right) = \ln\left(\frac{1}{2/3}\right) \tag{10}$$

The probability associated with the final entropy of the unopen door is 2/3, as shown in eq. (10). Therefore, the contestant should switch to the remaining unopened door to increase his chance of winning the car from 1/3 to 2/3.

Suppose the game involved four doors instead of three and after the contestant chooses one door, Monty Hall opens a nonwinning door. Should the contestant switch to one of the two unopened doors? Once again, the problem can be solved using information theory. The probability of the car being behind an unopened door is ½, since we have two available doors out of a total of four doors. The associated entropy for the ½ probability is ln(2). When Monty Hall opens one door, he actually chose two available doors out of the remaining three doors for a probability of 2/3. The information associated with the 2/3 probability is -ln(3/2), so the final entropy is obtained by adding the information to the original entropy, as shown in eq. (11).

$$\mathbf{S_F} = \mathbf{S_i} + \mathbf{I} = \ln(2) - \ln\left(\frac{3}{2}\right) = \ln\left(\frac{4}{3}\right) = \ln\left(\frac{1}{3/4}\right) \tag{11}$$

The probability associated with the entropy in eq. (11) is ¾**,** which is shared between the two doors. Therefore, each of the remaining two doors has a probability of 3/8 of containing the car. Since 3/8 is greater than the original door probability of ¼, the contestant should switch doors.

If Monty Hall opened two of the remaining three doors instead of just one door, he actually chose just one available door out of the original three doors. This 1/3 probability has an associated information of -ln(3). The probability of one remaining available door is ¼**,** so the associated entropy is ln(4). Using these changes in eq.(11) yields eq. (12).

$$\mathbf{S_F} = \mathbf{S_i} + \mathbf{I} = \ln(4) - \ln(3) = \ln\left(\frac{4}{3}\right) = \ln\left(\frac{1}{3/4}\right) \tag{12}$$

From eq. (12), the probability of the remaining door is ¾. Therefore, the contestant can increase his chances of winning the car from ¼ to ¾ by switching doors.

The Monty Hall Problem can be generalized by considering **N** doors with only one door having the desired car. The contestant chooses one door with a probability of 1/**N** of containing the car. Monty Hall then reveals the content of **M** doors that only contain goats. The contestant is then given the choice of switching doors with any of the remaining **N–1-M** doors. Does the contestant increase his chances of winning the car by switching doors? The probability that the car is contained in one of the **N-1-M** doors is given by (**N-1-M)/N**. The associated entropy is given by eq. (13).

$$\mathbf{S} = \ln\left(\frac{1}{p}\right) = \ln\left(\frac{\mathbf{N}}{\mathbf{N-1-M}}\right) \tag{13}$$

By opening **M** of the remaining **N-1** unopened doors, Monty Hall provides information associated with the probability (**N-1-M**)/(**N-1**), which has an information value given by eq. (14).

$$\mathbf{I} = -\ln\left(\frac{\mathbf{N-1}}{\mathbf{N-1-M}}\right) = \ln\left(\frac{\mathbf{N-1-M}}{\mathbf{N-1}}\right) \tag{14}$$

The final entropy, $\mathbf{S_F}$, is found by adding the information in eq. (14) to the entropy in eq. (13).

$$\mathbf{S_F} = \mathbf{S} + \mathbf{I} = \ln\left[\left(\frac{\mathbf{N}}{\mathbf{N-1-M}}\right)\left(\frac{\mathbf{N-1-M}}{\mathbf{N-1}}\right)\right] = \ln\left(\frac{\mathbf{N}}{\mathbf{N-1}}\right) = \ln\left(\frac{\mathbf{1}}{(\mathbf{N-1})/\mathbf{N}}\right) \tag{15}$$

Eq. (15) indicates that the final probability of the car being contained in the group of **N-1-M** doors is given by **(N-1)/N**. Since it is assumed that the doors in the group have equal probability, one can find that probability of a single door by dividing by the number of doors, as shown in eq. (16).

$$\mathrm{p} = \left(\frac{\mathbf{1}}{\mathbf{N-1-M}}\right)\left(\frac{\mathbf{N-1}}{\mathbf{N}}\right) = \left[\frac{\mathbf{N-1}}{\mathbf{N(N-1-M)}}\right] \tag{16}$$

The probability in eq. (16) is always greater than the probability of the initial choice, which is 1/**N**, as long as **M** is greater than zero. Therefore, the contestant should always switch doors in the generalized Monty Hall Problem. If **N** = 3 and **M** = 1, eq. (16) yields a probability of 2/3, which is in agreement with the original Monty Hall Problem in eq. (10). Eq. (16) also agrees with the results for the four door Monty Hall case which was examined above.

4. Conclusion

Information theory was used to solve the Monty Hall Problem and the generalized Monty Hall problem. The related probabilities were used to quantify the associated entropies and information. By always opening a nonwinning door, Monty Hall adds a quantity of information that lowers the entropy of the remaining unselected door. The lower entropy increases the probability that the car is behind the unselected door. It should be noted that the added information cannot change the probability of the contestant's original door selection, since it is not a door that Monty Hall is permitted to open. Therefore, the contestant should always switch his door choice to increase chances of winning the car from 1/3 to 2/3.

The probability for the generalized Monty Hall Problem was also developed. Results indicated that the information added by Monty Hall when he opened nonwinning doors served to decrease the entropy of the remaining unopened doors. The associated probability of each unopened door was found to be greater than the original selected door. Therefore, switching doors is always beneficial to the contestant.