Contribution to Goldbach's Conjectures

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Abstract: The internal structure of the natural numbers reveals the relation between the weak and strong Goldbach's conjectures. Explicitly, if the weak Goldbach's conjecture is true, the strong Goldbach's conjecture is, and Goldbach's conjectures are true.

Key words: Goldbach weak and strong conjectures, Harald A. Helfgott, Ivan Matveyevich Vinogradov, Goldbach's numbers.

The prime numbers are fundamental multiplication blocks of the natural integers. However, there is an indication that the prime numbers may be the fundamental addition operation blocks of the natural integers. The first such statement, known as Goldbach's strong and weak conjectures, was made by the Russian mathematician Christian Goldbach in 1742 year. The modern statements[1] of the conjectures are:

<u>WEAK CONJECTURE</u>: *Every odd integer greater than* 7 *is represented by a sum of three odd, not necessarily distinct primes.*
<u>STRONG CONJECTURE</u>: *Every even integer greater than* 4 *is represented by a sum of two odd, not necessarily distinct primes.*

After the work of Russian mathematician Ivan Matveyevich Vinogradov in 1937 the paper of Harald A. Helfgott "The Ternary Goldbach Conjecture is True", published in 2014, is recognized as the final proof of Goldbach's weak conjecture. The set of all odd natural integers, created by all possible sums of three, not necessarily distinct, primes are the Goldbach's 3G numbers. Apart from the 3G numbers, there are no other odd natural integers, and $2\mathbb{N}+1 = 3\mathrm{G}$. However, unless the strong Goldbach's conjecture is proven, the even integers Goldbach's set 2G will be only a subset of the even natural numbers $2\mathbb{N}$.

The basic idea is that the properties of the natural numbers are interrelated through their internal structure. This paper intends to interrelate the weak and strong Goldbach's conjectures.

Further underlined truth is that the weak Goldbach's conjecture is true. We specify the following notation, $|x\rangle$ stands for a column and $\langle y|$ for a row vector. The overline of an integer $\overline{z}$ indicates that z belongs to the column vector. The set of all primes is Π, $3a$ are elements of 3G and $2b$ are elements of 2G set. The pairing operation of two integer is $\hat{\wedge} : \hat{\wedge}(\xi,\eta)\xi \wedge \eta \sim \xi + \eta$. The operation $|\mathrm{X}\rangle\langle\mathrm{X}|$ creates two-dimensional objects by pairing objects. For example, the matrix $|\mathrm{X}\rangle\langle\mathrm{Y}|$ is the coupling of the column $|\mathrm{X}\rangle$ vector and row $\langle\mathrm{Y}|$ vector entries. The $\wedge$ symbol couples the arrays. The projection operation $\downarrow$ of a set A on the set B is $A \downarrow B = A \cap B$. The lift of a set A is $A \uparrow B = A \cup B$.
The set operations are used in the standard way and perhaps in a similar meaning. The operations $\oplus$ and $\ominus$ are the general objects addition operation.

[1]Integer 1 and 2 are excluded from the prime set.

Definition: *The pairing operation $\hat{\wedge}$ is the "onto complete" if the projection operation $3G \downarrow 2G$ and the lift operation $2G \uparrow 2G$ are onto. The operation is "distinct onto complete" if the onto complete is supported by the all set $2G$.*

Corollary 1. *Cardinal numbers of the sets $3G$ and $2G$ are identical.*

■ The proof, supported by the calculation in the Appendix, is done by construction in the following few logical steps.

1. *The pairing operation $\hat{\wedge}(3G \downarrow 2G)$ is the distinct onto complete.*

According to the weak Goldbach's conjecture, the 3G set is the 3-primes complete, and for each$3a \in$ 3G

$$\begin{aligned} 3a &= (\xi,\eta,\zeta) = ((\xi,\eta),\zeta) = (\xi,(\eta,\zeta)) = (\eta,(\xi,\zeta)) \;\Rightarrow\; \exists 2b \in \{(\alpha,\beta),\beta,\gamma),(\gamma,\alpha)\} \\ &\forall\, 3a \in \mathrm{3G}\ \exists 2b = (\alpha,\beta) \in \mathrm{2G} \quad \therefore\ \mathrm{3G} \downarrow \mathrm{2G} \subset \mathrm{2G}. \end{aligned}$$

Assume that the lift 2G $\uparrow$ 3G is not onto. Then there is a pair $2b \in$ 2G such that

$$\begin{aligned} &\forall\, 2b \in \mathrm{2G}\ \exists \gamma \in \Pi \therefore \hat{\wedge}(\alpha,\beta,\gamma) = 3a \in \mathrm{3G} \\ \Rightarrow\quad &3a \downarrow 2G = (\alpha,\beta) \in \mathrm{2G} \quad \therefore \mathrm{2G} \uparrow \mathrm{3G}, \end{aligned}$$

contradiction, and $3G \overset{\text{ONTO}}{\longleftrightarrow}$ 2G. The completeness implies that each 3G Goldbach's number is supported by at least one, not necessarily distinct, pair in 3G. To show that 2G are all even integers, we must show that there are sufficiently many distinct couples in the set 2G to support all odd integers. The following part is an explicit construction proof of the set of distinct odd integers supported by distinct prime pairs. The prime number ξ is the family prime of the triplet (ξ,η,ζ), and the prime η is the matrix row prime enumerator.

2. *All 3-prime integers of a prime ξ family are supported by the triangular fundamental matrix $\mathcal{B}^{\mathrm{D}}_{\overline{\eta}\Pi}$ of the prime pairs.*

All possible 3-prime integers of a prime η from the prime ξ the family are in the η row vector

$$(\eta,\Pi) \quad = \big\langle |\eta\rangle, \Pi\big| = \big|\eta\big\rangle\big\langle \Pi\big| = \big|\eta\big\rangle\big\langle \overline{\eta}; 1,3,5,7,\cdots\zeta\cdots\big|$$

of the matrix M1 in the table of matrices in the Appendix. The prime η is coupling to each, one by one prime $\zeta \in \Pi$, the distribution property of the prime η, to form the pair (η,ζ). The collection of all η rows is forming the matrix of the pairs $\big\langle \overline{\eta},\Pi\big|$. Since $\big\langle \overline{\eta},\Pi\big| = \big|\overline{\eta}\big\rangle\big\langle \Pi\big|$ the coupling operation has the multiplication property. While $\big\langle \overline{\eta},\Pi\big|$ is the coupling of the primes the $\big|\overline{\eta}\big\rangle\big\langle \Pi\big|$ is the coupling of the arrays. The matrix of the pairs M1 $= \big\langle \overline{\eta},\Pi\big|$ is essential, and will be called the fundamental matrix of the pairs $\mathcal{B}_{\overline{\eta}\Pi}$.

The simple inspection of the matrix $M1$ shows the redundancy of the fundamental matrix, the characteristic of all matrices in the construction. The first case of redundancy is the couple multiplicity due to the matrix's main diagonal symmetry, and the second case is the pair multiplicity based on the pair equivalence. Else two pairs are equivalent if they contribute the same value even integer. The goal is to construct the matrices without multiplicities. The Appendix shows the explicit calculation.

Notice that the duplicates of the identical symmetric pairs in the matrix M1 are shaded. The identical pair multiplicity eliminates by the removal of the left lower triangular sub-matrix of the fundamental matrix. Exactly

$$\hat{\mathsf{D}}\mathcal{B}_{\overline{\eta}\Pi} \quad = \hat{\mathsf{D}}\big\langle |\overline{\eta}\rangle, \Pi\big| = \big|\overline{\eta}\big\rangle\big\langle \hat{\mathsf{D}}\Pi\big| = \big|\overline{\eta}\big\rangle\big\langle \Pi^{\mathrm{D}}\big| = \quad \mathcal{B}^{\mathrm{D}}_{\overline{\eta}\Pi},$$

and the reduced fundamental matrix $\mathcal{B}^{\mathrm{D}}_{\overline{\eta}\Pi}$ is the unshaded triangular matrix of the matrix M1 in the table of the matrices in the Appendix. The multiplication property of the coupling induces the reduced

upper right triangular prime matrix Π^{D} in the matrix M2 in the Appendix.
The reduction operator $\hat{\mathsf{R}}$ removes the equivalence multiplicity from the matrix M2. A pair (η, ζ) in a current row η cancels with an equivalent pair in any of the previous rows, which is the corresponding ζ prime is canceled in the reduced prime matrix Π^{D}. Exactly

$$\begin{aligned}
\hat{\mathsf{R}}\mathcal{B}_{\overline{\eta}\Pi^{\mathrm{D}}} &= \big|\eta\big\rangle\big\langle\hat{\mathsf{R}}\Pi^{\mathrm{D}}\big| = \big|\eta\big\rangle\big\langle\Pi^{\mathrm{DR}}\big| = \quad \mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi} \\
\Pi^{\mathrm{DR}}_{\eta} &= \hat{\mathsf{R}}\Pi^{\mathrm{D}}_{\eta} = \Pi^{\mathrm{D}}_{\eta} \ominus \sum_{1<\eta'<\eta}^{\eta} \Pi^{\mathrm{D}}_{\eta} \cap \Pi^{\mathrm{D}}_{\eta'} \\
\Rightarrow \quad \mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi} &= \bigcup_{\eta} \big|\overline{\eta}\big\rangle\big\langle\Pi^{\mathrm{DR}}\big|.
\end{aligned}$$

The matrices M2 and M3 in the Appendix show the calculation. Unshaded entries of the matrices M2 and M3 are the primes and even integers of the unit multiplicities in the reduced matrices of the prime and the even numbers.

Remark: The distinct primes in the matrix M2 and distinct couples in the fundamental matrix M3 are all possible distinct primes of the reduced prime matrix Π^{DR} and all possible couples of the reduced fundamental matrix $\mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}$. By construction, these two matrices are in one-to-one correspondence. Moreover, the fundamental matrix $\mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}$, once created, is unique for all the family representatives ξ.

3. *There are exactly as many distinct prime pairs as there are odd numbers.*

The matrix $\mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}$ is a fundamental matrix unique for all family primsξ. Each single family prime ξ couples to the same fundamental reduced matrix $\mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}$ to create all the family prime ξ odd integers $\mathcal{T}_{\xi} = \bar{\xi} \wedge \mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}$. Since 2-prime integers of the matrix $\mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}$ are distinct by the construction the odd integers $\mathcal{T}_{\xi}$ are distinct $3\mathrm{G}_{\xi}$ Goldbach's numbers. While each of the matrices M3.1, M3.2 M3.3, … is the family of the distinct 3-prime integers their intersections are not empty. Inherited multiplicity of the $3\mathrm{G}_{\xi}$ numbers eliminates by the family multiplicity reduction operator $\hat{\Psi}$.
Further, the sets $3\mathrm{G}(1), 3\mathrm{G}(3), 3\mathrm{G}(5), \cdots, 3\mathrm{G}(\xi)\cdots$ are distinct 3G families of the odd integers with the intersections $3\mathrm{G}(\xi) \cap_{1<\xi'<\xi} 3\mathrm{G}(\xi') \neq \emptyset$. Then for each family prime ξ

$$\hat{\Psi}(3\mathrm{G}(\xi)) \quad = 3\mathrm{G}(\xi) \ominus \sum_{1<\xi'<\xi} 3\mathrm{G}(\xi) \ominus (3\mathrm{G}(\xi) \cap 3\mathrm{G}(\xi')) = \quad 3\mathrm{G}_{\xi} \quad \Rightarrow \quad 3\mathrm{G} = \bigcup_{\xi} 3\mathrm{G}_{\xi}.$$

The Goldbach's set 3G rests on the collection of the distinct prime pairs by the construction, it is distinct onto complete, and the number of the distinct 3G integers is the same as the number of the distinct pairs in the set 2G, or the sets 3G and 2G are distinct onto complete with respect to the pairing operation. The matrix M4 in the Appendix shows the calculation. ■

Corollary 2. *If the weak Goldbach's conjecture is true the strong Goldbach's conjecture is.*

■ All above is obtained under condition that the weak Goldbach's conjecture is true. All possible Goldbach's numbers 3G are supported by the fundamental matrix $\mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}$ and $3\mathrm{G} = |\overline{\Pi}\rangle\big\langle\mathcal{B}^{\mathrm{DR}}_{\overline{\eta}\Pi}\big|$. According to the first part of Corollary 1 sets 3G and 2G are "onto complete" with respect to the pairing operation, and according to Corollary 2 they are the "distinct onto complete" with respect to the same operation. Thus, the Goldbach's numbers 2G, 3G and the odd $2\mathbb{N}+1$ and even$2\mathbb{N}$ integers have the same cardinal numbers. Consequently, the strong Goldbach's conjecture is true.
In conclusion Goldbach's conjectures are true. ■

APPENDIX

The following table is the collection of the matrices supporting the construction of the all 3G Goldbach's integers on the set of all 2G Goldbac;s integers to show the one-to-one correspondence between two sets, all under condition that the weak Goldbah's conjecture is true.

Table 1. Construction of the 3G Integers

MATRIX M1: Fundamental Matrix $\mathcal{B}_{\overline{\Pi}\Pi}$											
$\forall\,\xi$	$\zeta \rightarrow$	**1**	**3**	**5**	**7**	**11**	**13**	**17**	**19**	**23**	$\cdots$
	(η,ζ)	(η,ζ)	(η,ζ)	(η,ζ)	(η,ζ)	(η,ζ)	(η,ζ)	(η,ζ)	(η,ζ)		
	1	(1,1)	(1,3)	(1,5)	(1,7)	(1,11)	(1,13)	(1,17)	(1,19)	(1,23)	$\cdots$
	3	(3,1)	(3,3)	(3,5)	(3,7)	(3,11)	(3,13)	(3,17)	(3,19)	(3,23)	$\cdots$
	5	(5,1)	(5,3)	(5,5)	(5,7)	(5,11)	(5,13)	(5,17)	(5,19)	(5,23)	$\cdots$
	7	(7,1)	(7,3)	(7,5)	(7,7)	(7,11)	(7,13)	(7,17)	(7,19)	(7,23)	$\cdots$
	11	(11,1	(11,3)	(11,5)	(11,7)	(11,11)	(11,13)	(11,17)	(11,19)	(11,23)	$\cdots$
	13	13,1)	(13,3)	(13,5)	(13,7)	(13,11)	(13,13)	(13,17)	(13,19)	(17,23)	$\cdots$
	17	(17,1)	(17,3)	(15,5)	(17,7)	(17,11)	(17,13)	(17,17)	(17,19)	(19,23)	$\cdots$
	19	(19,1)	(19,3)	(17,5)	(19,7)	(19,511	(19,13)	(19,17)	(19,19)	(23,23)	$\cdots$

MATRIX M2: Diagonal Symmetric Primes $\Pi^{\text{D}} = \hat{\text{D}}\,\Pi$																	
$\forall\,\xi$ (η,ξ)	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	
$(1,\xi)$	1	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	$\cdots$
$(3,\xi)$		3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	$\cdots$
$(5,\xi)$			5	7	11	13	17	19	23	29	31	37	41	43	47	53	$\cdots$
$(7,\xi)$				7	11	13	17	19	23	29	31	37	41	43	47	53	$\cdots$
$(11,\xi)$					11	13	17	19	23	29	31	37	41	43	47	53	$\cdots$
$(13,\xi)$						13	17	19	23	29	31	37	41	43	47	53	$\cdots$
$(17,\xi)$							17	19	23	29	31	37	41	43	47	53	$\cdots$
$(19,\xi)$								19	23	29	31	37	41	43	47	53	$\cdots$

MATRIX M3: Unique Matrix $2\text{G} = \lvert\overline{\Pi}\rangle\langle\Pi^{\text{DR}}\rvert$ of All Distinct Prime Couples																	
$\forall\,\xi$ (η,ξ)	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	
$(1,\xi)$	2	4	6	8	12	14	18	20	24	30	32	38	42	44	48	54	$\cdots$
$(3,\xi)$				10		16		22	26		34	40		46	50	56	$\cdots$
$(5,\xi)$									28		36				52	58	$\cdots$
$(7,\xi)$																	$\cdots$
$(11,\xi)$																	$\cdots$
$(13,\xi)$																	$\cdots$
$(17,\xi)$																	$\cdots$
$(19,\xi)$																	$\cdots$

MATRIX M3.1: Distinct 3G(1) $= 1 + \langle \Pi^{\mathbf{dr}} \rvert$ Integers for $\xi = 1$

$\xi = \mathbf{1}$	(η,ξ)	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	
	$(1,\xi)$	3	5	7	9	13	15	19	21	25	31	33	39	43	45	49	55	$\cdots$
	$(3,\xi)$				11		17		23	27		35	41		47	51	57	$\cdots$
	$(5,\xi)$									29		37				53	59	$\cdots$
	$(7,\xi)$																	$\cdots$
	$(11,\xi)$																	$\cdots$
	$(13,\xi)$																	$\cdots$
	$(17,\xi)$																	$\cdots$
	$(19,\xi)$																	$\cdots$

MATRIX M3.2: Distinct 3G(3) $= 3 + \langle \Pi^{\mathrm{DR}} \rvert$ Integers for $\xi = 3$

$\forall\, \xi = \mathbf{3}$	(η,ξ)	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	
	$(1,\xi)$	5	7	9	11	15	17	21	23	27	33	35	41	45	47	51	57	$\cdots$
	$(3,\xi)$				13		19		25	29		37	43		49	53	59	$\cdots$
	$(5,\xi)$									31		39				55	61	$\cdots$
	$(7,\xi)$																	$\cdots$
	$(11,\xi)$																	$\cdots$
	$(13,\xi)$																	$\cdots$
	$(17,\xi)$																	$\cdots$
	$(19,\xi)$																	$\cdots$

MATRIX M3.3:Distinct 3G(5) $= 5 + \langle \Pi^{\mathrm{DR}} \rvert$ Integers for $\xi = 5$

$\xi = \mathbf{5}$	(η,ξ)	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	
	$(1,\xi)$	7	9	11	13	17	19	23	25	29	35	37	43	47	49	53	59	$\cdots$
	$(3,\xi)$				15		21		27	31		39	45		51	55	61	$\cdots$
	$(5,\xi)$									33		41				57	63	$\cdots$
	$(7,\xi)$																	$\cdots$
	$(11,\xi)$																	$\cdots$
	$(13,\xi)$																	$\cdots$
	$(17,\xi)$																	$\cdots$
	$(19,\xi)$																	$\cdots$

MATRIX M4: All Distinct 3G $=$ 3G(1) $\ominus \sum_{1<\xi'<\xi}$ 3G(ξ) $\ominus$ [3G(ξ) $\cap$ 3G(ξ')] Integers

$\xi = \mathbf{1}$	(η,ξ)	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	ζ	
	$(1,\xi)$	3	5	7	9	13	15	19	21	25	31	33	39	43	45	49	55	$\cdots$
	$(3,\xi)$				11		17		23	27		35	41		47	51	57	$\cdots$
	$(5,\xi)$									29		37				53	59	$\cdots$
	$(7,\xi)$																	$\cdots$
	$(11,\xi)$																	$\cdots$
	$(13,\xi)$																	$\cdots$
	$(17,\xi)$																	$\cdots$
	$(19,\xi)$																	$\cdots$

References

[1] H. A. Helfgott, *The ternary Goldbach Conjecture is true*, arXiv: 1312.7748v2 [math.NT] 17 Jan 2014.

[2] George E. Andrews, *Number Theory,* Dower Publications, Inc. New York.

[3] W. E. Deskins, *Abstract Algebra,* The MacMilan Company, New York.