A ‘trinionic’ representation of a classical group

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Abstract

We apply ‘trinions’ put forward in viXra:1712.0131 [v1] to the Lie group SU(3) to discuss some physical matters.

1 Glossary

0 or $\vec{0}$: zero vector .

$a \in A$: a is a member of the set A.

$A := B$: A is defined as B.

A^{T}: transpose of a matrix A .

$\mathbb{C}$: the set of complex numbers .

CP or $\times$: cross product .

$\det$: determinant .

DP: dot product .

$\imath$: imaginary unit .

I_n: $n \times n$ identity matrix .

LHS: left-hand side .

MI: mathematical induction .

MT: multiplication table .

$\mathbb{N}$: $\{1, 2, 3, \ldots\}$.

$\mathbb{N}_0$: $\mathbb{N} \cup 0$.

O: the origin $O(0, 0, 0)$.

O_n: $n \times n$ null matrix .

$\mathbb{R}$: the set of real numbers .

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RHS: right-hand side .

$\mathbb{R}^n$: the vector space of n-tuples $x = (x_1, \ldots, x_n)$ with each $x_i \in \mathbb{R}$.

SU(n): special unitary group of degree n .

tr: trace .

VTP: vector triple product .

wrt: with respect to .

$|x|$: absolute value of x .

$\vec{x} \perp \vec{y}$: vector x is perpendicular to vector y .

$\vec{x} \not\perp \vec{y}$: vector x is not perpendicular to vector y.

2 Introduction and preliminary computation

The Lie group SU(3) , a classical group , has been known to be very useful physically [1], whereas we don't know what to do about its waywardness [2]. So we try applying 'trinions' (t_r's [1]) [3] to it to get some insights. At the outset, we write a 3×3 'trinionic' matrix A explicitly:

$$A = \begin{pmatrix} a_{11} + b_{11}i + c_{11}j & a_{12} + b_{12}i + c_{12}j & a_{13} + b_{13}i + c_{13}j \\ a_{21} + b_{21}i + c_{21}j & a_{22} + b_{22}i + c_{22}j & a_{23} + b_{23}i + c_{23}j \\ a_{31} + b_{31}i + c_{31}j & a_{32} + b_{32}i + c_{32}j & a_{33} + b_{33}i + c_{33}j \end{pmatrix},$$

where $a_{mn}, b_{mn}, c_{mn} \in \mathbb{R}$ with $m, n \in \mathbb{N}$ and $1 \le m, n \le 3$. Recalling the definitions of special unitarity, we demand A satisfy

$$A^\dagger A = I_3 \ , \tag{1}$$

and

$$\det A = 1 \ . \tag{2}$$

Then, we compute the LHS of (1).

$$A^\dagger A = \begin{pmatrix} a_{11} + b_{11}i + c_{11}j & a_{12} + b_{12}i + c_{12}j & a_{13} + b_{13}i + c_{13}j \\ a_{21} + b_{21}i + c_{21}j & a_{22} + b_{22}i + c_{22}j & a_{23} + b_{23}i + c_{23}j \\ a_{31} + b_{31}i + c_{31}j & a_{32} + b_{32}i + c_{32}j & a_{33} + b_{33}i + c_{33}j \end{pmatrix}^{\dagger} {}^{2}$$

[1] Not to be confused with tr.

[2] The character '†' is defined like the case of quantum mechanics .

$$
\cdot \begin{pmatrix} a_{11}+b_{11}i+c_{11}j & a_{12}+b_{12}i+c_{12}j & a_{13}+b_{13}i+c_{13}j \\ a_{21}+b_{21}i+c_{21}j & a_{22}+b_{22}i+c_{22}j & a_{23}+b_{23}i+c_{23}j \\ a_{31}+b_{31}i+c_{31}j & a_{32}+b_{32}i+c_{32}j & a_{33}+b_{33}i+c_{33}j \end{pmatrix}
$$

$$
= \begin{pmatrix} a_{11}-b_{11}i-c_{11}j & a_{21}-b_{21}i-c_{21}j & a_{31}-b_{31}i-c_{31}j \\ a_{12}-b_{12}i-c_{12}j & a_{22}-b_{22}i-c_{22}j & a_{32}-b_{32}i-c_{32}j \\ a_{13}-b_{13}i-c_{13}j & a_{23}-b_{23}i-c_{23}j & a_{33}-b_{33}i-c_{33}j \end{pmatrix}
$$

$$
\cdot \begin{pmatrix} a_{11}+b_{11}i+c_{11}j & a_{12}+b_{12}i+c_{12}j & a_{13}+b_{13}i+c_{13}j \\ a_{21}+b_{21}i+c_{21}j & a_{22}+b_{22}i+c_{22}j & a_{23}+b_{23}i+c_{23}j \\ a_{31}+b_{31}i+c_{31}j & a_{32}+b_{32}i+c_{32}j & a_{33}+b_{33}i+c_{33}j \end{pmatrix},
$$

which we rewrite as

$$
\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = B.
$$

For convenience, we reproduce the MT of t_r's [3, **Table 1**]:

$\times$	1	i	j
1	1	i	j
i	i	0	0
j	j	0	0

From now on, t_r-related calculations will be performed according to the above table.

We compute each entry of B explicitly.

$$
\begin{aligned}
b_{11} &= (a_{11}-b_{11}i-c_{11}j)\cdot(a_{11}+b_{11}i+c_{11}j)+(a_{21}-b_{21}i-c_{21}j)\cdot(a_{21}+b_{21}i+c_{21}j) \\
&\quad +(a_{31}-b_{31}i-c_{31}j)\cdot(a_{31}+b_{31}i+c_{31}j) \\
&= a_{11}^2+a_{21}^2+a_{31}^2,
\end{aligned}
$$

$$
\begin{aligned}
b_{12} &= (a_{11}-b_{11}i-c_{11}j)\cdot(a_{12}+b_{12}i+c_{12}j)+(a_{21}-b_{21}i-c_{21}j)\cdot(a_{22}+b_{22}i+c_{22}j) \\
&\quad +(a_{31}-b_{31}i-c_{31}j)\cdot(a_{32}+b_{32}i+c_{32}j) \\
&= a_{11}a_{12}+(a_{11}b_{12}-a_{12}b_{11})i+(a_{11}c_{12}-a_{12}c_{11})j \\
&\quad +a_{21}a_{22}+(a_{21}b_{22}-a_{22}b_{21})i+(a_{21}c_{22}-a_{22}c_{21})j \\
&\quad +a_{31}a_{32}+(a_{31}b_{32}-a_{32}b_{31})i+(a_{31}c_{32}-a_{32}c_{31})j,
\end{aligned}
$$

$$
\begin{aligned}
b_{13} &= (a_{11} - b_{11}i - c_{11}j) \cdot (a_{13} + b_{13}i + c_{13}j) + (a_{21} - b_{21}i - c_{21}j) \cdot (a_{23} + b_{23}i + c_{23}j) \\
&\quad + (a_{31} - b_{31}i - c_{31}j) \cdot (a_{33} + b_{33}i + c_{33}j) \\
&= a_{11}a_{13} + (a_{11}b_{13} - a_{13}b_{11})i + (a_{11}c_{13} - a_{13}c_{11})j \\
&\quad + a_{21}a_{23} + (a_{21}b_{23} - a_{23}b_{21})i + (a_{21}c_{23} - a_{23}c_{21})j \\
&\quad + a_{31}a_{33} + (a_{31}b_{33} - a_{33}b_{31})i + (a_{31}c_{33} - a_{33}c_{31})j, \\
b_{21} &= (a_{12} - b_{12}i - c_{12}j) \cdot (a_{11} + b_{11}i + c_{11}j) + (a_{22} - b_{22}i - c_{22}j) \cdot (a_{21} + b_{21}i + c_{21}j) \\
&\quad + (a_{32} - b_{32}i - c_{32}j) \cdot (a_{31} + b_{31}i + c_{31}j) \\
&= a_{11}a_{12} + (a_{12}b_{11} - a_{11}b_{12})i + (a_{12}c_{11} - a_{11}c_{12})j \\
&\quad + a_{21}a_{22} + (a_{22}b_{21} - a_{21}b_{22})i + (a_{22}c_{21} - a_{21}c_{22})j \\
&\quad + a_{31}a_{32} + (a_{32}b_{31} - a_{31}b_{32})i + (a_{32}c_{31} - a_{31}c_{32})j, \\
b_{22} &= (a_{12} - b_{12}i - c_{12}j) \cdot (a_{12} + b_{12}i + c_{12}j) + (a_{22} - b_{22}i - c_{22}j) \cdot (a_{22} + b_{22}i + c_{22}j) \\
&\quad + (a_{32} - b_{32}i - c_{32}j) \cdot (a_{32} + b_{32}i + c_{32}j) \\
&= a_{12}^2 + a_{22}^2 + a_{32}^2, \\
b_{23} &= (a_{12} - b_{12}i - c_{12}j) \cdot (a_{13} + b_{13}i + c_{13}j) + (a_{22} - b_{22}i - c_{22}j) \cdot (a_{23} + b_{23}i + c_{23}j) \\
&\quad + (a_{32} - b_{32}i - c_{32}j) \cdot (a_{33} + b_{33}i + c_{33}j) \\
&= a_{12}a_{13} + (a_{12}b_{13} - a_{13}b_{12})i + (a_{12}c_{13} - a_{13}c_{12})j \\
&\quad + a_{22}a_{23} + (a_{22}b_{23} - a_{23}b_{22})i + (a_{22}c_{23} - a_{23}c_{22})j \\
&\quad + a_{32}a_{33} + (a_{32}b_{33} - a_{33}b_{32})i + (a_{32}c_{33} - a_{33}c_{32})j, \\
b_{31} &= (a_{13} - b_{13}i - c_{13}j) \cdot (a_{11} + b_{11}i + c_{11}j) + (a_{23} - b_{23}i - c_{23}j) \cdot (a_{21} + b_{21}i + c_{21}j) \\
&\quad + (a_{33} - b_{33}i - c_{33}j) \cdot (a_{31} + b_{31}i + c_{31}j) \\
&= a_{11}a_{13} + (a_{13}b_{11} - a_{11}b_{13})i + (a_{13}c_{11} - a_{11}c_{13})j \\
&\quad + a_{21}a_{23} + (a_{23}b_{21} - a_{21}b_{23})i + (a_{23}c_{21} - a_{21}c_{23})j \\
&\quad + a_{31}a_{33} + (a_{33}b_{31} - a_{31}b_{33})i + (a_{33}c_{31} - a_{31}c_{33})j, \\
b_{32} &= (a_{13} - b_{13}i - c_{13}j) \cdot (a_{12} + b_{12}i + c_{12}j) + (a_{23} - b_{23}i - c_{23}j) \cdot (a_{22} + b_{22}i + c_{22}j) \\
&\quad + (a_{33} - b_{33}i - c_{33}j) \cdot (a_{32} + b_{32}i + c_{32}j) \\
&= a_{12}a_{13} + (a_{13}b_{12} - a_{12}b_{13})i + (a_{13}c_{12} - a_{12}c_{13})j \\
&\quad + a_{22}a_{23} + (a_{23}b_{22} - a_{22}b_{23})i + (a_{23}c_{22} - a_{22}c_{23})j \\
&\quad + a_{32}a_{33} + (a_{33}b_{32} - a_{32}b_{33})i + (a_{33}c_{32} - a_{32}c_{33})j, \\
b_{33} &= (a_{13} - b_{13}i - c_{13}j) \cdot (a_{13} + b_{13}i + c_{13}j) + (a_{23} - b_{23}i - c_{23}j) \cdot (a_{23} + b_{23}i + c_{23}j) \\
&\quad + (a_{33} - b_{33}i - c_{33}j) \cdot (a_{33} + b_{33}i + c_{33}j) \\
&= a_{13}^2 + a_{23}^2 + a_{33}^2.
\end{aligned}
$$

Equating the above with entries of I_3, one gets the following equations.

$$
\begin{cases}
a_{11}^2 + a_{21}^2 + a_{31}^2 = 1, \\
a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, \\
a_{11}b_{12} - a_{12}b_{11} + a_{21}b_{22} - a_{22}b_{21} + a_{31}b_{32} - a_{32}b_{31} = 0, \\
a_{11}c_{12} - a_{12}c_{11} + a_{21}c_{22} - a_{22}c_{21} + a_{31}c_{32} - a_{32}c_{31} = 0, \\
a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0, \\
a_{11}b_{13} - a_{13}b_{11} + a_{21}b_{23} - a_{23}b_{21} + a_{31}b_{33} - a_{33}b_{31} = 0, \\
a_{11}c_{13} - a_{13}c_{11} + a_{21}c_{23} - a_{23}c_{21} + a_{31}c_{33} - a_{33}c_{31} = 0, \\
a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, \\
a_{12}b_{11} - a_{11}b_{12} + a_{22}b_{21} - a_{21}b_{22} + a_{32}b_{31} - a_{31}b_{32} = 0, \\
a_{12}c_{11} - a_{11}c_{12} + a_{22}c_{21} - a_{21}c_{22} + a_{32}c_{31} - a_{31}c_{32} = 0, \\
a_{12}^2 + a_{22}^2 + a_{32}^2 = 1, \\
a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, \\
a_{12}b_{13} - a_{13}b_{12} + a_{22}b_{23} - a_{23}b_{22} + a_{32}b_{33} - a_{33}b_{32} = 0, \\
a_{12}c_{13} - a_{13}c_{12} + a_{22}c_{23} - a_{23}c_{22} + a_{32}c_{33} - a_{33}c_{32} = 0, \\
a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0, \\
a_{13}b_{11} - a_{11}b_{13} + a_{23}b_{21} - a_{21}b_{23} + a_{33}b_{31} - a_{31}b_{33} = 0, \\
a_{13}c_{11} - a_{11}c_{13} + a_{23}c_{21} - a_{21}c_{23} + a_{33}c_{31} - a_{31}c_{33} = 0, \\
a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, \\
a_{13}b_{12} - a_{12}b_{13} + a_{23}b_{22} - a_{22}b_{23} + a_{33}b_{32} - a_{32}b_{33} = 0, \\
a_{13}c_{12} - a_{12}c_{13} + a_{23}c_{22} - a_{22}c_{23} + a_{33}c_{32} - a_{32}c_{33} = 0, \\
a_{13}^2 + a_{23}^2 + a_{33}^2 = 1.
\end{cases}
$$

We notice a certain kind of duplication in the above. For example, the equations $a_{11}c_{12} - a_{12}c_{11} + a_{21}c_{22} - a_{22}c_{21} + a_{31}c_{32} - a_{32}c_{31} = 0$ and $a_{12}c_{11} - a_{11}c_{12} + a_{22}c_{21} - a_{21}c_{22} + a_{32}c_{31} - a_{31}c_{32} = 0$, which seem different, are essentially the same, since multiplying the LHS and RHS of the former by -1 amounts to the latter. Omitting such duplication, one simplifies these equations to

$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1, & (3) \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1, & (4) \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1, & (5) \\ a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, & (6) \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0, & (7) \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, & (8) \\ a_{12}b_{11} - a_{11}b_{12} + a_{22}b_{21} - a_{21}b_{22} + a_{32}b_{31} - a_{31}b_{32} = 0, & (9) \\ a_{11}c_{12} - a_{12}c_{11} + a_{21}c_{22} - a_{22}c_{21} + a_{31}c_{32} - a_{32}c_{31} = 0, & (10) \\ a_{13}b_{11} - a_{11}b_{13} + a_{23}b_{21} - a_{21}b_{23} + a_{33}b_{31} - a_{31}b_{33} = 0, & (11) \\ a_{13}c_{11} - a_{11}c_{13} + a_{23}c_{21} - a_{21}c_{23} + a_{33}c_{31} - a_{31}c_{33} = 0, & (12) \\ a_{12}b_{13} - a_{13}b_{12} + a_{22}b_{23} - a_{23}b_{22} + a_{32}b_{33} - a_{33}b_{32} = 0, & (13) \\ a_{12}c_{13} - a_{13}c_{12} + a_{22}c_{23} - a_{23}c_{22} + a_{32}c_{33} - a_{33}c_{32} = 0. & (14) \end{cases}$$

3 Managing to get an example

Solving $(3) - (14)$ in a resounding manner *does* seem a daunting task. So we would like to rely on intuition to some extent. [3] Starting with $(3) - (5)$, we intuitively set

$$(a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}) = (1, 0, 0, 0, 1, 0, 0, 0, 1). \tag{15}$$

Fortunately, (15) happens to satisfy $(6) - (8)$, giving us some clues. Next, using it, we make $(9) - (14)$ simpler:

$$\begin{cases} -b_{12} + b_{21} = 0, \\ c_{12} - c_{21} = 0, \\ -b_{13} + b_{31} = 0, \\ -c_{13} + c_{31} = 0, \\ b_{23} - b_{32} = 0, \\ c_{23} - c_{32} = 0. \end{cases}$$

Again, we intuitively set

$$(b_{12}, b_{13}, b_{21}, b_{23}, b_{31}, b_{32}, c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1). \tag{16}$$

[3] Of course, we are aware that we *can* miss something important due to the very imperfection of our intuition.

Unknowns we haven't considered yet include $b_{11}, b_{22}, b_{33}, c_{11}, c_{22}$, and c_{33}, all of which we set to be 0 for the sake of simplicity. Taken together, we are led to the matrix

$$C = \left(\begin{array}{ccc} 1 & i+j & i+j \\ i+j & 1 & i+j \\ i+j & i+j & 1 \end{array} \right).$$

We immediately notice the following.

Property 3.1. $\mathrm{tr} C = 3$.

What about $\det C$, then? Using the Leibniz formula , we compute

$$1 \cdot \begin{vmatrix} 1 & i+j \\ i+j & 1 \end{vmatrix} - (i+j) \cdot \begin{vmatrix} i+j & i+j \\ i+j & 1 \end{vmatrix} + (i+j) \cdot \begin{vmatrix} i+j & 1 \\ i+j & i+j \end{vmatrix}$$
$$= 1 \cdot \{1 \cdot 1 - (i+j) \cdot (i+j)\} - (i+j) \cdot \{(i+j) \cdot 1 - (i+j) \cdot (i+j)\} + (i+j) \cdot \{(i+j) \cdot (i+j) - 1 \cdot (i+j)\}$$
$$= 1 - (i+j)^2 - (i+j)^2 + (i+j)^3 + (i+j)^3 - (i+j)^2$$
$$= 1.$$

So we point out

Property 3.2. $\det C = 1$.

The above *Property* is found to satisfy (2). By the way, is (1) satisfied like (2)? Since the entries of C come from the values satisfying (1), actually, we don't have to check whether $C^\dagger C = I_3$ holds. That said, we compute

$$C^\dagger C = \left(\begin{array}{ccc} 1 & i+j & i+j \\ i+j & 1 & i+j \\ i+j & i+j & 1 \end{array} \right)^\dagger \left(\begin{array}{ccc} 1 & i+j & i+j \\ i+j & 1 & i+j \\ i+j & i+j & 1 \end{array} \right)$$
$$= \left(\begin{array}{ccc} 1 & -i-j & -i-j \\ -i-j & 1 & -i-j \\ -i-j & -i-j & 1 \end{array} \right) \left(\begin{array}{ccc} 1 & i+j & i+j \\ i+j & 1 & i+j \\ i+j & i+j & 1 \end{array} \right)$$
$$= \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) = I_3$$

just for the sake of confirmation. As expected, C has been shown to satisfy (1). So we can say we have obtained the below example of A (rather) intuitively.

Example 3.3.

$$\left(\begin{array}{ccc} 1 & i+j & i+j \\ i+j & 1 & i+j \\ i+j & i+j & 1 \end{array} \right).$$

Talking of the computation of DP of rows (or columns) of such a matrix, two ways are thinkable. One is to regard them as (usual) real vectors . For example, we compute the DP of the first and second rows of the above example in this way:

$$1 \cdot (i + j) + (i + j) \cdot 1 + (i + j) \cdot (i + j) = 2(i + j).$$

The other is to treat them as if they were complex vectors . For example, we compute the DP of the first and second columns of the same example in this way: [4]

$$1 \cdot \overline{i + j} + (i + j) \cdot \overline{1} + (i + j) \cdot \overline{i + j} = 1 \cdot -(i + j) + (i + j) \cdot 1 + (i + j) \cdot -(i + j) = 0.$$

Taking footnote 2 into consideration, we will adopt the latter.

Incidentally, since matrices spanning the Lie algebra of SU(3) are Hermitian , we try to know whether it is also the case with C. Since

$$C^\dagger = \begin{pmatrix} 1 & -i-j & -i-j \\ -i-j & 1 & -i-j \\ -i-j & -i-j & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & i+j & i+j \\ i+j & 1 & i+j \\ i+j & i+j & 1 \end{pmatrix}, \begin{pmatrix} -1 & -i-j & -i-j \\ -i-j & -1 & -i-j \\ -i-j & -i-j & -1 \end{pmatrix},$$

it turns out that $C^\dagger \neq C, -C$. We thus point out the following.

Property 3.4. C is neither Hermitian nor skew-Hermitian .

4 Some decomposition

We decompose C by writing

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i+j & i+j \\ i+j & 0 & i+j \\ i+j & i+j & 0 \end{pmatrix} = I_3 + D \tag{17}$$

and compute the DP's of each row of D as mentioned earlier. [5]

<u>First and second rows</u>

$$(0, i + j, i + j) \cdot \overline{(i + j, 0, i + j)} = (0, i + j, i + j) \cdot (-i - j, 0, -i - j) = 0.$$

<u>Second and first rows</u>

$$(i + j, 0, i + j) \cdot \overline{(0, i + j, i + j)} = (i + j, 0, i + j) \cdot (0, -i - j, -i - j) = 0.$$

[4] In what follows, $\bar{x}$ denotes the conjugate of x. *Cf.* [3, *Def.* 2.1.4].

[5] Similar computations wrt columns are left to the reader as an exercise.

Second and third rows

$(i+j, 0, i+j) \cdot \overline{(i+j, i+j, 0)} = (i+j, 0, i+j) \cdot (-i-j, -i-j, 0) = 0.$

Third and second rows

$(i+j, i+j, 0) \cdot \overline{(i+j, 0, i+j)} = (i+j, i+j, 0) \cdot (-i-j, 0, -i-j) = 0.$

Third and first rows

$(i+j, i+j, 0) \cdot \overline{(0, i+j, i+j)} = (i+j, i+j, 0) \cdot (0, -i-j, -i-j) = 0.$

First and third rows

$(0, i+j, i+j) \cdot \overline{(i+j, i+j, 0)} = (0, i+j, i+j) \cdot (-i-j, -i-j, 0) = 0.$

So we point out the following.

Property 4.0.1. Rows of D are orthogonal like those of a unitary matrix .

We then make the following claim.

Claim 4.0.2. $C^n = I_3 + nD$, where $n \in \mathbb{N}_0$. (18)

Proof. MI on n. First, we compute

$$D^2 = \begin{pmatrix} 0 & i+j & i+j \\ i+j & 0 & i+j \\ i+j & i+j & 0 \end{pmatrix} \begin{pmatrix} 0 & i+j & i+j \\ i+j & 0 & i+j \\ i+j & i+j & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (19)$$

Next, we note when $n = 0$, the LHS of (18) is $C^0 = I_3$, so is its RHS, which means that (18) holds for $n = 0$. We now assume that (18) holds for $n = k$, that is, we assume we have

$$C^k = I_3 + kD.$$

It follows from (17) that

$$C^k \cdot C = (I_3 + kD) \cdot (I_3 + D).$$

That is, $C^{k+1} = I_3^2 + I_3 \cdot D + kD \cdot I_3 + kD \cdot D = I_3 + D + kD + kD^2$. Using (19), we get

$$C^{k+1} = I_3 + (k+1)D,$$

which means that (18) holds also for $n = k + 1$.

□

4.1 Another kind of decomposition

We can also write

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (i+j) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + (i+j) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = I_3 + (i+j)E + (i+j)F.$$

Remark 4.1.1. $F = E^{\mathrm{T}}$, and $E = X + Y + Z$ in terms of Heisenberg algebra , where

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 4.1.2. Using E and F, (18) can be rewritten as $C^n = I_3 + n(i+j)E + n(i+j)F$.

4.2 Yet another kind of decomposition

C can also be written as

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i & j \\ j & 0 & i \\ i & j & 0 \end{pmatrix} + \begin{pmatrix} 0 & j & i \\ i & 0 & j \\ j & i & 0 \end{pmatrix} = I_3 + H + J.$$ [6]

5 On $n \times n$ 'special trace matrix', ST_n

Inspired by *Properties* 3.1 and 3.2, we get the idea of $n \times n$ 'special trace matrix', which is abbreviated as ST_n and defined as follows.

Definition 5.1. $\det(\mathrm{ST}_n) = 1$.

Definition 5.2. $\mathrm{tr}(\mathrm{ST}_n) = n$.

Example 5.3. By the above definitions, C is a kind of ST_3.

Example 5.4. Likewise, $\mathrm{ST}_1 = 1$, if we can think of the natural number 1 as a 1×1 matrix whose $\det$ and tr are 1.

[6] We refrain from using the character 'I', in case it should be confused with I_3.

N.B. In what follows, '$\imath$' needs to be differentiated from 'i'.

Examples 5.5. Likewise,

1) I_2 is a kind of ST_2;

2) Likewise, $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ is a kind of ST_2;

3) $\begin{pmatrix} 1+\imath & -\imath \\ \imath & 1-\imath \end{pmatrix}$ is a kind of ST_2.

Notation 5.6. We write $\mathrm{ST}_{n,\mathbb{R}}$ instead of ST_n, when we wish to put an emphasis on the fact that each entry of ST_n is a real number .

Example 5.7. $\begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & -2 \\ 5 & 2 & 3 \end{pmatrix}$ is a kind of $\mathrm{ST}_{3,\mathbb{R}}$.

Notation 5.8. Likewise, we can write $\mathrm{ST}_{n,\mathbb{C}}$ instead of ST_n.

Example 5.9. $\begin{pmatrix} 2+\imath & 1+\imath & 1-\imath \\ \imath & 2-2\imath & -1+\imath \\ -1+\imath & \frac{7}{4}-\imath & -1+\imath \end{pmatrix}$ is a kind of $\mathrm{ST}_{3,\mathbb{C}}$.

6 $X^3+Y^3+Z^3-3XYZ = x^3+y^3+z^3-3xyz$: 'distance-preservation' by coincidence?

Consider the transformation given by $I_3 + H$: [7]

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & i & j \\ j & 1 & i \\ i & j & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Remark 6.1. $\det$ of the above matrix is 1, tr of it being 3.

We then compute

$$\begin{aligned} X^3+Y^3+Z^3-3XYZ &= (x+iy+jz)^3+(jx+y+iz)^3+(ix+jy+z)^3 \\ &\quad -3(x+iy+jz)\cdot(jx+y+iz)\cdot(ix+jy+z) \\ &= x^3+3x^2(yi+jz)+y^3+3y^2(zi+jx)+z^3+3z^2(xi+yj) \\ &\quad -3\{xyz+i(x^2y+y^2z+z^2x)+j(x^2z+y^2x+z^2y)\} \\ &= x^3+y^3+z^3-3xyz. \end{aligned}$$

Letting *dist*$:=x^3+y^3+z^3-3xyz$ [8] , we note that *dist* remains the same after such a transformation.

[7] For a similar transformation, see **Appendix 9.2**, in which $I_3 + J$ plays a role.

[8] *Cf.* [4].

Here we recall 'distance-preserving' examples such as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

under which $X^2 + Y^2 = x^2 + y^2$ holds and

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

under which $X^2 - Y^2 = x^2 - y^2$ holds.

Moreover, we let $\vec{a} = (x, y, z)$, $\vec{b} = (y, z, x)$, $\vec{c} = (z, y, x)$ and consider the parallelepiped, whose volume V is given by $|(\vec{a} \times \vec{b}) \cdot \vec{c}|$, where the character '$\cdot$' denotes DP . Then, $V = |((x, y, z) \times (y, z, x)) \cdot (z, x, y)| = |3xyz - x^3 - y^3 - z^3| = |-(x^3 + y^3 + z^3 - 3xyz)| = |-dist|$.

We have thus caught a glimpse of the relevance of t_r's to two-/three-dimensional spaces, even if it is coincidence. This prompts us to seek for their physical significance.

7 Physical application(s) of t_r's: when CP is associative wrt multiplication

We have dealt with SU(3), a Lie group , whereas its corresponding Lie algebra is $\mathfrak{su}(3)$. Since $\mathbb{R}^3$ equipped with the Lie bracket given by CP, which we encountered in the previous section, is one of the examples of such algebras [5], we examine whether t_r's have something to do with CP.

7.1 Checking whether t_r's are associative wrt multiplication

First, we check if t_r's are associative wrt multiplication. Let

$$\begin{cases} t_{r1} = a_1 + b_1 i + c_1 j, \\ t_{r2} = a_2 + b_2 i + c_2 j, \\ t_{r3} = a_3 + b_3 i + c_3 j, \end{cases}$$

where a_i, b_i, $c_i \in \mathbb{R}$ with $i \in \mathbb{N}$ and $1 \leq i \leq 3$. Next, we compute

$$\begin{aligned} (t_{r1} \cdot t_{r2}) \cdot t_{r3} &= \{(a_1 + b_1 i + c_1 j) \cdot (a_2 + b_2 i + c_2 j)\} \cdot (a_3 + b_3 i + c_3 j) \\ &= \{a_1 a_2 + (a_1 b_2 + a_2 b_1) i + (a_1 c_2 + a_2 c_1) j\} \cdot (a_3 + b_3 i + c_3 j) \\ &= a_1 a_2 a_3 + (a_1 a_2 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2) i + (a_1 a_2 c_3 + a_2 a_3 c_1 + a_3 a_1 c_2) j. \end{aligned}$$

Likewise, we compute

$$\begin{aligned} t_{r1} \cdot (t_{r2} \cdot t_{r3}) &= (a_1 + b_1 i + c_1 j) \cdot \{(a_2 + b_2 i + c_2 j) \cdot (a_3 + b_3 i + c_3 j)\} \\ &= (a_1 + b_1 i + c_1 j) \cdot \{a_2 a_3 + (a_2 b_3 + a_3 b_2) i + (a_2 c_3 + a_3 c_2) j\} \\ &= a_1 a_2 a_3 + (a_1 a_2 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2) i + (a_1 a_2 c_3 + a_2 a_3 c_1 + a_3 a_1 c_2) j. \end{aligned}$$

So we have

$$(t_{r1} \cdot t_{r2}) \cdot t_{r3} = t_{r1} \cdot (t_{r2} \cdot t_{r3}),$$

which means that t_r's are associative wrt multiplication.

7.2 By the way, is CP *always* non-associative wrt multiplication?: getting a non-example

Since in the preceding subsection, t_r's have been shown to be associative wrt multiplication, we search for a case in which associativity and non-associativity wrt multiplication coexist [6]. For example, it is known that three-dimensional Euclidean space equipped with CP operation exemplifies a non-associative algebra . If we interpret the adjective 'non-associative' as "not necessarily associative" , we literally come across the coexistence of associativity and non-associativity. So we 'poke around' in CP for a while, raising a (naive) question about the vectors u, v, w in $\mathbb{R}^3$.

Question 7.2.1. Perchance $u \times (v \times w)$ equals $(u \times v) \times w$?

Even intuitively, one can present the following, answering in the affirmative.

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Indeed,

$$u \times (v \times w) = (u \times v) \times w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So

Answer 7.2.2. Yes, at least in a certain case.

With the above non-example [9] , we will be conscious of this kind of subtlety for a while.

7.3 Other non-examples

Though presenting just one non-example has proven sufficient for answering *Question* 7.2.1, we show some more.

Non-example 7.3.1. $u = (1, -1, 0)$, $v = (0, 0, 1)$, and $w = (1, 1, 0)$.

Remark 7.3.2. In the above non-example, $v \times w = (-1, 1, 0) = -u$, and $u \times v = (-1, -1, 0) = -w$.

[9] By 'non-example', we mean an example in which CP shows multiplicative associativity like the vectors u, v, and w.

So we compute $u \times (v \times w) = \ u \times (-u) \ = -(-u \times u) \ = u \times u$. Recalling the formula

$$\mathbf{a} \times \mathbf{a} = \mathbf{0} \,, \tag{20}$$

we get $u \times (v \times w) = \mathbf{0}$. Likewise, we get $(u \times v) \times w = \mathbf{0}$. So we can say $u \times (v \times w) = (u \times v) \times w$, that is, u, v, w are associative wrt multiplication. Explicitly, we compute $u \times (v \times w) = (1, -1, 0) \times ((0, 0, 1) \times (1, 1, 0)) = (1, -1, 0) \times (-1, 1, 0) = (0, 0, 0)$, and $(u \times v) \times w = ((1, -1, 0) \times (0, 0, 1)) \times (1, 1, 0) = (-1, -1, 0) \times (1, 1, 0) = (0, 0, 0)$. In any event, we have shown that u, v, w are associative wrt multiplication again.

Remark 7.3.3. We notice that $u \perp v$, $v \perp w$, and $w \perp u$, since the DP's of u and v, v and w, and w and u are all 0.

Although we obtained another non-example, we ended up with $\vec{0}$ again. And some might find those examples 'trivial', just because they are $\vec{0}$'s. For those who are fond of something nonzero, we introduce the following.

Definition 7.3.4. When VTP amounts to $\vec{0}$, we call it 'trivial'.

Example 7.3.5. $u \times (v \times w)$ is called 'trivial', since it equals $\vec{0}$.

By the way, if *Non-example* 7.3.1 seems to have come out of nowhere, looking too intuitive, we refer to a known identity

$$u \times (v \times w) - (u \times v) \times w = (w \times u) \times v. \text{ [7]} \tag{21}$$

This seems to say u, v, w are non-associative wrt multiplication, since unless its RHS amounts to $\mathbf{0}$, we have $u \times (v \times w) \neq (u \times v) \times w$. However, we would like to raise another (naive) question.

Question 7.3.6. What if the RHS of (21) amounts to $\mathbf{0}$?

This question can be answered easily:

Answer 7.3.7. If it equals $\mathbf{0}$, one immediately gets $u \times (v \times w) = (u \times v) \times w$, which means that multiplication of u, v, w is associative.

Notation 7.3.8. In what follows, we write *e.g.*, $\vec{u}$ for u to differentiate vectors from scalars.

Trying to get yet another non-example, we make the following claim and prove it.

Claim 7.3.9. If we have $\vec{w} = k\vec{u}$, where $k \in \mathbb{R}$, in (21), then $\vec{u}$, $\vec{v}$, and $\vec{w}$ are associative wrt multiplication.

Proof. Since $\vec{w} = k\vec{u}$, the RHS of (21) becomes $(k\vec{u} \times \vec{u}) \times \vec{v} = k(\vec{u} \times \vec{u}) \times \vec{v} = k \cdot \vec{0} \times \vec{v} = \vec{0}$. Hence, we have $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$, which means that $\vec{u}, \vec{v}, \vec{w}$ are associative wrt multiplication.

□

With this proven claim, we present yet another non-example:

Non-example 7.3.10. $\vec{u} = (1, 1, 1)$, $\vec{v} = (1, 2, 0)$, $\vec{w} = (2, 2, 2)$.

In the above non-example, we note $\vec{w} = 2\vec{u}$, which reflects *Claim* 7.3.9. Then, we explicitly compute $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (1, 1, 1) \times ((1, 2, 0) \times (2, 2, 2)) = (1, 1, 1) \times (4, -2, -2) = (0, 6, -6)$, and $(\vec{u} \cdot \vec{v}) \cdot \vec{w} = ((1, 1, 1) \times (1, 2, 0)) \times (2, 2, 2) = (-2, 1, 1) \times (2, 2, 2) = (0, 6, -6)$. Since we have shown that $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}\ (= (0, 6, -6))$, we can say u, v, w are associative wrt multiplication [10] , confirming the validity of *Claim* 7.3.9.

Remark 7.3.11. $\vec{u} \not\perp \vec{v}$, $\vec{v} \not\perp \vec{w}$, and $\vec{w} \not\perp \vec{u}$, since the DP's of u and v, v and w, and w and u are all nonzero.

Remark 7.3.12. Since $\vec{u} \cdot (\vec{v} \cdot \vec{w}), (\vec{u} \cdot \vec{v}) \cdot \vec{w} \neq \vec{0}$, we call such VTP's 'nontrivial'. [11]

Having obtained some non-examples, we make preparation for dealing with things in a more general way.

Preparation 7.3.13. Writing $\vec{u} = (a_1, a_2, a_3)$, $\vec{v} = (b_1, b_2, b_3)$, and $\vec{w} = (c_1, c_2, c_3)$, we compute

$$\begin{aligned}\vec{u} \times (\vec{v} \times \vec{w}) &= (a_1, a_2, a_3) \times ((b_1, b_2, b_3) \times (c_1, c_2, c_3)) \\ &= (a_1, a_2, a_3) \times (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1) \\ &= (a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3, \\ &\quad a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1, \\ &\quad a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2),\end{aligned} \tag{22}$$

and

$$\begin{aligned}(\vec{u} \times \vec{v}) \times \vec{w} &= ((a_1, a_2, a_3) \times (b_1, b_2, b_3)) \times (c_1, c_2, c_3) \\ &= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \times (c_1, c_2, c_3) \\ &= (a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2, \\ &\quad a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3, \\ &\quad a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1).\end{aligned} \tag{23}$$

Equating (22) with (23), one gets

[10] *Cf. Non-example* 7.3.1.
[11] See *Def.* 7.3.4.

$$\begin{cases} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 = a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2, \\ a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1 = a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3, \\ a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 = a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1, \end{cases}$$

which simplify to

$$\begin{cases} c_1(a_2b_2 + a_3b_3) = a_1(b_3c_3 + b_2c_2), & (24) \\ c_2(a_3b_3 + a_1b_1) = a_2(b_1c_1 + b_3c_3), & (25) \\ c_3(a_1b_1 + a_2b_2) = a_3(b_2c_2 + b_1c_1). & (26) \end{cases}$$

Finishing preparation and regarding $(24) - (26)$ as Diophantine equations , we perform a Ruby search to get

Non-example 7.3.14. $\vec{u} = (5, 2, -2)$, $\vec{v} = (2, -2, 3)$, $\vec{w} = (4, 1, -2)$. [12]

Using the above non-example, we compute $\vec{u} \times (\vec{v} \times \vec{w}) = (5, 2, -2) \times ((2, -2, 3) \times (4, 1, -2)) = (5, 2, -2) \times (1, 16, 10) = (52, -52, 78)$ and $(\vec{u} \times \vec{v}) \times \vec{w} = ((5, 2, -2) \times (2, -2, 3)) \times (4, 1, -2) = (2, -19, -14) \times (4, 1, -2) = (52, -52, 78)$, confirming that $\vec{u}, \vec{v}, \vec{w}$ are associative wrt multiplication.

Remark 7.3.15. $u \perp v$, $v \perp w$, and $w \not\perp u$. [13]

Remark 7.3.16. The above non-example is'nontrivial'. [14]

Through such a search, we also got something looking quite 'trivial':

Non-example 7.3.17. $\vec{u} = (-2, -2, -2)$, $\vec{v} = (-2, -2, -2)$, $\vec{w} = (-2, -2, -2)$.

In the above non-example, we note $\vec{u} = \vec{v} = \vec{w}$. From (20), it is clear that $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times \vec{0} = \vec{0}$, $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{0} \times \vec{w} = \vec{0}$. That said, we explicitly compute $\vec{u} \times (\vec{v} \times \vec{w}) = (-2, -2, -2) \times ((-2, -2, -2) \times (-2, -2, -2)) = (-2, -2, -2) \times (0, 0, 0) = (0, 0, 0)$, and $(\vec{u} \times \vec{v}) \times \vec{w} = ((-2, -2, -2) \times (-2, -2, -2)) \times (-2, -2, -2) = (0, 0, 0) \times (-2, -2, -2) = (0, 0, 0)$, just confirming that $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w} = \vec{0}$. So $\vec{u}, \vec{v}, \vec{w}$ are associative wrt multiplication.

Remark 7.3.18. $u \not\perp v$, $v \not\perp w$, and $w \not\perp u$. [15]

Remark 7.3.19. Sure enough, this non-example has proven to be 'trivial'. [16]

[12] For computational details, see **Appendix 9.4**.
[13] *Cf. Remarks* 7.3.3 and 7.3.11.
[14] See *Def.* 7.3.4.
[15] *Cf. Remarks* 7.3.3, 7.3.11, and 7.3.15.
[16] See *Def.* 7.3.4.

7.4 Wrapping up non-examples

We tabulate the non-examples we obtained in **7.3**:

Table

Non-examples [17]	$\vec{u} \perp \vec{v}$? [18]	$\vec{v} \perp \vec{w}$? [19]	$\vec{w} \perp \vec{u}$? [20]
$\vec{u} = (1, -1, 0), \vec{v} = (0, 0, 1), \vec{w} = (1, 1, 0).$	Yes	Yes	Yes
$\vec{u} = (1, 1, 1), \vec{v} = (1, 2, 0), \vec{w} = (2, 2, 2).$	No	No	No
$\vec{u} = (5, 2, -2), \vec{v} = (2, -2, 3), \vec{w} = (4, 1, -2).$	Yes	Yes	No
$\vec{u} = (-2, -2, -2), \vec{v} = (-2, -2, -2), \vec{w} = (-2, -2, -2).$	No	No	No

Table (cont'd)

$\vec{u} \times (\vec{v} \times \vec{w})$ [21]	Is $\vec{u} \times (\vec{v} \times \vec{w})$'trivial'? [22]
$(0, 0, 0)$	Yes
$(0, 6, -6)$	No
$(52, -52, 78)$	No
$(0, 0, 0)$	Yes

7.5 Representing CP by t_r's yields multiplicative associativity

Here we note if we introduce t_r's to CP computation, that is, if we rewrite *e.g.*, $\vec{u} = (a_1, a_2, a_3)$ with $a_1, a_2, a_3 \in \mathbb{R}$ as

$$\vec{u} = (a_1, a_2, a_3) \longrightarrow t_{r1} = a_1 + a_2 i + a_3 j, \tag{27}$$

we can make our computation associative because of the multiplicative associativity shown in **7.1** More concretely, we make further replacement:

$$\begin{cases} \vec{v} = (b_1, b_2, b_3) \longrightarrow t_{r2} = b_1 + b_2 i + b_3 j, \\ \vec{w} = (c_1, c_2, c_3) \longrightarrow t_{r3} = c_1 + c_2 i + c_3 j, \end{cases}$$

where $b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$, and

$$\begin{cases} (\vec{u} \times \vec{v}) \times \vec{w} \longrightarrow (t_{r1} \cdot t_{r2}) \cdot t_{r3}, \\ \vec{u} \times (\vec{v} \times \vec{w}) \longrightarrow t_{r1} \cdot (t_{r2} \cdot t_{r3}). \end{cases}$$

[17] On the other hand, examples, in which multiplicative associativity doesn't hold, include $\vec{u} = (1, 3, 0)$, $\vec{v} = (-4, 5, 1)$, $\vec{w} = (0, -1, 0)$; $\vec{u} = (1, -2, 3)$, $\vec{v} = (-1, 4, 5)$, $\vec{w} = (0, 1, 3)$, etc.

[18] If DP of $\vec{u}$ and $\vec{v}$ equals 0, 'Yes'. Otherwise, 'No'.

[19] If DP of $\vec{v}$ and $\vec{w}$ equals 0,'Yes'. Otherwise, 'No'.

[20] If DP of $\vec{w}$ and $\vec{u}$ equals 0,'Yes'. Otherwise, 'No'.

[21] We have $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$, because we collected such non-examples.

[22] Ditto.

Then, as we have already done in **7.1**, we get

$$\begin{aligned}(\vec{u}\times\vec{v})\times\vec{w} &= (t_{r1}\cdot t_{r2})\cdot t_{r3}\\ &= a_1a_2a_3+(a_1a_2b_3+a_1a_3b_2+a_2a_3b_1)i+(a_1a_2c_3+a_1a_3c_2+a_2a_3c_1)j,\end{aligned} \tag{28}$$

and

$$\begin{aligned}\vec{u}\times(\vec{v}\times\vec{w}) &= t_{r1}\cdot(t_{r2}\cdot t_{r3})\\ &= a_1a_2a_3+(a_1a_2b_3+a_1a_3b_2+a_2a_3b_1)i+(a_1a_2c_3+a_1a_3c_2+a_2a_3c_1)j.\end{aligned} \tag{29}$$

Since $(28) = (29)$, one can say the equation $(\vec{u}\times\vec{v})\times\vec{w} = \vec{u}\times(\vec{v}\times\vec{w})$ *always* holds, which means that a 'trinionic' representation of CP has 'effaced' the subtlety of which we got conscious in **7.2**. This seems significant in terms of mathematical clarity. What about *physical* side, then? By 'reversing' [23] (28) and/or (29), one gets the vector

$$(a_1a_2a_3, a_1a_2b_3+a_1a_3b_2+a_2a_3b_1, a_1a_2c_3+a_1a_3c_2+a_2a_3c_1), \tag{30}$$

which seems different from (22) and/or (23). Thus, one might imagine physical contents the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ originally entailed have been entirely changed by 'trinionic' replacement. However, VTP *can* actually be immutable after such replacement [24]. Therefore, our response to [3, *Question* 2.1.5] is

Answer 7.5.1. Maybe.

8 Discussion

We would like to discuss the results we have obtained mainly from a physical point of view. First, a t_r-related transformation was shown to 'preserve *dist*':$=x^3+y^3+z^3-3xyz$. If one is allowed to draw a (rough) parallel between such '*dist*-preservation' and invariance of arclength under coordinate transformations , one can say t_r's are related to physics, recalling the relevance of arclength to physics [25] . As for a 'trinionic' representation of CP, since formulae comprising CP's are known to be very useful in simplifying vector calculations in physics , it is likely that such a representation has something to do with physics.

Next, we discuss mathematical side. From $(3)-(5)$, it is clear that (a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}), and (a_{13}, a_{23}, a_{33}) are the points on $x^2+y^2+z^2=1$. In regard to $(6)-(8)$, writing $\vec{x}=(a_{11}, a_{21}, a_{31})$, $\vec{y}=(a_{12}, a_{22}, a_{32})$, $\vec{z}=(a_{13}, a_{23}, a_{33})$, we have

$$\vec{x}\cdot\vec{y}=\vec{x}\cdot\vec{z}=\vec{y}\cdot\vec{z}=0, \tag{31}$$

[23] We mean by 'reversing' that for example, we get the vector (a_1, a_2, a_3) from $a_1+a_2i+a_3j$ the other way around. For example, 'reverse' the direction of the arrow in (27).

[24] More specifically, if we set *e.g.*, $\vec{u}=(0,0,1)$, $\vec{v}=(0,0,2)$, $\vec{w}=(0,0,3)$, we have $(22)=(23)=(30)=\vec{0}$, ending up with the same, $\vec{0}$. So 'trinionic' replacement does not always results in the change of vector.

[25] However, we are unaware whether some go so far as to remember $c^2t^2-x^2-y^2-z^2=c^2t'^2-x'^2-y'^2-z'^2$.

where the character '$\cdot$' stands for DP [8]. (31) geometrically means that the vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ intersect perpendicularly to each other at O.

We pay some attention to chemical side, for that matter. One can 'decompose' the RHS of (15) into the 'vectors' $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ reflecting (31). [26] If we regard them as primitive translation vectors used in the description of crystal structures , it seems that four copies of $(1,1,1)$, which are likewise obtained from the RHS of (16), are relevant to plane (111) in crystallography .

Taken together, the stuff we have so far discussed comes from our application of t_r's to the Lie group SU(3) [9], which is why we believe they are not without physical significance, taking into consideration the known role of SU(3) in physics . Although our arguments have been far from exhaustiveness in terms of 'trinionic' matrices, we would like to content ourselves with just one example, or *Example* 3.3, for the moment.

Acknowledgment. We would like to thank the developers of Ruby for their indirect help, which enabled us to perform the computation in **Appendix 9.4**.

References

[1] Chow, T. L., "Mathematical Methods for Physicists," Cambridge University Press 2000 p453.

[2] Morris, D., "Lie Groups and Lie Algebras," Abane & Right 2016 p50.

[3] Suzuki, K., "Sketching 'trinions' and 'heptanions'," viXra:1712.0131 [v1] .

[4] Morris, D., "Non-commutative Differentiation and the Commutator," Createspace Independent Publishing 2018 p66.

[5] Howe, M. R., "An Invitation to Representation Theory," Springer 2022 p82.

[6] Gürlebeck, K., Habetha, K., and Sprößig, W., "Holomorphic Functions in the Plane and n-dimensional Space," Birkhäuser Verlag AG 2008 p42.

[7] D'Angelo, J. P., "Linear and Complex Analysis for Applications," CRC Press 2023 p75.

[8] Voight, J., "Quaternion Algebras," Springer 2021 p28.

[9] Morris, D., "Lie Groups and Lie Algebras," Abane & Right 2016 p115.

[26] See also three vectors mentioned in **7.2**.

9 Appendix

9.1 Noting some similarity with Latin square

In brief, the Latin square

A	B	C
C	A	B
B	C	A

seems to underlie *e.g.*,

0	i	j
j	0	i
i	j	0

,

or MT-like rewriting of H. [27]

9.2 A similar transformation that 'preserves *dist*'

We can also consider the following.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & j & i \\ i & 1 & j \\ j & i & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then, we compute

$$\begin{aligned} X^3 + Y^3 + Z^3 - 3XYZ &= (x + jy + iz)^3 + (ix + y + jz)^3 + (jx + iy + z)^3 \\ &\quad -3(x + jy + iz) \cdot (ix + y + jz) \cdot (jx + iy + z) \\ &= x^3 + 3x^2(jy + iz) + y^3 + 3y^2(ix + jz) + z^3 + 3z^2(jx + iy) \\ &\quad -3\{xyz + i(xy^2 + z^2y + zx^2) + j(x^2y + y^2z + z^2x)\} \\ &= x^3 + y^3 + z^3 - 3xyz, \end{aligned}$$

[27] Unfortunately, we don't have a deep understanding about this relevance at the time of writing. . ..

which is the same as *dist* defined in Section **6**.

9.3 Matrix representation of i and j

We think of representing i and j by some matrices. Writing *e.g.*,

$$i = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad j = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

we have $I_2 i = i$, $i^2 = j^2 = ij = ji = O_2$, etc. Identifying I_2 and O_2 with 1 and 0, respectively, one sees that i and j satisfy [3, **Table 1**]. [28]

9.4 Ruby computation

We provide the Ruby code used in **7.3**: [29] , [30] , [31]

```
% zsh --version
zsh 5.9 (x86_64-redhat-linux-gnu)
% type ruby
ruby is /usr/bin/ruby
% cat --version
cat (GNU coreutils) 9.1
Copyright (C) 2022 Free Software Foundation, Inc.
% cat dio_cross_prod.rb
#!/usr/bin/ruby
eval "
a=-3
while a<=4
a +=1
b=-3
while b<=4
b +=1
c=-3
while c<=4
c +=1
d=-3
while d<=4
d +=1
e=-3
```

[28] We might discuss the case in which we deal with O_n, $n = 3, 4, 5\dots$ elsewhere.

[29] Computation is performed on 8-core AMD processors of a Fedora Linux 38 machine.

[30] For the sake of simplicity, unknowns $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ in (24) – (26) have been rewritten as `a`, `b`, `c`, `d`, `e`, `f`, `g`, `h`, `i`, respectively.

[31] ‘Raw’ output is not always kept intact. For instance, most lines following the command ‘head - -version’ have been deleted for simplicity.

```
while e<=4
e +=1
f=-3
while f<=4
f +=1
g=-3
while g<=4
g +=1
h=-3
while h<=4
h +=1
i=-3
while i<=4
i +=1
sum1=a*(e*h+f*i)
sum2=g*(b*e+c*f)
sum3=b*(d*g+f*i)
sum4=h*(a*d+c*f)
sum5=c*(d*g+e*h)
sum6=i*(a*d+b*e)
if(sum1==sum2)
if(sum3==sum4)
if(sum5==sum6)
print('(',a,',',b,',',c,'), (',d,',',e,',',f,'), (',g,',',h,',',i,')',''\n')
end end end end
end end end end
end end end end
"
```

Then, we run the above code.

```
% ruby -v
ruby 3.2.2 (2023-03-30 revision e51014f9c0) [x86_64-linux]
% ruby dio_cross_prod.rb>ruby_dio_cross_prod.txt&
```

We try groping for the 'head' of the data we obtained:

```
% head --version
head (GNU coreutils) 9.1
Copyright (C) 2022 Free Software Foundation, Inc.
% head ruby_dio_cross_prod.txt
(-2,-2,-2), (-2,-2,-2), (-2,-2,-2)
(-2,-2,-2), (-2,-2,-2), (-1,-1,-1)
(-2,-2,-2), (-2,-2,-2), (0,0,0)
(-2,-2,-2), (-2,-2,-2), (1,1,1)
(-2,-2,-2), (-2,-2,-2), (2,2,2)
(-2,-2,-2), (-2,-2,-2), (3,3,3)
(-2,-2,-2), (-2,-2,-2), (4,4,4)
(-2,-2,-2), (-2,-2,-2), (5,5,5)
(-2,-2,-2), (-2,-2,-1), (-2,-2,-2)
(-2,-2,-2), (-2,-2,-1), (-1,-1,-1)
```

We now get *Non-example* 7.3.17 from

```
(-2,-2,-2), (-2,-2,-2), (-2,-2,-2)
```

shown above.

What about the 'whole body'?

```
% wc --version
wc (GNU coreutils) 9.1
Copyright (C) 2022 Free Software Foundation, Inc.
% cat ruby_dio_cross_prod.txt|wc -l
1623962
```

This last output suggests that the numerical data we obtained are somewhat 'bulky'. However, if we manage to open ruby_dio_cross_prod.txt using *e.g.,* Emacs , we can get *Non-example* 7.3.14 as shown below. (see the highlighted line.)

```
% emacs --version
GNU Emacs 28.3
Copyright (C) 2022 Free Software Foundation, Inc.
GNU Emacs comes with ABSOLUTELY NO WARRANTY.
You may redistribute copies of GNU Emacs
under the terms of the GNU General Public License.
For more information about these matters, see the file named COPYING.
% emacs ruby_dio_cross_prod.txt
```

ruby_dio_cross_prod.txt - GNU Emacs at lochove

File Edit Options Buffers Tools Text Help

Save Undo

```
(5,2,-2), (2,-2,3), (-2,1,2)
(5,2,-2), (2,-2,3), (-2,4,4)
(5,2,-2), (2,-2,3), (-1,-1,0)
(5,2,-2), (2,-2,3), (-1,2,2)
(5,2,-2), (2,-2,3), (-1,5,4)
(5,2,-2), (2,-2,3), (0,0,0)
(5,2,-2), (2,-2,3), (0,3,2)
(5,2,-2), (2,-2,3), (1,-2,-2)
(5,2,-2), (2,-2,3), (1,1,0)
(5,2,-2), (2,-2,3), (1,4,2)
(5,2,-2), (2,-2,3), (2,-1,-2)
(5,2,-2), (2,-2,3), (2,2,0)
(5,2,-2), (2,-2,3), (2,5,2)
(5,2,-2), (2,-2,3), (3,0,-2)
(5,2,-2), (2,-2,3), (3,3,0)
(5,2,-2), (2,-2,3), (4,1,-2)
(5,2,-2), (2,-2,3), (4,4,0)
(5,2,-2), (2,-2,3), (5,2,-2)
(5,2,-2), (2,-2,3), (5,5,0)
(5,2,-2), (2,-2,4), (0,0,0)
(5,2,-2), (2,-2,4), (5,2,-2)
(5,2,-2), (2,-2,5), (0,0,0)
(5,2,-2), (2,-2,5), (5,2,-2)
(5,2,-2), (2,-1,-2), (0,0,0)
(5,2,-2), (2,-1,-2), (5,2,-2)
(5,2,-2), (2,-1,-1), (0,0,0)
(5,2,-2), (2,-1,-1), (5,2,-2)
(5,2,-2), (2,-1,0), (0,0,0)
(5,2,-2), (2,-1,0), (5,2,-2)
(5,2,-2), (2,-1,1), (0,0,0)
(5,2,-2), (2,-1,1), (5,2,-2)
(5,2,-2), (2,-1,2), (0,0,0)
(5,2,-2), (2,-1,2), (5,2,-2)
(5,2,-2), (2,-1,3), (0,0,0)
-:---  ruby_dio_cross_prod.txt    97% L1559610   (Text)
Mark set
```