Investigation on Brocard-Ramanujan Problem

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Abstract

Exploring $n! + 1 = m^2$[3] for natural number[2] solutions beyond $n = 4, 5, 7$ confirms no further solutions exist, validated by using GCD Linear Combination Theorem.

Key Words: Proof,GCD,Brocard-Ramanujan Function

1 Introduction

Brocard's problem, introduced in 1876[2] and 1885[3], seeks to determine the values of n such that $n!\ + 1$ forms a perfect square that is m^2.In 1913[4], Srinivasa Ramanujan also investigated the identical problem.Yet solutions known only for n=4,5, and 7.
In this article I have proved that for equation $n! + 1 = m^2$ where the n,m are natural number[4],except 4,5,7 no further natural number solution exist.Using a very common technique of proof in pure mathematics known as the proof by contradiction.I have proceeded in the following way to investigate the problem.First of all I have shown that $n! + 1 = m^2$[3] where n,m are natural number have solution is equivalent to there exist factor of $\frac{n!}{4}$as $\frac{n!}{4} = oe$ and $|e - o| = 1$ where o and e are odd and even natural number

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respectively.Then I have shown using GCD linear combination theorem that $|e-o|=1$ implies gcd(o,e)=1.So as the gcd(o,e)=1 becomes necessary condition for $|e-o|=1$,Then I have shown that for $n \geq 8$ as considering the $\frac{n!}{4}=oe$ and gcd(o,e)=1 implies $|e-o| \neq 1$ and as there not exist factor of $\frac{n!}{4}$ such that $\frac{n!}{4}=oe$ and $|e-o|=1$ so there is no natural number solution for $n \geq 8$ to $n!+1=m^2$ [2]where n,m are natural number which is proof for the Brocard-Ramanujan problem.

Remark: If nothing is stated about number then consider it as natural number.

2 Equivalent Statement of The Brocard-Ramanujan Problem

Lemma 1: *$n!+1=m^2$[4] where n,m are natural number if and only if $\frac{n!}{4}=k(k+1)$ where 2k=m-1,k is a natural number.*
Proof: As $n!+1=m^2$ implies $n!=m^2-1=(m-1)(m+1)$ here m-1 and m+1 are either both even or both odd but for $n \geq 2$, n! is even as the product of two odd number is odd number implies m-1 and m+1 are both even number so let 2k=m-1 implies 2(k+1)=m+1,k is a natural number implies $n!=2k.2(k+1)$ implies $\frac{n!}{4}=k(k+1)$.
For converse $\frac{n!}{4}=k(k+1)$ implies $n!=2k.2(k+1), 2k=m-1$ implies $n!=(m-1)(m+1)$ implies $n!+1=m^2$.

Remark: For the Problem $n!+1=m^2$ where n,m are integer[4]can be solved by solving for natural number,we can find negative solution as the m^2 (means as both positive and negative number square is positive)and as typically factorial is defined only for natural number,so n is a natural number,we so considering only natural number.

Lemma 2:$\frac{n!}{4}=k(k+1)$ *where n,k is a natural number if and only if* $\frac{n!}{4}=o.e$ *and* $|e-o|=1$ *where o is odd number and e is even number.*
Proof: So as in $\frac{n!}{4}=k(k+1)$, k and k+1 are consecutive positive integer so one is even then other must be odd and vice versa.Lets say o is odd number and e is even number then $\frac{n!}{4}=o.e$ and $|e-o|=1$.
For converse if e-o=1 then e=o+1 implies $\frac{n!}{4}=o(o+1)$ and if o-e=1 then o=e+1 implies $\frac{n!}{4}=e(e+1)$.

Theorem 1(Equivalent Theorem on Brocard-Ramanujan Problem): *$n!+1=m^2$ have a natural number solution exist if and only if there exists integer factor o,e of $\frac{n!}{4}$ such that $\frac{n!}{4}=oe$ and $|e-o|=1$.*
*Proof:*As from Lemma 1 $n!+1=m^2$ where n,m are [4] natural number if and only if $\frac{n!}{4}=k(k+1)$ where 2k=m-1,k is a natural number. From this lemma we conclude if there exist solution for the one then there exist for other. Similarly from lemma 2 if there exist solution for the one then there exist for other So,$n!+1=m^2$ have a integer solution if and only if there exists integer factor o,e of $\frac{n!}{4}$ such that $\frac{n!}{4}=oe$ and $|e-o|=1$.
In conclusion finding o and e such that $\frac{n!}{4}=o.e$ and $|e-o|=1$ is equivalent to finding n,m in $n!+1=m^2$ from above theorem. The solving Brocard-Ramanujan problem is equivalent to finding factor of $\frac{n!}{4}$ o and e such that $\frac{n!}{4}=oe$ and $|e-o|=1$.

Definition:

1. Odd factor set denoted by O and defined as $O=\{o:\text{o is an odd factor of }\frac{n!}{4}\}$.
2. Even factor set denoted by E and defined as $E=\{e:\text{e is an even factor of }\frac{n!}{4}\}$.
3. The Greatest Common Divisor (GCD):Largest shared divisor of two integers, dividing both without remainder, termed Greatest Common Divisor (GCD).

The GCD Linear Combination Theorem states for any non-zero integers a and b the greatest common divisor of a and b, denoted as gcd(a,b), can be expressed as linear combination with integer coefficients x and y as $ax+by$. The gcd(a,b) is the smallest positive integer that can be written in the form $ax+by$[1]. If gcd(a,b)=1 that is a and b are relatively prime then there exist integer x and y such that ax+by=1[1].

Corollary:*If $|e-o|=1$ implies gcd(e,o)=1 that is e and o are relatively prime number but converse of the statement is not true that is gcd(o,e)=1 does not implies $|e-o|=1$.*
*Proof:*As $|e-o|=1$ implies o-e=1 or e-o=1 from GCD Linear Combination Theorem implies gcd(o,e)=1. If gcd(o,e)=1 implies there exist x,y such that

ox+ey=1 as it is not necessary to x,y be 1,-1 or -1,1.

3 Brocard-Ramanujan Theorem

Yet we have shown $\frac{n!}{4} = oe$ and $|e - o| = 1$ where o $\in$ O and e $\in$ E this is equivalent to $\frac{n!}{4} = k(k+1)$ where there exist natural number n, k this is equivalent to $n! + 1 = m^2$ where there exist natural number n,m.

Lemma 3:$\frac{n!}{4}$ *= oe where o ∈ O, e ∈ E and gcd(o,e)=1 implies* $ox + ey = 1$ *there exist solution for the Brocard-Ramanujan Problem if and only if x=1,y=-1 or x=-1,y=1 that is* $|e - o| = 1$.
*Proof:*As $|e - o| = 1$ that is x=1,y=-1 and x=-1, y=1 so lemma 1 and lemma 2 implies there exist solution to $n! + 1 = m^2$.This statement is just equivalent statement of Theorem 1 with the help of the x,y as parameter in it which is related to the gcd(o, e).As there for every odd factor o and even factor e of $\frac{n!}{4}$ such that gcd(o, e)=1 but $|e - o| \neq 1$ then there is no solution.Using the GCD Linear Combination Theorem as $|e - o| = 1$ implies gcd(o,e)=1 as it is not necessary that if gcd(o,e)=1 implies $|e - o| = 1$.So o is odd factor and e is even factor of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and gcd(o, e)=1 implies $ox + ey = 1$ there exist solution for the Brocard-Ramanujan problem if and only if x=1,y=-1 or x=-1,y=1 that is $|e - o| = 1$. If *gcd*$(o, e) \neq 1$ implies $ox + ey \neq 1$ implies there not exist any integer x,y such that ox+ey=1 implies there not exist x=1,y=-1 or x=-1,y=1 as well.So, gcd(o,e)=1 is necessary condition for the existence of the natural number solution to Brocard-Ramanujan Problem.

So,check as not true for n=1,2,3 as the $\frac{n!}{4}$ is fraction so there is no existence of odd and even numberfactor a fraction. So, condition start to apply after $n \geq 4$. So we check for n=4,5,7 the condition is hold but n=6,8,9,......there not exist odd factor o and even factor e of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and $|e - o| = 1$ but there exist odd factor o and even factor e of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and gcd$(o.e)$=1. For example in case of n=6 as $\frac{6!}{4} = 1.2.3.5.6$ there not exist odd factor o and even factor e of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and $|e - o| = 1$ but there exist odd factor o and even factor e of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and gcd(o, e)=1 as $o = 5, e = 36$ and in case of n=7 as $\frac{7!}{4} = 1.2.3.5.6.7$ there exist odd factor o and even factor e of such that gcd(o, e)=1 and $|e - o| = 1$ as $o = 35, e = 36$. Similarly we can check for every integer greater than 7.But we need to prove it that for all $n \geq 8$ there not exist odd factor o and even factor e of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and $|e - o| = 1$ but there exist odd factor o and even factor e of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and gcd(o, e)=1.

Lemma 4: *The odd factor of* $\frac{n!}{4} = 1.2.3.5.6.7.8.9.....n$ *such that gcd(o,e)=1 is only consists of product of prime numbers P such that* $[\frac{n}{2}] < P \leq n$ *here [ ] denote the greatest integer function.*
Proof: Product of natural number is odd number if and only if all component of the product is odd. Product of natural number is even number if and only if at least one component is even number. Let say fac(n)=$\{x : x \text{ divides } n\}$ and if element of set is odd then it is odd factor if even then even factor,So the odd factor of $\frac{n!}{4} = 1.2.3.5.6......n$ is only formed by the odd component of this product but even factor can formed using all component if it is taken with an even component so nearly half component of the given product form odd factor that are 3,5,7,9,.... if we apply condition that gcd(o,e)=1 and $\frac{n!}{4} = oe$ as if $x < [\frac{n}{2}]$ then for all such x there exist an even component $2x < n$ as the gcd(o,e)=1 the only odd component x such that $[\frac{n}{2}] < x \leq n$ and as all $x < [\frac{n}{2}]$ are part of the even factor e such that gcd(o,e)=1 so any number not relatively prime with x such that $[\frac{n}{2}] < x \leq n$ then it is also the part of even factor e so as at most only prime number p such that $[\frac{n}{2}] < P \leq n$ as remain that contributing to the odd factor.

Definition:(Brocard Ramanujan function) *Denoted by the F(n) and defined as F(n) equal to difference of Least even factor(e) and Greatest odd factor(o) such that gcd(o,e)=1 and* $\frac{n!}{4} = oe$.

Lemma 5: *The Greatest odd factor such that gcd(o,e)=1 and* $\frac{n!}{4} = oe$ *is given as* $o = P_1.P_2.P_3.P_4.....P_q$ *and the Brocard-Ramanujan function is given as* $F(n) = \frac{n!}{4.P_1.P_2.P_3.P_4.....P_q} - P_1.P_2.P_3.P_4.....P_q$*,here* P_i *is the prime number such that* $[\frac{n}{2}] < P_i \leq n$ *here [ ] denote the greatest integer function.*
Proof: As in lemma 4 shown that odd factor is only formed from all prime P_i such that $[\frac{n}{2}] < P_i \leq n$ so the greatest odd factor of $\frac{n!}{4}$ is

$$o = P_1.P_2.P_3.P_4.....P_q.$$

where $[\frac{n}{2}] < P_i \leq n$. So the least even factor such that gcd(o,e)=1 and

$\frac{n!}{4} = oe$ is

$$e = \frac{n!}{4.P_1.P_2.P_3.P_4...P_q}.$$

So the Brocard-Ramanujan function is given as

$$F(n) = \frac{n!}{4.P_1.P_2.P_3.P_4.....P_q} - P_1.P_2.P_3.P_4.....P_q$$

here $[\frac{n}{2}] < P_i \leq n$ here [ ] denote the greatest integer function.

Lemma 6: *For natural number $n \geq 8$, $F(n) > 1$.*
Proof: As $F(n) = \frac{n!}{4.P_1.P_2.P_3.P_4.....P_q} - P_1.P_2.P_3.P_4.....P_q$ here P_i are prime number such that $[\frac{n}{2}] < P_i \leq n$ here [ ] denote the greatest integer function.

$$G(n) = \frac{F(n)}{P_1.P_2.P_3.P_4....P_q} = \frac{n!}{4.(P_1.P_2.P_3.P_4.....P_q)^2} - 1$$

As all prime P_i such that $[\frac{n}{2}] < P_i \leq n$ are relatively prime to all $x < [\frac{n}{2}]$, if we consider only relatively prime with the 6 the number of component of product $\frac{n!}{4}$ that are $[\frac{n}{2}] < x \leq n$ is $[\frac{n}{6}]$ so number of P_i is less than $[\frac{n}{6}]$ as we only consider relatively prime with the 6.The mininimum number of composite number between $[\frac{n}{2}]$ and n is $[\frac{n}{6}]$ as we are considering that the number relatively prime with 6 is a prime number as this is not always the case and as the prime number P_i are odd then then $P_i + 1$ is a even number which is between $[\frac{n}{2}]$ and n except when n is itself the largest prime as we are considering the number relatively prime with 6 is a prime number but that number may not be relatively prime with other number.

$$\text{So, } G(n) = \frac{[\frac{n}{2}]!}{4}.Q - 1$$

$$\text{where } Q = \frac{P_1 + 1}{P_1}.\frac{P_2 + 1}{P_2}.......\frac{\text{Remaining Factor product}}{\text{Greatest prime factor}(P_q)}.$$

$$\text{So as } Q > 1 \text{ for } n \geq 8 \text{ so } G(n) > \frac{[\frac{n}{2}]!}{4} - 1$$

$$\text{As we can observe for } n \geq 8, G(n) > 1, \text{ So the } F(n) > P_1.P_2.P_3.P_4.....P_q$$

implies $F(n) > 1$.

Theorem 2(Brocard-Ramanujan Theorem): *The $n! + 1 = m^2$[4] have no other natural number solution other than n=4,5,7.*

Proof: As $n! + 1 = m^2$ have solution if and only if there exist o,e the factor of $\frac{n!}{4}$ such that $\frac{n!}{4} = oe$ and $|e - o| = 1$ from the equivalent theorem on Brocard-Ramanujan problem.As $|e - o| = 1$ implies gcd(o,e)=1 ,As from corollary of gcd linear combination theorem its converse is not true ,So the gcd(o,e)=1 becomes the necessary condition for the $|e - o| = 1$. So,$n! + 1 = m^2$[4] have a solution if and only if gcd(o,e)=1 and $\frac{n!}{4} = oe$ implies $|e - o| = 1$.We have check up to n=8 manually and found that condition is satisfy for n=4,5,7. For $n \geq 8$ we have defined a function known the Brocard-Ramanujan function F(n) as the difference of least even factor(e) and greatest odd factor(o) such that $\frac{n!}{4} = oe$ and gcd(o,e)=1, so $F(n) = e - o$. We have found out in the lemma 6 that for $n \geq 8$, $F(n) > 1$ implies $|e - o| > 1$ implies $|e - o| \neq 1$ implies contradicting the initial assumption that $|e - o| = 1$,so there is no natural number solution to the $n! + 1 = m^2$[4] except n=4,5,7.Hence Brocard-Ramanujan Theorem is proved.Finally we can say that n=4,5,7 are the only solution to the Brocard-Ramanujan Problem.

References

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