Understanding the Mathematics of the CMB Power Spectrum

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The CMB Power Spectrum is one of the most important concepts in Big Bang theory. However, the mathematics of power spectrum analysis are complex and less than intuitive. This paper discusses the mathematics of the power spectrum analysis for CMB, with a focus on bringing clarity to this topic. We discuss the concepts involved and provide examples of the calculations that build towards the CMB power spectrum chart. We examine topics such as the C_l angular power spectrum calculation and associated a_{lm} calculation. We discuss the Y_l^m and Y_l^{m*}spherical harmonic equations and the P_l^m Legendre functions. We provide example calculations to assist readers in understanding the mathematics behind developing the power spectrum chart.

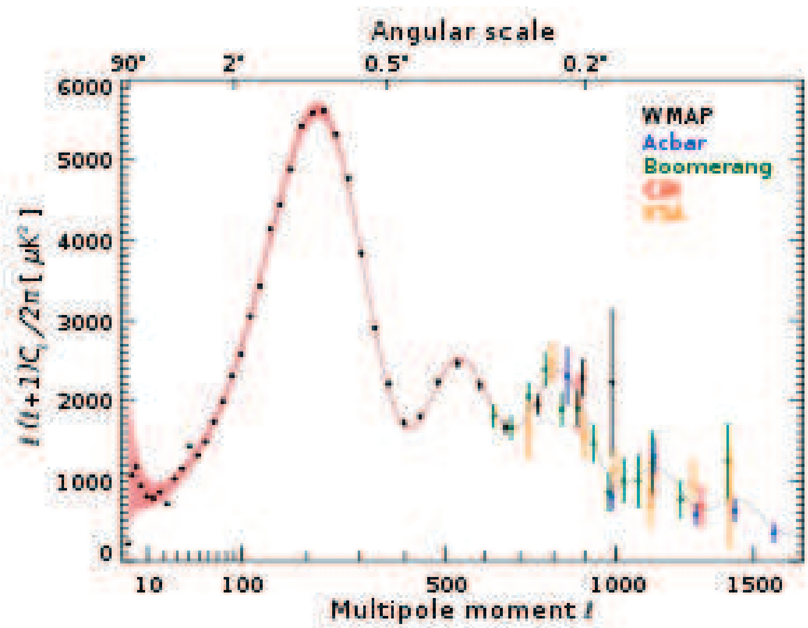
Our audience is those individuals that are interested in cosmology and the Big Bang, and who want to have a better understanding of the mathematics behind the CMB power spectrum analysis.

Understanding the Mathematics of the CMB Power Spectrum

One of the most important concepts in Big Bang theory is the CMB power spectrum. It is an analytical concept, founded on complex mathematics that are not intuitive. This article will attempt to shed some light on the mathematics of the power spectrum for CMB.

Let's start with what the CMB power spectrum represents. The CMB power spectrum is an analysis of the CMB temperature data. It looks at the temperature anisotropies, examining the temperature fluctuations across progressively smaller regions on the surface of last scattering sphere. The familiar power spectrum chart below is a plot of this analysis. Looking at this chart, we observe that the x-axis is labelled Multipole moment l. The parameter l is associated with the size of the regions being examined, with the region size inversely proportional to the value of l, although the relationship is not precise. Note that l has a positive integer value.

The power spectrum in the chart has data points for progressively larger values of l . Note that the x-axis is presented in log form. The y axis is the measured angular power spectrum for a given value of l. This is designated as C_l. The actual scale in the chart is $\frac{l(l+1)}{2\pi}C_l$. You should also note that C_l is measured in $(\mu K)^2$ or micro-Kelvin degrees, squared.



[1]

CMB Power Spectrum

[1] Source – Wikipedia Cosmic microwave background article

Spherical Harmonics

As you read the literature about the CMB power spectrum, you will see references to spherical harmonics analysis. We will expand on this topic below, but for now, the format of the spherical harmonic equation is.

$$Y_l^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}$$

$Y_l^m(\theta,\varphi)$ is used in conjunction with a function that maps values on a sphere. Spherical harmonics is a form of data analysis that is analogous to Fourier analysis. If we were examining a musical tone, for example, we would be able to use Fourier analysis techniques to determine the specific notes that are in a musical chord, and we could also determine the strength of the individual notes.

Spherical harmonics analysis examines data mapped on a sphere by an associated function of two variables, $f(\theta,\varphi)$. In the CMB analysis, spherical harmonics techniques are used to identify the region sizes that exhibit the greatest fluctuation in temperature. Spherical harmonic analysis examines the impact of increasing values of l, with the sphere divided into increasingly smaller regions.

The l and m subscripts (superscripts) of $Y_l^m(\theta,\varphi)$ are integer values that define the region size that is examined. All values of l with their associated m values are calculated to produce the power spectrum value associated with a particular l . We note that in Fourier analysis there is an equivalent m parameter as in spherical harmonics, but it can be any value. In spherical harmonics analysis m is an integer value, and this difference leads to the concept of regions in spherical harmonic analysis. The spherical harmonic equation generates complex numbers, with a complex number for each unique (θ,φ) and (l,m).

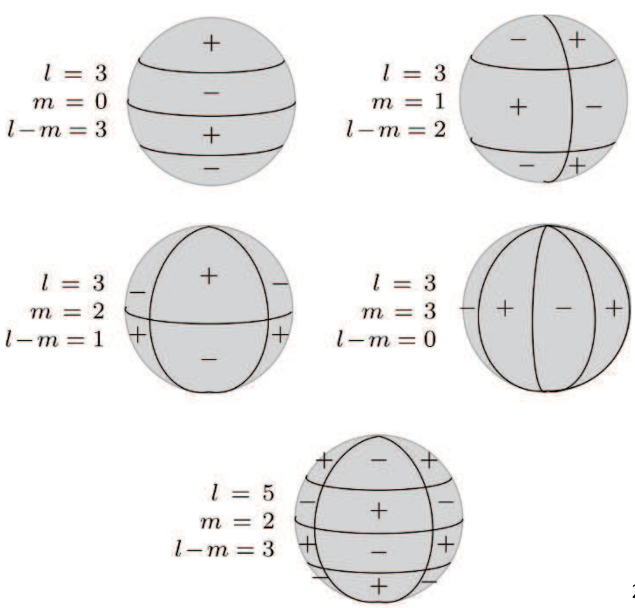
$Y_l^m(\theta,\varphi)$ has a scalar component $\left((-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\right)$, a Legendre $P_l^m(\cos\theta)$ function, and a complex exponentiation component $e^{im\varphi}$. The $e^{im\varphi}$component is a complex number in the format $cosm\varphi$ + $isinm\varphi$.

$P_l^m(\cos(\theta)$ is a Legendre function in terms of $cos\theta\ and\ sin\theta$. Adrien-Marie Legendre was a French mathematician who lived from 1752-1833. He was a contemporary of Piere-Simon Laplace and Joseph Fourier. He developed Legendre polynomials as a two variable solution that is similar to a Taylor series expansion. Assuming the two variables are θ and φ, Legendre polynomials allow for the creation of an infinite series of terms with coefficients in θ and powers in φ. In subsequent work he expanded his Legendre polynomial concept into Legendre functions. Much later, Lord Kelvin utilized Legendre functions in the development of the spherical harmonics' equation.

Appendix A discusses Legendre polynomials and Legendre functions.   There are well defined Legendre functions for each l, m combination. Examples of Legendre functions are listed, and we will use these functions in our calculations. The combination of Legendre functions in one variable (θ) and the complex exponentiation in another (φ) provides for separation of variables, simplifying calculations. In spherical harmonic calculations, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$.

The l parameter in spherical harmonics is a positive integer that represents the total number of nodes for a particular analysis. Nodes divide the sphere into different regions. A node is either a longitudinal line, or a latitudinal line on the sphere. l is the total count of nodes for a particular calculation. m is a subset of l, and so $-l \leq m \leq l$. m is the number of longitudinal nodes, and $l - m$ is the number of latitudinal nodes. For example, if $l = 2,\ m = 1$ the sphere is divided by 1 longitudinal line through the poles, and 1 latitudinal line through the equator, for a total of 2 nodal lines that divide the sphere into 4 regions. If $l = 3, m = 2$ then the sphere has 2 longitudinal nodes, and 1 latitudinal node, dividing the sphere into 8 regions. Note that m can have a negative value. If $m = -2$ for example, there are still 2 longitudinal nodes, but the φ computation orientation switches from counterclockwise to clockwise.

The graphic below illustrates how l and m values divide the sphere into regions for analysis. Points on a nodal band have $Y_l^m(\theta, \varphi)$ values of zero. In the illustration, points in + regions have positive values, and in – regions have negative values. You may also see illustrations of this concept with regions of various colors, with red indicating positive values and blue indicating negative values.



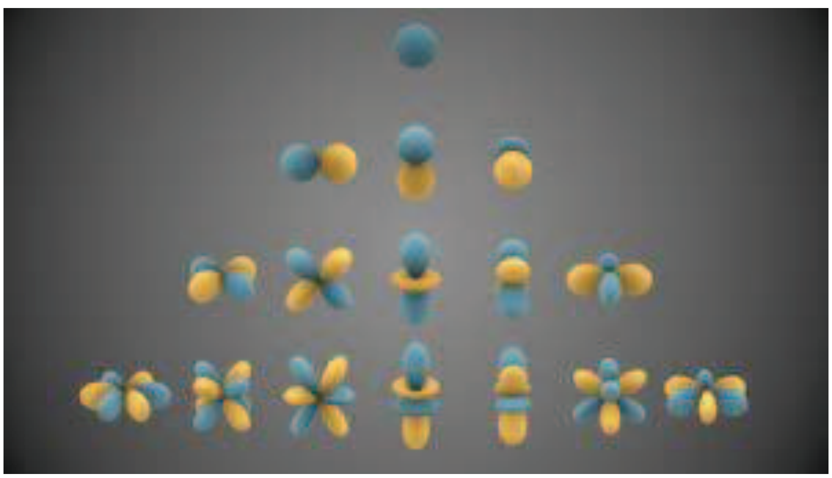
[2]

[2] Source Wikipedia Spherical Harmonics article

Spherical Harmonics – Nodal Bands

Nodes are bands with zero values for either the P_l^m component or the $e^{im\varphi}$ component. For example, if $l = 3, m = 2$ we have 2 longitudinal nodes and 1 latitudinal node. Since $P_3^2(cos\theta) = 15sin^2\theta cos\theta$ (from Appendix A) we have a zero value latitudinal band at $\theta = 90$. We also see that we have two longitudinal bands for $(cos2\varphi \pm isin2\varphi)$ at either $\varphi = 45° \; or \; 135°$ for the real component, or at $\varphi = 90° \; or \; 180°$ for the imaginary component. If $l = 4, m = 2$ then $P_4^2(cos\theta) = 15/2sin^2(7cos^2\theta - 1)$ (from Appendix A) and we have 2 longitudinal nodal bands for the θ values that are the solutions for $7cos^2\theta - 1 = 0$, and we again have 2 longitudinal bands for $(cos2\varphi \pm isin2\varphi)$ at either $\varphi = 45° \; or \; 135°$ for the real component, or at $\varphi = 90° \; or \; 180°$ for the imaginary component.

A common alternate depiction of spherical harmonics is the following set of images. We include this for your information but have focused on the previous images which we believe are more intuitive. The two sets are equivalent.



Visual representations of the first few real spherical harmonics.

Blue portions represent regions where the function is positive,

and yellow portions represent where it is negative. [3]

[3] Source – Wikipedia Spherical Harmonics article

The Mathematics of Power Spectrum Analysis

We will approach this discussion by starting with the above Power Spectrum chart and working from there to provide insight into the mathematics, one layer at a time. C_l is computed for each integer value of l. C_l is referred to as the angular power spectrum. $C_l = \langle |a_{lm}|^2 \rangle = \frac{1}{l}(\sum_{m=-l}^{l} |a_{lm}|^2)$ where m is also an integer, ranging from $-l \leq m \leq l$. To get a particular C_l you would compute $(2l+1)$ a_{lm} equations, one for each value of m in the range, and then take the average of the squared absolute values of each computed a_{lm}.

The a_{lm} function measures the amplitude of the temperature fluctuations. It is used to determine the region sizes that produce the greatest fluctuation in amplitude. In the power spectrum above we see peaks in amplitude at approximately $l = 220, 550, \text{and } 850$.

In the literature, a_{lm} is defined as $a_{lm} = \int \frac{\Delta T}{T}(\theta,\varphi) Y_l^{m*}(\theta,\varphi) sin\theta d\theta d\varphi$. Here (θ,φ) defines a point on the sphere that is the surface of last scattering. $\frac{\Delta T}{T}(\theta,\varphi)$ is the temperature function, with a value at (θ,φ) that equals the difference between the CMB temperature at the point (θ,φ), and the CMB average temperature of 2.7255 K, with this value divided by the CMB average temperature of 2.7255 K. CMB temperatures deviate is in a very narrow range of $\pm .0006°$ K from the average CMB temperature od $2.7255°$ K. a_{lm} is the amplitude of the temperature variations for a particular l, m pair, and C_l is the average of the squares of the absolute values of all the a_{lm} calculations for a specific l.

Spherical Harmonic Equation Conjugate form

Note that the a_{lm} equation above includes $Y_l^{m*}(\theta,\varphi)$. This is referred to as the conjugate form of $Y_l^m(\theta,\varphi)$. In the literature you will see $Y_l^{m*}(\theta,\varphi)$ equated with $(-1)^m Y_l^{-m}(\theta,\varphi)$ and so if

$$Y_l^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos(\theta)\, e^{im\varphi} \text{ then}$$

$$Y_l^{m*}(\theta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}} P_l^{-m}(\cos(\theta)\, e^{-im\varphi} = (-1)^m Y_l^{-m}(\theta,\varphi)$$

The conjugate format has several differences from the basic spherical harmonic equation. Note that the factorial term is inverted. P_l^m becomes P_l^{-m}. We also note that $e^{im\varphi}$ becomes $e^{-im\varphi}$, changing the orientation of the φ calculation from counterclockwise to clockwise.

As an example, $Y_3^{2*}(\theta,\varphi) = (-1)^2 \sqrt{\frac{7}{4\pi}\frac{5!}{1!}}\, P_3^{-2}(\cos\theta) e^{-2im}$

$$= (-1)^2 \sqrt{\frac{7}{4\pi}\frac{5!}{1!}} 1/8\, sin^2\theta cos\theta (cos2\varphi - isin2\varphi)$$

$$= \sqrt{\frac{105}{32\pi}}\; sin^2\theta cos\theta (cos2\varphi - isin2\varphi)$$

Calculating a_{lm}

As stated above, $a_{lm} = \int \frac{\Delta T}{T}(\theta,\varphi) Y_l^{m*}(\theta,\varphi) sin\theta d\theta d\varphi$.

In the literature you will see $\frac{\Delta T}{T}(\theta,\varphi)$ defined as $\frac{\Delta T}{T}(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\, Y_l^m(\theta,\varphi)$

and that $a_{lm} = \int \frac{\Delta T}{T}(\theta,\varphi) Y_l^{m*}(\theta,\varphi) sin\theta d\theta d\varphi$.

We see this as a circular definition, for which we have no explanation. You will also see $\frac{\Delta T}{T}(\theta,\varphi)$ defined as the variance from the CMB average temperature for the point (θ,φ) on the surface of last scattering temperature, divided by the average CMB temperature.

To illustrate the a_{lm} calculation we will propose a temperature function for $\frac{\Delta T}{T}(\theta,\varphi)$. The CMB temperature data set can be viewed as an isotropic Gaussian random distribution. Additionally, we note that $\int \frac{\Delta T}{T}(\theta,\varphi) Y_l^{m*}(\theta,\varphi) sin\theta d\theta d\varphi = 0$, unless the temperature function contains a φ factor that prevents this from happening. This is true since $\int_0^{2\pi}\left(\cos(m\varphi) - isin(m\varphi)\right) d\varphi = 0$ for all integer values of m. Therefore we must have a φ based component.

Considering the above comments, we propose that we set the temperature equation to

$$\frac{\Delta T}{T}(\theta,\varphi) = \frac{.0002°}{2.7255} \cos\left(\frac{\pi\theta}{2}\right) sin^3(\pi\varphi).$$

This honors the temperature deviation range, and emphasizes differences close to the average. This equation also leads to integrals for θ and φ that are non-zero. This allows us to see the full impact of the spherical harmonic calculation. We simplify the coefficients and use $7.3381\text{x}10^{-5} \cos\left(\frac{\pi\theta}{2}\right) sin^3\pi\varphi$ as our temperature equation.

We will show the calculation for several of the m values and then provide a table that summarizes the full set of calculations for C_3. We emphasize that this equation is used solely to facilitate the calculation of a sample angular power spectrum value, and in no way do we imply that this is the "correct" temperature function for CMB.

C_3.

We will calculate C_3. Since $C_l = \langle |a_{lm}| \rangle^2 = \frac{1}{l}\left(\sum_{m=-l}^{l} |a_{lm}|^2\right)$ we will need to calculate $a_{3,3}$, $a_{3,2}$, $a_{3,1}$, $a_{3,0}$, $a_{3,-1}$, $a_{3,-2}$, and $a_{3,-3}$

$$a_{3m} = \int \frac{\Delta T}{T}(\theta,\varphi) Y_3^{m*}(\theta,\varphi) sin\theta d\theta d\varphi =$$

$$= \int 7.3381\text{x}10^{-5} \cos\left(\frac{\pi\theta}{2}\right) sin^3(\pi\varphi)(-1)^m \sqrt{\frac{7}{4\pi}\frac{(3+m)!}{(3-m)!}}\, P_3^{-m} e^{-im\varphi} sin\theta d\theta d\varphi$$

This integral can be rearranged as a scalar component outside the integral, an integral of the θ components, and an integral of the φ components. We note that we have used an integral calculator for the integration calculations.

We define the scalar component as the product of $\sqrt{\frac{7}{4\pi}\frac{(3+m)!}{(3-m)!}}$, the co-efficient $7.3381\text{x}10^{-5}$ from the temperature equation, and the co-efficient of the P_l^m Legendre function.

Calculating $a_{3,0}$, $a_{3,2}$ and $a_{3,-2}$

We will calculate $a_{3,0}$, $a_{3,2}$ and $a_{3,-2}$ to illustrate the mathematics involved in spherical harmonics.

In the case where $l = 3\ and\ m = 0$ we have

$a_{30} = \int \frac{\Delta T}{T}(\theta,\varphi)Y_3^{0^*}(\theta,\varphi)sin\theta d\theta d\varphi$ and so

$$a_{3,0} = \int_0^{2\pi}\int_0^{\pi} 7.3381\text{x}10^{-5}\cos\left(\frac{\pi\theta}{2}\right) sin^3(\pi\varphi)(-1)^0\sqrt{\frac{7}{4\pi}}(1/2(5\cos^3\theta - 3\cos\theta))\ sin\theta d\theta d\varphi$$

$$= .7.3381\text{x}10^{-5}\sqrt{\frac{7}{4\pi}\frac{1}{2}}\int_0^{2\pi}\int_0^{\pi}\left(\cos\left(\frac{\pi\theta}{2}\right)\right)(5\cos^3\theta - 3\cos\theta)(\ sin\theta)(sin^3(\pi\varphi)\ d\theta d\varphi$$

Calculating the scalar component, the integral of the θ components, and the integral of the φ components, gives us.

$$7.3381\text{x}10^{-5}\sqrt{\frac{7}{4\pi}\frac{1}{2}} = .0000274$$

$$\int_0^{\pi}(\ \cos\left(\frac{\pi\theta}{2}\right))(5\cos^3\theta - 3\cos\theta)(\ sin\theta)d\theta = -.1103$$

$$\int_0^{2\pi} sin^3(\pi\varphi)\ d\varphi = .0383$$

Consequently $a_{3,0}$ = .0000274 x -.1103 x .0383 = -.00000012 or .12 in μKelvin and $|a_{30}|^2$ = .01

$a_{3,2}$

For the case where $l = 3\ and\ m = 2$ we have

$$a_{32} = \int_0^{2\pi}\int_0^{\pi} 7.3381\text{x}10^{-5}\ (\cos\frac{\pi\theta}{2})(sin^3(\pi\varphi))\ (-1)^2\sqrt{\frac{7}{4\pi}\frac{5!}{1!}}(\frac{1}{8}sin^2\theta cos\theta)\ (\cos(2\varphi) - isin(2\varphi))\ sin\theta d\theta d\varphi$$

$$= 7.3381\text{x}10^{-5}\sqrt{\frac{7}{4\pi}\frac{5!}{1!}\frac{1}{8}}\int_0^{2\pi}\int_0^{\pi}(\cos\frac{\pi\theta}{2})(sin^3\theta cos\theta)(sin^3(\pi\varphi))(\cos(2\varphi) - isin(2\varphi))\ d\theta d\varphi$$

Calculating the individual segments, we get

$$7.3381\text{x}10^{-5}\sqrt{\frac{7}{4\pi}\frac{5!}{1!}}\frac{1}{8} = .00007503$$

$$\int_0^{\pi}(\cos\frac{\pi\theta}{2})(sin^3\theta cos\theta)d\theta = .2255$$

$$\int_0^{2\pi}(sin^3(\pi\varphi))(\cos(2\varphi) - isin(2\varphi))\,d\varphi = .09615 + .1958i$$

This gives us $a_{32} = 1.627 + 3.313i$ in μKelvin and $|a_{32}|^2$ = 13.6

$a_{3,-2}$

In the case where $l = 3\ and\ m = -2$ we have

$a_{3,-2} = \int_0^{2\pi}\int_0^{\pi} 7.3381\text{x}10^{-5}\,(\cos\frac{\pi\theta}{2}sin^3(\pi\varphi))(-1)^2\sqrt{\frac{7}{4\pi}\frac{1!}{5!}}(15sin^2\theta cos\theta)\,(\cos(2\varphi) + isin(2\varphi))\,sin\theta d\theta d\varphi$ =

$$7.3381\text{x}10^{-5}\sqrt{\frac{7}{4\pi}\frac{1!}{5!}}\,15\int_0^{2\pi}\int_0^{\pi}(\cos\frac{\pi\theta}{2})(sin^3\theta cos\theta)(sin^3(\pi\varphi))\,(\cos(2\varphi) + isin(2\varphi))\,d\theta d\varphi$$

Calculating the individual segments, we get

$$7.3381\text{x}10^{-5}\sqrt{\frac{7}{4\pi}\frac{1!}{5!}}15 = .00007503$$

$$\int_0^{\pi}(\cos\frac{\pi\theta}{2})(sin^3\theta cos\theta)d\theta = .2255$$

$$\int_0^{2\pi}(sin^3(\pi\varphi))(\cos(2\varphi) + isin(2\varphi))\,d\varphi = .09615 - .1958i$$

Giving us $a_{3,-2}$ = 1.627 -3.313i in μKelvin and $|a_{3,-2}|^2$ = 13.6.

Note that the calculation of a_{32} an $a_{3,-2}$ produce the same result. This holds in all cases of $\pm m$. Also note that C_l is measured in μK or micro-Kelvin degrees squared.

Table of Values

The following table contains all the calculation results for $l = 3$, given our temperature equation of $\frac{\Delta T}{T}(\theta, \varphi) = 7.3381\text{x}10^{-5} \cos\left(\frac{\pi\theta}{2}\right) sin^3(\pi\varphi)$.

m	Scalar	θ integral	φ integral	raw a_{3m}value in μKelvin	$\lvert a_{3m}\rvert^2$	
0	.00002740	-.1103	.0383	-.1157	.01	
1	-.00002373	.3515	.04766 + .0644i	-.3975 - .5371i	.45	
-1	-.00002373	.3515	.04766 - .0644i	-.3975 + .5371i	.45	
2	.00007503	.2255	.09615 + .1958i	1.627 + 3.313i	13.62	
-2	.00007503	.2255	.09615 - .1958i	1.627 - 3.313i	13.62	
3	-.00003063	-.7185	.9476 + 2.006i	+20.854 + 44.14i	2383.2	
-3	-.00003063	-.7185	.9476 -2.0057i	+20.854 - 44.14i	2383.2	
				C3	=	685
				$l(l+1)C3/2\pi$	=	1309

C_l = $\langle\lvert a_{lm}\rvert\rangle^2$ and so $C_3 = \frac{1}{7}(2383.2 + 13.62 \ +.45 + .01 + .45 + 13.62 + 2383.2) = 685.$

Since the vertical scale of the Power Spectrum is $\frac{l(l+1)}{2\pi}C_l$ the plot value at $l = 3$ would be 1309. We also calculated the C_4 value to be 288, and that the (20/2π)C_4 plot value is 918. We see that these values are in reasonable agreement with the power spectrum chart. We leave the C_4 calculation as an exercise for our readers.

The work above has been done to show the process of calculating the CMB Power Spectrum. We reiterate that our temperature equation is not represented as an accurate CMB temperature equation, but rather is used to facilitate the above calculation process. We hope this has clarified Power Spectrum analysis.

Looking back at the Power Spectrum chart above we see peaks in amplitude at approximately $l = 220, 550, \text{and } 850$. Calculating C_l for these values would require 441, 1101, 1701 a_{lm} calculations respectively, making this a great candidate for supercomputer processing power.

Conclusion

Our goal in this paper was to clarify the mathematics behind the CMB power spectrum analysis. In the process we discussed the power spectrum chart, spherical harmonics, and the calculations of angular power spectrum values, among other topics. To facilitate our discussion, we proposed a CMB temperature equation to illustrate the calculations and we then walked through examples of the calculations.

We hope that we bring clarity to the mathematics behind the CMB power spectrum. The mathematics involved here was developed over several hundred years, with major contributions from Legendre, Fourier, Kelvin, and others. The resulting spherical harmonics analysis provides a method for analyzing data mapped on a sphere. Spherical harmonics analysis is an objective analysis that knows nothing about the source of the data being analyzed. The calculation of the power spectrum for CMB does not tell us why the power spectrum chart has the specific shape that it does. That is left to theoretical physicists who propose factors such as the curvature of space time, the percentage of baryonic matter in the universe, or the percentage of dark matter as determining factors. These factors are used in the development of models that try to replicate the shape of the raw data power spectrum curve, to understand why the power spectrum has its particular shape that it does. We have not attempted to explain the shape of the curve but have merely focused on the mathematics used to create the power spectrum chart.

We hope we have helped clarify the mathematics and concepts involved in this topic. We invite your questions and comments.

Just for Fun

The following is a whimsical 1820 watercolor portrait of French mathematicians Adrien-Marie Legendre and Joseph Fourier, created by Julien-Leopold Boilly in (1820). Perhaps we will end up looking like these two if we spend too much time trying to understand the mathematics of Cosmology.

[4]

1820 watercolor portrait of French mathematicians Adrien-Marie Legendre and Joseph Fourier

[4] Reference Boilly, Julien-Leopold. (1820). Album de 73 Portraits-Charge Aquarelle's des Membres de l'Institute (watercolor portrait #29). Biliotheque de l'Institut de France

Appendix A

Adrien-Marie Legendre developed Legendre polynomials as a two-variable solution, similar to a Taylor series expansion. Legendre polynomials allow for the creation of an infinite series of terms with coefficients in one variable, and powers in the second variable. Chapters 3-4 of the University of Guelph PHYS*3130 Mathematical Physics Lecture Notes provide an excellent introduction to Legendre polynomials.

Legendre polynomials are the coefficients mentioned above, with $cos\theta$ as the variable. They are denoted as $P_l(cos\theta)$ where l is an integer form 0 to ∞.

The first 5 Legendre polynomials in trigonometric form are.

$$P_0(cos\theta) = 1$$

$$P_1(cos\theta) = cos\theta$$

$$P_2(cos\theta) = 1/2(3cos^2\theta - 1)$$

$$P_3(cos\theta) = 1/2(5cos^2\theta - 3)$$

$$P_4(cos\theta) = 1/8(35cos^4\theta - 30cos^2\theta + 3)$$

Legendre functions are derived from Legendre polynomials. The following relationship allows for the calculation of Legendre functions.

$$P_l^m = sin^m(\theta)\frac{d^m}{d(cos\theta)^m}\ P_l(cos\theta)$$

As an example

$$P_1^1 = sin(\theta)\frac{d}{d(cos\theta)}\ P_1(cos\theta) = sin(\theta)\frac{d}{d(cos\theta)}\ cos\theta\ = sin\theta$$

$$P_2^1 = sin(\theta)\frac{d}{d(cos\theta)}\ P_2(cos\theta) = sin(\theta)\frac{d}{d(cos\theta)}\ 1/2(3cos^2\theta - \ 1) = 3sin\theta cos\theta$$

The following is a list of the Legendre functions for $l = 1\ to\ 4$

$$P_1^0(cos\theta) = \cos\theta$$

$$P_1^1(cos\theta) = \sin\theta$$

$$P_2(cos\theta) = 1/2(3\cos^2\theta - 1)$$

$$P_2^1(cos\theta) = 3sin\theta cos\theta$$

$$P_2^2(cos\theta) = 3sin^2\theta$$

$$P_3(cos\theta) = 1/2(5cos^3\theta - 3cos\theta)$$

$P_3^1(cos\theta) = 3/2sin\theta(5cos^2\theta - 1)$

$P_3^2(cos\theta) = 15sin^2\theta cos\theta$

$P_3^3(cos\theta) = 15sin^3\theta$

$P_4(cos\theta) = 1/8(35cos^4\theta - 30cos^2\theta + 3)$

$P_4^1(cos\theta) = 5/2sin\theta(7cos^3\theta - 3cos\theta)$

$P_4^2(cos\theta) = 15/2sin^2\theta(7cos^2\theta - 1)$

$P_4^3(cos\theta) = 105sin^3\theta cos\theta$

$P_4^4(cos\theta) = 105sin^4\theta$

The $P_l^{-m}(cos\theta)$ Legendre functions can be derived from the Legendre functions above using the following relationship.

$$P_l^{-m}(cos\theta) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\cos(\theta)$$

The following are the $-m$ Legendre functions for $l = 3, 4$. We will use these in our calculation examples.

$$P_3^{-1}(cos\theta) = (-1)^1 \frac{2!}{4!}\ P_3^1(cos\theta) = -\frac{2!}{4!}(3/2\sin\,(5cos^2\theta - 1) = -\frac{1}{8}\ sin\theta cos\theta$$

$$P_3^{-2}(cos\theta) = (-1)^2 \frac{1!}{5!} P_3^2(\cos(\theta) = \frac{1!}{5!} 15sin^2\theta cos\theta = 1/8\ sin^2\theta cos\theta$$

$$P_3^{-3}(cos\theta) = (-1)^3 \frac{0!}{6!} P_3^3(cos\theta) = -\frac{1}{6!} 15sin^3\theta = -\frac{1}{48}\ sin^3\theta$$

$$P_4^{-1}(cos\theta) = (-1)^1 \frac{3!}{5!} P_4^1(cos\theta) = -\frac{1}{8} sin\theta(7cos^2\theta - 3cos\theta)$$

$$P_4^{-2}(cos\theta) = (-1)^2 \frac{2!}{6!} P_4^2(cos\theta) = \frac{1}{48} sin^2\theta(7cos^2\theta - 1)$$

$$P_4^{-3}(cos\theta) = (-1)^3 \frac{1!}{7!} P_4^3(cos\theta) = -\frac{1}{48} sin^3\theta cos\theta$$

$$P_4^{-4}(cos\theta) = (-1)^4 \frac{0!}{8!} P_4^4(cos\theta) = \frac{1}{384} sin^4\theta$$