On the Nonexistence of Solutions to a Diophantine Equation Involving Prime Powers

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Abstract

This paper investigates the Diophantine equation $p^r + (p+1)^s = z^2$ Where $p > 3$, $s \geq 3$, z is an even integer. The focus of the study is to establish rigorous results concerning the existence of solutions within this specific parameter space. The main result presented in this paper demonstrates the absence of solutions under the stated conditions. The proof employs mathematical techniques to systematically address the case when the prime p exceeds 3, and the exponent s is equal to or greater than 2, while requiring the solution to conform to the constraint of an even z. This work contributes to the understanding of the solvability of the given Diophantine equation and provides valuable insights into the interplay between prime powers and the resulting solutions.

1 Introduction

The study of Diophantine equations, which seek integer solutions to polynomial expressions, has been a subject of profound mathematical exploration. In this context, the Diophantine equation $p^r + (p+1)^s = z^2$ stands as an intriguing challenge, Where $p > 3$, $s \geq 3$, and z is constrained to be an even integer. Our investigation delves into the intricacies of this specific mathematical problem, aiming to establish rigorous results regarding the existence of solutions within the defined parameter space.The primary objective of this paper is to rigorously prove the absence of solutions to the Diophantine equation under the specified conditions. By focusing on the case where the prime p exceeds 3, and the exponent b is equal to or greater than 2, we systematically address the inherent challenges presented by this particular form of the equation. The additional constraint that the solution must conform to the evenness of z further enriches the complexity of the problem.Our work contributes significantly to the broader understanding of the solubility of the given Diophantine equation. By exploring the interplay between prime powers and the resulting solutions, we aim to provide valuable insights that extend beyond the immediate scope of this specific equation. The results obtained in this paper contribute to the ongoing dialogue in Diophantine equation theory and align with recent efforts to deepen our understanding of mathematical structures. Notably, recent research has seen a surge in interest in Diophantine equations of the form $x^r + y^s = z^2$, where x, y are positive integers, and r, s , z are non-negative integers. Pioneering work by Sroysang [[6], [7],[1],[2]] has successfully solved this equation for various pairs (x,y), offering solutions for cases such as (3,5),(3,17), (3,17), (3,85), (3,45),(143,145) and others. Sroysang has also addressed the equation for the pairs (7,8) and (31,32) [[7], [5]], along with posing a conjecture related to the equation $p^r + (p+1)^s = z^2$ .Building upon this foundation, Chotchaisthit [3] made significant strides in 2013 by determining all solutions of the equation where p is the Mersenne prime, and specific forms of solutions were elucidated in [4] .

2 Theorem

Let p be a prime number. We have that
If $p > 3$, then the equation

$$p^r + (p+1)^s = z^2$$

no have solution in r, s, z with $s \geq 2$ and z even.
And
If $p > 2$, then the equation

$$p^r + (p+1)^s = z^2$$

no have solution in r, s, z if s has at least one prime factor ≥ 7
For this we consider two Cases

2.1 For case 1

If $p > 3$, then the equation

$$p^r + (p+1)^s = z^2$$

no have solution in r, s, z with $s \geq 2$ and z even.

Solution
Now consider two parts

Let part(i)

$p \equiv$1(mod3),then by considering case (1) (mod 3) we have

$$1 + 2^s \equiv z^2 \equiv 0, 1 (mod 3)$$

Let $s = 2s_1$, now take the factor of

$$p^r + (p+1)^s = z^2$$

or

$$p^r = z^2 - (p+1)^s$$
$$p^r = z^2 - (p+1)^{2s_1}$$
$$p^r = (z - (p+1)^{s_1})(z + (p+1)^{s_1}).$$

so we consider two factor

$$p^u = (z - (p+1)^{s_1})......(i)$$

and

$$p^v = (z + (p+1)^{s_1})......(ii)$$

and u+v=r .By subtracting $(i) and (ii)$ we get

$$2(p+1)^{s_1} = p^v - p^u$$

Now here $v > u$ Since p does not divide 2(p + 1), then u = 0 and the equation becomes

$$2(p+1)^{s_1} = p^r - 1$$

From this relation (mod p), we obtain (By Binomial theorem use for) that $3 \equiv 0(mod p)$ and so p = 3 contradicting of this $p > 3$.
Let

$$p \equiv 2(mod 3)$$

now case (i) (mod 3) we have

$$2^r \equiv z^2 \equiv 0, 1(mod 3).$$

Let part(ii)
Let we take, $r = 2r_1$ we make the factor

$$(p+1)^s = z^2 - p^{2r_1}$$
$$(p+1)^s = (z - p^{r_1})(z + p^{r_1}).$$

We use that the gcd(m + n, m-n) divides 2 gcd(m, n) so that we take $gcd(z - p^{r_1}, z + p^{r_1}))$ divides 2 $\gcd(z, p^{r_1})$. However, that $p \neq z$ and so $gcd(z - p^{r_1}, z + p^{r_1})$=1 or 2.Since z is even and p is odd, so gcd is equal to 1. we obtain

$$z - p^{r_1} = x^s$$

and

$$z + p^{r_1} = y^s$$

where xy = p + 1 on the Diophantine Equation $p^r + (p+1)^s = z^2$ By subtracting these equations, we obtain $2p^{r_1} = y^s - x^s$ and then y and x have the same parity. On other way we get xy = p + 1 is even and where p is odd and x,y are even.In this case, the 2-adic order of $y^s - x^s$ is at least $b \geq 2$ while the 2-adic order of $2p^{r_1}$ is obviously 1 contradicting the equality $2p^{r_1} = y^s - x^s$ so no solution for this equation.

2.2 For case 2

If $p > 2$, then the equation

$$p^r + (p+1)^s = z^2$$

no have solution in r, s, z if s has at least one prime factor ≥ 7

Solution
The proof for case 2 is similar by the same steps we get that does not have solution for

$$p \equiv 1(mod 3)$$

and

$$p \equiv 2(mod 3)$$

for z is even.
If z is odd
so the relation is

$$(p+1)^s = (z^2 - p^{r_1})(z^2 + p^{r_1}).$$

for gcd $(z^2 - p^{r_1}, z^2 + p^{r_1}) = 2$
we have two way,
one way is

$$z^2 - p^{r_1} = 2x^s \ ; \ 2(z^2 + p^{r_1}) = y^s$$

2nd way is

$$2(z^2 - p^{r_1}) = x^s \ ; \ (z^2 + p^{r_1}) = 2y^s$$

where gcd(x, y) = 1 and xy = p + 1.
so we avoid unnecessary repetitions,
For the way one we get

$$z^2 - p^{r_1} = 2x^s \ and \ 2(z^2 + p^{r_1}) = y^s$$

First, we proof that x, y, z are pairwise co-prime. In fact, gcd (x, y) =1. Suppose that w is a prime number dividing x and z, then by the relation

$$z^2 - p^{r_1} = 2x^s$$

we get that gcd w=p(since w / p).we following the contradiction $p = q|x|p + 1$.Now, suppose that q is a prime number dividing y and z, then by the relation

$$2(z^2 + p^{r_1}) = y^s$$

,we get

$$q/2p^{r_1}$$

if q/p we arrive at the same contradiction as before.So q / 2 we obtain 2 = q / z which is a contradiction so z is odd. So following contradiction $p = q|x|p + 1$. By way one

$$z^2 - p^{r_1} = 2x^s \ ; \ 2(z^2 + p^{r_1}) = y^s$$

we get

$$4z^2 = y^s + 4x^s$$

and than y=2n for some positive integer n.Then, the previous way can be rewritten as

$$z^2 = x^s + 2^{s-2}n^s$$

so s prime factor $t \geq 7$.
So we written as s=tm

$$z^2 = {x_1}^t + 2^{mt-2}{n_1}^t$$

So here $x_1 = x^m$ and $n_1 = n^m$, $mt - 2 \geq t - 2 \geq 5$ this equation have no solution (Its proof depends on the Modular approach to prove the Fermat Last Theorem).

3 Theorem 2

The Diophantine equation

$$z^2 = x^p + 2^q y^p$$

does not have solution in pairwise co-prime integers x, y, z when $q \geq 4$ and $p \geq 7$ is prime. This completes our proof.

4 Conclusion

In conclusion, our investigation into the Diophantine equation $p^r + (p+1)^s = z^2$ has yielded rigorous and compelling results, particularly when considering the case where the prime p exceeds 3 and the exponent b is equal to or greater than 2, with the additional constraint of z being an even integer. By systematically addressing the complexities inherent in this specific form of the equation, we have successfully proven the absence of solutions within the defined parameter space. This work significantly contributes to the broader understanding of the solvability of the given Diophantine equation, shedding light on the intricate interplay between prime powers and resulting solutions. Our findings extend beyond the immediate scope of this specific equation, providing valuable insights that align with recent efforts to deepen our understanding of mathematical structures, particularly in the realm of Diophantine equation theory.

While recent research, exemplified by the pioneering work of Sroysang and subsequent advancements by Chotchaisthit, has successfully tackled related Diophantine equations, our paper adds a new dimension to this evolving mathematical exploration. By building upon the existing body of knowledge, our main result establishes a noteworthy contribution to the ongoing discourse surrounding Diophantine equations and their solutions. As mathematics continues to progress, our findings pave the way for further exploration and refinement of these fascinating mathematical phenomena.

References

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