Comprehensive systemization of weak KAM theory and some open problems

Jong In Jae[1*], Jong U Hwan[1], Ra Ju Gwang[1]

1-Pyongyang Information Technology University,

Pyongyang, The Democratic People's Republic of Korea

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In this paper, we performed comprehensive systemization of weak KAM theory, the ultramodern theory in mathematics domain that is being regarded as important in theoretical and application aspect and is being studied actively in the world in present. Moreover we also systemized comprehensively the conjectures, the open problems, and the point at issue that are proposed in weak KAM theory. They contain 17 of the points at the issue that are newly proposed in this paper.

Contents

1. Introduction

1.1. Aim of the paper

The the fundamental research objects of weak KAM theory are Hamilton ordinary differential equations and Hamilton-Jacobi partial differential equations. Since Hamilton ordinary differential equation equivalent to the Newton equation that is the second low of motions $ma = m\ddot{x} = F(x, \dot{x})$ they are used often in various mechnical and enginnerring problems. On the other hand, since the value functions in optimal control theory satisfy Hamilton-Jacobi equation (see [Bardi, 1997]) Hamilton-Jacobi equations are also used in various engineering problems such as optimal control problems and differential game problems. Therefore weak KAM theory that bridges between viscosity solution theory of Hamilton-Jacobi equations and qualitative theory of Hamilton equations is applied to various mechanical problems and enginnering problems. In fact weak KAM theory are applied to vasious mechanical problems and engineering problems such as large time behaviour problem of evolutionary Hamilton-Jacobi equations, the hogenization problems of Hamilton-Jacobi equations, optimal transportation problems, optimal switching problems, optimal control problems, problems related to weakly coupled system, problems related to mean field game.

To review the whole features of weak KAM theory that is being studied in various fields actively on the world scale in present is a bit difficult problem. Although many review papers related to weak KAM theory such as c[Gomes 2002], a[Evans 2004], [Evans 2005], [Kaloshin 2005], [Siconolfi 2006], [Bernard 2012], [Fathi 2012] [Siconolfi 2012], [Rifford 2013], [Fathi 2014], [Siconolfi 2016], [Sorrentino 2016], [Sorrentino-Bernardi, 2017] are proposed in the past, they review an aspect of weak KAM theory. So the comprehensive systemization of weak KAM theory has not been done yet.

Therefore we performed the comprehensive systemization of weak KAM theory for one to review the whole of fields and detailed fields of weak KAM theory. Moreover we sytemized comprehensively the conjectures, the open problems, and points at the issue proposed in detailed fields of weak KAM theory. They contain 17 problems such as problem Ⅰ.1-1, problem Ⅰ.1-2, problem Ⅰ.3-2, problem Ⅰ.4-5, problem Ⅰ.5-4, problem Ⅰ.5-5, problem Ⅰ.6-1, problem Ⅰ.6-2, problem Ⅱ.1-2, problem Ⅱ.4-1, problem Ⅱ.4-2, problem Ⅱ.4-3, problem Ⅱ.4-4, problem Ⅱ.5-1, Problem II.5-2, problem Ⅱ.7-1, problem Ⅱ.7-2 that we set up newly in this paper.

1.2. Outline of KAM theory, Aubry-Mather theory, and weak KAM theory

Assume that M is n-dimensional C^∞ compact connected manifold. T^*M denotes cotangent bundle of M and $\pi_*: T^*M \to M$ denotes standard projection. We call a continuous function $T^*M \to \mathbf{R}$ Hamiltonian on M. We denote point of T^*M by (x, p) with $x \in M$ and $p \in T_x^*M = \pi_*^{-1}(x)$. Here $T_x^*M = L(T_xM, \mathbf{R})$ denotes the linear space of linear forms $T_xM \to \mathbf{R}$.

Let us $H_0: T^*M \to \mathbf{R}; (x, p) \mapsto H_0(p)$ and $H_1: T^*M \to \mathbf{R}; (x, p) \mapsto H_1(x, p)$ are real analytic function and $\varepsilon > 0$. Then we put

$$H(x, p) = H_0(p) + \varepsilon H_1(x, p). \tag{1.0}$$

We call the Hamilton equation with Hamiltonian (1.0)

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}(x, p), \frac{dp}{dt} = -\frac{\partial H}{\partial x}(x, p) \tag{1.1}$$

the nearly integrable Hamiltonian system. If $\varepsilon = 0$, then (1.1) become

$$\frac{dx}{dt} = \frac{\partial H_0}{\partial p}(p) =: \omega(p), \frac{dp}{dt} = -\frac{\partial H_0}{\partial x}(p) = 0 \tag{1.2}$$

and therefore the solution of (1.1) satisfying initial condition $x(0) = x_0,\ p(0) = p_0$ is

$$x(t) = x_0 + \omega(p_0)t, (t \in \mathbf{R}),\ \ p(t) = p_0. \tag{1.3}$$

Thus non-perturbating system (1.2) is integrable.

If $M = \mathbf{T}^n = \mathbf{R}^n / \mathbf{Z}^n$ (n-dimensional torus), then from (1.3) any solution of (1.2) is conditionally periodic and $\mathcal{T}(p_0) = \mathbf{T}^n \times \{p_0\}$ is the invariant torus of (1.2). Then the vector $\omega(p_0) \in \mathbf{R}^n$ is called the frequency vector of the torus $\mathcal{T}(p_0)$. Since $\bigcup_{p_0 \in \mathbf{R}^n} \mathcal{T}(p_0) = \mathbf{T}^n \times \mathbf{R}^n = T^*\mathbf{T}^n$ the phase space of (1.2) $T^*\mathbf{T}^n$ is represented as non-intersection sum of invariant tori.

Initial point $(x_0, p_0) \in T^*\mathbf{T}^n$ lies on the invariant torus $\mathcal{T}(p_0) = \mathbf{T}^n \times \{p_0\}$ and from invariantness of $\mathcal{T}(p_0)$ the solution orbit $\{(x(t), p(t))\,;\ t \in \mathbf{R}\} \subset T^*\mathbf{T}^n$ of (1.2) lie on $\mathcal{T}(p_0)$. Since $\mathcal{T}(p_0)$ is compact the solution orbit $\{(x(t), p(t))\,;\ t \in \mathbf{R}\}$ is bounded. Therefore any solution of (1.2) is Lagrange stable.

Here the problem that invariant tori $\mathcal{T}(p_0), (p_0 \in \mathbf{R}^n)$ of the unperturbed Hamiltonian system for $\varepsilon = 0$ persist when $\varepsilon > 0$ is raised. Originally Poincaré himself called the problem of studying perturbations of quasi-periodic motions in nearly integrable Hamiltonian systems, the fundamental problem of dynamics([Arnold 1989], pp. 400).

The celebrated KAM theory by [Kolmogorov 1954], [Arnold 1963], [Moser 1962] clarified that under some condition if ε is sufficiently small, then almost of invariant tori $\mathcal{T}(p_0), (p_0 \in \mathbf{R}^n)$ in unperturbed persist as invariant tori with same frequency vector $\omega(p_0)$. Then the invariant tori of perturbed Hamiltonian system (1.2) are called the KAM tori.

Then the KAM theory proposed the problem that how is fate of the invariant tori when perturbation parameter becomes large. Tne Aubry-Mather theory by [Aubry-Daeron 1983], [Mather 1982] provided the answer to the problem.

Aubry-Mather theory clarified that for monotone twist area preserving diffeomorphism of two-dimensional annulus $\mathcal{A} = S^1 \times (0,1)$ (See [Katok-Hasselblatt 1995]) there exist invariant sets called Aubry-Mather sets with any rotation number and they are the graphs of a Lipsitz functions difined of the circle S^1.

On the other hand, the symplectic map that transfer the Hamiltonian to integrable Hamiltonian in KAM theory is given from the generation function which is a differentiable solution of the corresponding Hmilton-Javobi equation. Then the graph of the derivative of the solution of the Hamilton-Jacobi equation becomes the invariant torus of the Hamilton equation.

Let us explain this fact more in detai. Let us consider the Hamilton equation (1.1) with the Hamiltonian (1.0) defined on the cotangent bundle $T^*\mathbf{T}^n = \mathbf{T}^n \times \mathbf{R}^n = \{(x, p)\}$ on $M = \mathbf{T}^n$ and Hamilton-Jacobi equation for Hamilton equation (1.1)

$$H(x, \frac{\partial u}{\partial x}) = \overline{H}(P). \tag{1.4}$$

Here $\overline{H}: \mathbf{R}^n \to \mathbf{R}; P \mapsto H(P)$ is a certain C^2 function.

Assume that $u: \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{R}; (x, P) \mapsto u(x, P)$ is a C^1 solution of (1.4). Then let us compose symplectic map $\chi: \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{T}^n \times \mathbf{R}^n$; $(X, P) \mapsto \chi(X, P) = (x, p)$ based on implicite function relation

$$p = \frac{\partial u}{\partial x}(x, P),\ X = \frac{\partial u}{\partial P}(x, P). \tag{1.5}$$

For any $P \in \mathbf{R}^n$ we define $F_P: \mathbf{T}^n \to \mathbf{T}^n$ by $X = F_P(x) = \frac{\partial u}{\partial P}(x, P)$. Now we assume that for any $P \in \mathbf{R}^n$ there exists the inverse map $F_P^{-1}: \mathbf{T}^n \to \mathbf{T}^n; X \mapsto x = F_P^{-1}(X)$ of F_P. We define $\psi: \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{T}^n$ by $\psi(X, P) = F_P^{-1}(X)$ and define $\varphi: \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{R}^n$ by

$$p = \varphi(X, P) = \frac{\partial u}{\partial q}(\psi(X, P), P).$$

Next we define $\chi: \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{T}^n \times \mathbf{R}^n$ by $\chi(X, P) = (x, p) = (\psi(X, P), \varphi(X, P))$. Then χ is a symplectic map by theory of generationg function with symplectic map (see [Arnold 1989], Chapter 9, Section 47). Let us $\phi: \mathbf{T}^n \times \mathbf{R}^n \to \mathbf{T}^n \times \mathbf{R}^n$ is the inverse map of χ. Then ϕ is also a symplectic map. If $\phi(x, p) = (X, P)$, then $(X, P) = \chi^{-1}(x, p)$ and $(x, p) = \chi(X, P) = (\psi(X, P), \varphi(X, P))$.

Let us transfer Hamilton equation (1.1) by sympletic map $\phi: (x, p) \to (X, P)$. Then the Hamiltonian in new coordinate system is given by

$$K(X, P) = H(\phi^{-1}(x, p)) = H(\psi(X, P), \frac{\partial u}{\partial q}(\psi(X, P), P)) = H(x, \frac{\partial u}{\partial x}(x, P)) = \overline{H}(P).$$

Therefore Hamilton equation (1.1) is transered to

$$\frac{dX}{dt} = \frac{\partial K}{\partial P} = \frac{\partial \overline{H}}{\partial P}(P) =: \omega(P),\ \frac{dP}{dt} = -\frac{\partial K}{\partial X} = 0. \tag{1.1$'$}$$

The solution of $(1.1)'$ satisfies initial condition $X(0) = X,\ P(0) = P$ is given by

$$(\omega(P)t + X,\ P) \in \mathbf{T}^n \times \mathbf{R}^n,\ (t \in \mathbf{R}). \tag{1.6}$$

Therefore $\mathbf{T}^n \times \{P\}, (P \in \mathbf{R}^n)$ are invariant tori of $(1.1)'$. Consequently

$$\begin{aligned} \mathcal{T}(P) &:= \phi^{-1}(\mathbf{T}^n \times \{P\}) = \bigcup_{X \in \mathbf{T}^n} \{(\psi(X, P), \varphi(X, P))\} = \\ &= \bigcup_{x \in \mathbf{T}^n} \{(\psi(X, P), \frac{\partial u}{\partial q}(\psi(X, P), P))\} = \bigcup_{X \in \mathbf{T}^n} \{(F_P^{-1}(X), \frac{\partial u}{\partial q}(F_P^{-1}(X), P))\} = \\ &= \bigcup_{x \in \mathbf{T}^n} \{(x, \frac{\partial u}{\partial q}(x, P))\} = graph\, \frac{\partial u}{\partial q}(\cdot, P) \subset T^*\mathbf{T}^n,\ (P \in \mathbf{R}^n) \end{aligned} \tag{1.7}$$

are invariant tori of (1.1) and these invariant tori are the graph of the derivatives of the solution of the Hamilton-Jacobi equation (1.4).

However as it happens, it is usually impossible to find global C^1 solutions of Hamilton-Jacobi equations (see [Fathi 2012]). So [Crandall-Lions 1983] developed the theory of viscosity solution, weak solution of Hamilton-Javobi equation that is not differentiable in general. A. Fathi clarified the connection between the theory of viscosity solution of Hamilton-Javobi equation and Aubry-Mather theory in a numer of papers such as a[Fathi 1997], b[Fathi 1997], a[Fathi 1998], b[Fathi 1998] etc and initiated the so-called weak KAM theory. A. Fathi developed his theory for the Tonelli Hamiltonian systems. $H: T^*M \to \mathbf{R}$; $(x, p) \mapsto H(x, p)$ is called Tonnelli if and only if 1) H is C^2, 2) $H_{pp}(x, p)$ is definite positive as quadratic form, 3) $\lim_{\|p\| \to +\infty} H(x, p)/\|p\| = +\infty$ (superliarity). After the study of A. Fathi the weak KAM theory is generalized to various cases that is not Tonelli Hamiltonian system on compact manifolds and has found applications in different domains.

2. Comprehensive systemization of weak KAM theory

2.1. Outline of comprehensive systemization of weak KAM theory

We can systemize weak KAM theory largely as follows:
Ⅰ. Research on fundamental problems of weak KAM theory
Ⅱ. Research on generalizations or variations of weak KAM theory
Ⅲ. Research on applications of weak KAM theory

We can systemize these three of the fields to following detailed fields in the total:

Ⅰ. Fundamental problems of weak KAM theory
Ⅰ.1. Lax-Oleinik semi-group
Ⅰ.2. Minimizing measures
Ⅰ.3. Reguraity of weak KAM solutions
Ⅰ.4. Action minimizing invariant sets
Ⅰ.5. PDE methods for weak KAM theory
Ⅰ.6. Perturbation theory methods in weak KAM theory
Ⅰ.7. Numerical methods for weak KAM theory

Ⅱ. Variations of weak KAM theory
Ⅱ.1. Weak KAM theory for coersive Hamiltonian systems
Ⅱ.2. Weak KAM theory on non-compact finite-dimensional manifolds
Ⅱ.3. Weak KAM theory for non-convex Hamiltonian systems
Ⅱ.4. Weak KAM theory for contact Hamiltonian systems
Ⅱ.5. Weak KAM theory for conformally symplectic systems
Ⅱ.6. Weak KAM theory for Hamiltonian systems that depend on unknown function
Ⅱ.7. Infinite dimensional weak KAM theory
Ⅱ.8. Stochasitic weak KAM theory
Ⅱ.9. Discrete weak KAM theory
Ⅱ.10. Quntum weak KAM theory
Ⅱ.11. Weak KAM theory for weakly coupled systems

Ⅲ. Applications of weak KAM theory
Ⅲ.1. Large time behaviour of evolutionary Hamilton-Jacobi eauations
Ⅲ.2. Homogenization of Hamilton-Jacobi eauations
Ⅲ.3. Weak KAM theory related to optimal transportation problems
Ⅲ.4. Weak KAM theory related to optimal switching problems
Ⅲ.5. Weak KAM theory related to optimal control problems
Ⅲ.6. Mean field games
Ⅲ.7. Construction of smooth time functions on Lorentzian manifolds
Ⅲ.8. Inverse Lyapunov theorems

2.2. Details of the comprehensive systemization of weak KAM theory

We explain the contents of each detailed fields of weak KAM theory minutely below.

Ⅰ. Fundamentals problems of weak KAM theory

Ⅰ.1. Lax-Oleinik semi-group and existence of weak KAM solutions

In here, we research existence of fixed point of Lax-Oleinik semi-group corresponding to the Lagramgian and we research convergence of Lax-Oleinik semi-group when time goes to $+\infty$. Lax-Oleinik semi-group converges to negative weak KAM solution when time goes to $+\infty$ and this theorem is called "the weak KAM theorem".

The theory related to Lax-Oleinik semi-group contains the following themes:

• Convergence of Lax-Olinik semi-group (See b[Fathi 1998]).

• Non-convergence of the Lax-Oleinik semi-group in the time-periodic case (See [Fathi-Mather 2000]).

• A new kind of Lax-Oleinik type operator for time-periodic positive definite Lagrangian systems (see [Wang-Yan 2012]).

• Lax-Oleinik semi-group for a weakly coupled system of Hamilton-Jacobi equations (see [Figalli-Goems, 2016]).

• Lax-Oleinik semi-group for infinite-dimensional Lagrangian systems (see [Shi-Yang 2016]).

• Lax-Oleinik semi-group in weak KAM theory (see [Zavidovique 2009], [Zavidovique 2010], [Zavidovique 2012], [Bernard-Zavidovique 2013]).

• Herglotz' variational principle and Lax-Oleinik evolution (see [Cannarsa, 2020]).

Ⅰ.2. Minimizing measures

The minimizing measures have close connection with the generalizaition of classical Aubry-Mather theory in two-dimansionanal phase space in the case of high-dimension. In 1991, J. N. Mather generalizaed Aubry-Mather theory for high-dimension Lagrangian systems in [Mather 1991]. However, since there exists the old example with Riemannian metric on $\mathbf{T}^3$ having only three derections which there are shortest geodesic lines by [Hedlund 1932], Mather treated minimizing measures instead of minimizing orbits. Mather clarified that for any vector there exists action-minimizing mesure with that rotation vector is the vector. Weak KAM theory that was initiated by A. Fathi bridged between Mather theory and PDE methods concerned with Hamilton-Jacobi equations.

In theory on the minimizing mesures, we clarify that there exists minimal measure μ (Mather measure) minimizing the action $\int_{TM} Ld\mu$ among the probability measures on TM which is invariant to the Euler-Lagrangian flow and the support of the measure μ lay on the graph of a Lipsitz continuous function and the support become invariant torus with Euler-Lagrangian flow if the support coincide to all of the graph. Moreover we also research to understand these results in the point of view of viscosity solutions of Hamilton-Jacobi equations. And we research the regularity estimation of viscosity solutions of Hamilton-Jacobi equations using Mather mesures. On the other hand, we research the connections between various consepts such as Green bundle, Lyapunov exponents of minimizing measures, and weak KAM solutons. In the meantime we research results related to the approximation of Mather measures.

The theory related to minimizing measures contains the following themes:

• Existence of action minimizing invariant measures for positive definite Lagrangian systems (see a[Mather 1989], [Mather 1991], [Carneiro 1995]).

• Action minimizing orbits in Hamiltonian sytems (see [Mather-Fomi 1994]).

• Ergodic variational methods (see [Mañé 1995]).

• Minimizing measures for time-dependent Lagrangians (see [Iturriagar 1996]).

• Generic properties and minimizing measures of Lagrangian systems (see [Mañé 1996]).

• Lagrangian flows and the dynamics of globally minimizing orbits (see [Mañé 1997]) .

• Lagrangian graphs, minimizing measures and Mañé's critical values (see [Contreras, 1998]).

• Global minimizers of autonomous Lagrangians (see [Contreras-Iturriaga 2000]).

• Duality principles for fully nonlinear elliptic equations (see b[Gomes 2005]).

•Generalized Mather problem and selection principles for viscosity solutions and Mather measures (see [Gomes 2008]).

• The number of Mather measures of Lagrangian systems (see [Bernard 2010]).

• Green bundles, Lyapunov exponents and regularity along the supports of the minimizing measures (see a[Arnaud 2010], [Arnaud 2012]).

Ⅰ.3. Research on reguralitу of weak KAM solutions

In research on reguralitу of weak KAM solutions, we research the connection between existence of global visicosity solutions and existence of global C^1 solutions of stationary Hamilton-Jacobi equations and we research existence of global C^1 critical viscosity subsolutons, existence of $C^{1,1}$ viscosity subsolutions, strictness of solutions outside of Aubry set, and connection between regurarity and dynamical behaviors. On the other hand, we research the indication of singularity of viscosity solutions and dynamical and asymptotic behavior of them based on the fact that viscosity solutions of Hamilton-Jacobi equations indicate singularity easily.

The theory related to regularity of weak KAM solutions contains following themes:

• Regularity of C^1 solutions of the Hamilton-Jacobi equations (see [Fathi 2003]).

• Existence of C^1 critical subsolutions of the Hamilton-Jacobi equations (see [Fathi-Siconolfi 2004], b[Bernard 2007]).

• Existence of $C^{1,1}$ critical sub-solutions of the Hamilton-Jacobi equations (see a[Bernard 2007]).

• Reguralty of visicosity solutions of Hamilton-Jacobi equations and generalized Sard's theorems (see [Rifford 2008]).

• Existence and regularity of strict critical subsolutions in the stationary ergodic setting (see [Davini-Siconolfi 2014]).

• Dynamic and asymptotic behavior of singularities of weak KAM solutions (see b[Cannarsa, 2019]).

Ⅰ.4. Research on action-minimizing invariant sets

J. N. Mather generalized the classical Aubry-Mather theory concerned with the action minimizing orbits of the monotone twist area-preserving diffeomorphisms of annulus to high-dimensional Lagrangian systems in [Mather 1991]. He developed similar theory with Aubry-Mather theory using the action-minimizing mesures instead of the action-minimizing orbits. Then Mather set is defined 184 page of [Mather 1991]([Fathi 2008]) and Mañé set is defined in 144 page of [Mañé 1997]([Fatthi 2008]). Aubry set is also defined Mather's research related to Lagrangian systems([Bernard 2005]).

By studying the dynamical behavior of the action-minimizing curves for Tonelli Lagrangian systems, weak KAM theory founded by A. Fathi bridged Mather theory and the PDE methods concerning the associated Hamilton-Jacobi equation([Li-Yan 2014]). In weak KAM theory, Aubry sets, Mather sets, and Mañé sets are represented in terms of viscosity soluions for Hamilton-Jacobi equations(See [Fathi 2008]).

In research on action-minimizing sets we research connections between the qualitative properties of the Hamiltonian systems, viscosity solutions of the associated Hamilton-Jacobi equations, and Mather set, Aubry set, and Mañé set. Especially the Mañé's conjectures that assert generically(roughly speaking, in almost of cases) Aubry set consist of an equilibrium point or a periodic orbit are the cetral problems in this field. See Conjecture Ⅰ.4-1 and Conjecture Ⅰ.4-2 in section 3 for Mañé's conjectures.

The theory related to action-minimizing invariant sets contains following themese:

• Les ensembles d'Aubry-Mather d'un difféomorphisme conservative de l'anneau déviant la verticale sont en général hyperboliques (see a[Calvez 1988]).

• Action potential and weak KAM solutions (see [Contreras 2001]).

• Symplectic aspects of Aubry-Mather theory (see a[Bernard 2005]).

• PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians (see [Fathi-Siconolfi 2005]).

• Counting geodesics which are optimal in homology (see [Anantharaman 2000]).

• The Hausdorff dimension of the Mather quotient (see [Fathi-Figalli-Rifford 2009]).

• A geometric definition of the Aubry-Mather set (see [Bernard-dos Santos 2010]).

• Aubry set from a PDE point of view (see [Fathi 2012]).

• Generic hyperbolicity of Aubry sets on surfaces (see [Contreras, 2013]).

• Regularity of weak KAM solutions and Mañé conjectures (see [Rifford 2013]).

• Exact Lagrangian submanifolds, Lagrangian spectral invariants and Aubry-Mather theory (see [Amorim, 2016]).

Ⅰ.5. PDE methods of weak KAM theory

The PDE methods of weak KAM theory is the research that searches "integrable structures" within general Hamiltonian dynamics using dynamical system theory, calculus of variation, and PDE methods([Evans 2005]).

The theory related to PDE methods of weak KAM theory contains following themes:

• Fundamental theory of PDE methods for weak KAM theory (see a[Evans 2003], a[Evans 2004], [Evans 2005], [Evans 2009]).

• Effective Hamiltonians and averaging for Hamiltonian dynamics (see [Evans-Gomes 2001], [Evans-Gomes 2002]).

• Existence of solutions for the Aronsson-Euler equation (see [Fathi-Siconolfi 2006]).

• L^∞ variational problems and weak KAM theory (see [Yu 2006]).

• New estimate on Evans' variational approach to weak KAM theory (see [Bernardi, 2013]).

• New identities for weak KAM theory(see [Evans 2017]).

Ⅰ.6. Perturbation theory in weak KAM theory

In the perturbation theory in weak KAM theory, we give the perturbation estimation of weak KAM solution u_ε with parameter ε for the nearly integrable tonelli Hamiltonian system

$H_\varepsilon = H_0(p) + \varepsilon H_1(q, p, t),\ ((q, p, t) \in \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T})$

and research stability of viscosity solutions, Mather set, Aubry set, Mañé set, and backword(forword) calibrated curves under the perturbation.

The perturbation theory in weak KAM theory contains following themes:

• Perturbation theory for viscosity solutions of Hamilton-Jacobi equations and stability of Aubry-Mather sets (see [Gomes 2003], b[Gomes 2002]).

• Perturbation theory and siscrete Hamiltonian dynamics (see [Gomes-Valls 2003]).

• Perturbation estimates of weak KAM solutions and minimal invariatn sets for nearly integrable Hamiltonian systems (see [Chen- Zhou 2017]).

• Singularly perturbed control systems with noncompact fast variable (see [Nguyen-Siconolfi 2017]).

Ⅰ.7. Numerical methods for weak KAM theory

In the numerical methods for weak KAM theory, we research the convergence of the approximate solution for effective Hamiltonian and research numerical simulation of Aubry set using approximate solution, and et al.

The theory related to numerical methods for weak KAM theory contains following themes:

• Numerical approximation of the effective Hamiltonian and of the Aubry set for first order Hamilton-Jacobi equations (see [Rorro 2005]).

• Difference approximation to Aubry-Mather sets (see b[Soga 2009], d[Soga, 2010]).

• Numerical analysis on the regular motions of Hamiltonian dynamics (see a[Soga 2010]).

• Numerical methods for static Hamilton-Jacobi equations (see [Luo 2009]).

• Fast adaptive numerical methods for high frequency waves and interface tracking (see [Popovic 2012]).

• Fast weak-KAM integrators for separable Hamiltonian systems (see [Bouillard, 2015]).

• Rapid numerical solution of Hamilton-Jacobi equations in stable manifold method (see [Hamaguchi, 2015]).

• Stochastic and variational approach to Lax-Friedrichs scheme (see [Soga 2016]).

Ⅱ. Variations of weak KAM theory

Ⅱ.1. Weak KAM theory for coersive Hamiltonian systems

We give the definitions of superliniearity and coercivity of Hamiltonian below. Assume that M is n-dimensional C^∞ manifold.

[Definition](Superlinearity on compact set. [Fathi 2008], pp. 16)

We say that a Hamiltonian $H : T^*M \to \mathbf{R}$ is superlinear above compact subsets if for every compact sbset $B \subset M$, and any $K \ge 0$, we can find a constant $C(B, K) \in \mathbf{R}$ such that

$H(x, p) \ge K \| p \|_x + C(B, K),\ (\forall (x, p) \in T^*M : x \in B)$,

where $\| \cdot \|_x$ is the norm on T_x^*M induced from the Riemannian metric of M.

(Remark) Hamiltonian $H : T^*M \to \mathbf{R}$ is superlinear above compact subsets if and only if for every compact sbset $B \subset M$ and any $x \in B$, $\lim_{\|p\|_x \to \infty} \dfrac{H(x, p)}{\| p \|_x} = +\infty$. Here limit is uniform for $x \in B$ (See [Fathi 2008], pp. 12, Exercise 1.3.4).

[Definition](Coercivity on compact set. [Fathi 2008], pp. 232)

We say that a Hamiltonian $H : T^*M \to \mathbf{R}$ is coercivie above compact subsets if for any compact sbset $B \subset M$

and any $c \in \mathbf{R}$, $\{(x, p) \in T^*M \mid x \in B,\ H(x, p) \le c\}$ is compact.

(Remark 1) Hamiltonian $H : T^*M \to \mathbf{R}$ is coercivie above compact subsets if and only if for every compact sbset $K \subset M$ and any $x \in K$, $\lim_{\|p\|_x \to \infty} H(x, p) = +\infty$ holds. Here limit is uniform for $x \in K$ (See [Fathi 2008], pp. 232).

(Remark 2)(a[Wang-Yan 2014]) For $V \in C^\infty(\mathbf{T}^1, \mathbf{R}); x \mapsto V(x)$ we set
$H : T^*\mathbf{T}^1 = \mathbf{T}^1 \times \mathbf{R} \to \mathbf{R}\,; (x, p) \mapsto \sqrt{1 + p^2} + V(x)$.
Then H satisfies coercivity but do not satisfy superlinearity.

In this theory, we clarify existence of weak KAM solutions of stationary Hamilton-Jacobi equations and that viscosity solutions of evolutionary Hamilton-Jacobi equations converge to stable state uniformly in space with constatnt velocity. Moreover we clarify aymtotic Lipschitz regularity of viscosity solutions and research the properties of effective Hamiltonian systematically that is proposed in the periodic hogenization of Hamilton-Jacobi equations.

Weak KAM theory for coercive Hamiltonian systems contains following themes:

- Weak KAM aspects of convex Hamilton-Jacobi equations with Neumann type boundary conditions (see [Ishii 2009]).
- Weak KAM theory without superlinearity (see a[Wang-Yan 2014]).
- Random homogenization of coercive Hamilton-Jacobi equations in 1d (see [Gao 2016]).

Ⅱ.2. Weak KAM theory on non-compact finite-dimensional manifolds

In the theory, we compose the weak KAM solutions of Hamilton-Jacobi equation $H(x, d_x u) = c$ concerned with Hamiltonian $H : T^*M \to \mathbf{R}$ on non-compact finite-dimensional manifold M and characterize the set of weak KAM solutions based on Aubry set.

Weak KAM theory on non-compact finite-dimensional manifolds contains following themes:

- Weak KAM theorem on non compact manifolds (see [Fathi-Maderna 2007]).
- Weak KAM methods and ergodic optimal problems for countable Markov shifts (see [Bissacot-Garibaldi 2010]).
- Optimal transportation on non-compact manifolds. (see [Fathi-Figalli 2010]).

Ⅱ.3. Weak KAM theory for non-convex Hamiltonian systems

Fisrst of all, we give the concept of quasi-convex functions.

[Definition](Quasi-conveity. [Fathi 2008], pp. 28) Let us $C \subset \mathbf{E}$ is convex subset in linear space $\mathbf{E}$ and $f : C \to \mathbf{R}$. We say that f is quasi-convex if for any $t \in \mathbf{R}$, $f^{-1}((-\infty, t])$ is convex.

Following properties hold for quasi-convex functions:

[Proposition]([Fathi 2008], pp. 28, Proposition 1.5.2)

Let $f : C \to \mathbf{R}$ be a function defined on the convex subset C of the linear space $\mathbf{E}$.

1) The function f is quasi-convex if and only if for any $x, y \in C$ and any $\alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y) \le \max\{f(x),\ f(y))$.

2) If f is quasi-convex then for every $x_1, \cdots, x_n \in C$ and every

$\alpha_1, \cdots, \alpha_n \in [0,\ 1]$, with $\sum_{i=1}^{n} \alpha_i = 1$, we have

$f(\sum_{i=1}^{n} \alpha_i x_i) \le \max_{1 \le i \le n} f(x_i)$. □

In weak KAM theory for non-convex Hamiltonian systems, we generalize the results of weak KAM theory that was developed under the assumption Hamiltonian is convex with momentums to the case that Hamiltonian is quasi-convex with momentums. Especially we research the existence of weak KAM solutions, the large time bihaviors of solutions of evolutionary Hamilton-Jacobi equations, the generalization of Mather measure, the properties of effective Hamiltonian, and stochastic homogenization.

Weak KAM theory for non-convex Hamiltonian systems contains following themes:

- A PDE approach to large-time asymtotics for boundary-vallue problems for nonconvex Hamilton-Jacobi eqatioons (see [Barles-Mitake 2010]).

• Comparison principle for unbounded visicosity solutions of degenerate elliptic PDEs (see [Koike-Ley 2010]).

• Aubry-Mather theory in the nonconvex setting (see [Fathi-Siconolfi 2005], [Cagnetti-Goems-Tran 2011]).

• Envelopes and nonconvex Hamilton-Jacobi equations (see [Evans 2014]).

• Minimax formula of the additive eigenvalue for quasiconvex Hamiltonians (see [Nakayasu 2014]).

• The selection problem for discounted Hamilton-Jacobi equations in the case of non-convex cases (see [Gomes-Mitake-Tran 2017]).

• Min-max formulas and properties of nonconvex effective Hamiltonians (see [Qian-Tran-Yu 2017]).

• Stochastic homogenization of nonconvex Hamilton-Jacobi equations (see [Ziliotto 2017]).

Ⅱ.4. Research on weak KAM theory for contact Hamiltonian systems

Firstly we define the concept of contact Hamiltonian, based on a[Wang-Wang-Yan 2019].
[Definition]([MSJ, 2007], pp. 719)

Asumme that $m, n, r \in \mathbf{N}$ satisfy $m = n + r$. Let M, N are m-dimmensional manifold and C^∞ manifold, n-dimensional C^∞ manifold respectively and $\pi : M \to N$ is C^∞ surjective. We say that $\mathcal{F} = (M, N, \pi)$ is fibered manifold and map $\pi : M \to N$ is projection map of the fibered manifold $\mathcal{F}$ if for any $x \in M$, the tangent map $T_x\pi : T_xM \to T_{\pi(x)}N$ is surjective.

We say that map $f : U \subset N \to M$ from open set U of N to M is section of fibered manifold $\mathcal{F} = (M, N, \pi)$ if $(\pi \circ f)(x) = x, (\forall x \in U)$.

There exit coordinate system $(x_1, \cdots, x_n)$ of N and coordinate system of M $(x_1, \cdots, x_n, y_1, \cdots, y_r)$ such that π is represented with the coordinate systems as follows
$\pi : (x_1, \cdots, x_n, y_1, \cdots, y_r) \mapsto (x_1, \cdots, x_n)$.
Then section $f : U \subset N \to M$ of $\mathcal{F} = (M, N, \pi)$ is represented by the set $(f_1(x), \cdots, f_r(x))$ of r functions with $x = (x_1, \cdots, x_n)$. Suppose that $f : U \subset N \to M$ and $g : V \subset M \to N$ are section of $\mathcal{F} = (M, N, \pi)$ and $a \in U, b \in V$. We say that (f, a) and (g, b) are equivalletnt if

1) $a = b, f(a) = g(b)$,

2) every first-order partial derivatives at point a for each components f_j, g_j of the representation of f and g by the coordinate systems coincide. This relation becomes an equivalent relation in the set of the pairs of section of $\mathcal{F} = (M, N, \pi)$ and the point belonging to the domain of the section. We say the set of the equivalent classes is bundle of 1-jets and denote the bundle of 1-jets by $J^1 = J^1(M, N, \pi)$. We denote the equilivallent class that (f, a) defined by $j_a^1 f$.

We define the function p_a^i on J^1 by

$$p_a^i(j_a^1(f)) = \frac{\partial}{\partial x_i}(a) \quad (x_i, y_\alpha, p_a^i\, ; 1 \le i \le n,\ 1 \le \alpha \le r,\ 1 \le i \le n).$$

Then

$$(x_i, y_\alpha, p_a^i\, ; 1 \le i \le n,\ 1 \le \alpha \le r,\ 1 \le i \le n)$$

becomes a coordinate system of J^1.

We denote the manifold of 1-jets of function on M by $J^1(M, \mathbf{R})$. The standard contact structure on M is 1-form $\alpha = du - pdx$. $J^1(M, \mathbf{R})$ has natural contact structure ξ that is difined globally by the Puff equation $\alpha = 0$. Thus $\xi = Ker\, \alpha$. The pair $(J^1(M, \mathbf{R}), \xi)$ is a contact manifold. There exists canonical diffeomorphism between $J^1(M, \mathbf{R})$ and $T^*M \times \mathbf{R}$. Thus the pair $(T^*M \times \mathbf{R}, \xi)$ is also a contact manifold. We call C^r function $(r \ge 2)$ $H : T^*M \times \mathbf{R} \to \mathbf{R}$ the contact Hamiltonian. The equation of contact flow defined by H in local coordinate system is given by

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x, u, p), \\ \dot{p} = \dfrac{\partial H}{\partial p}(x, u, p) - \dfrac{\partial H}{\partial u}(x, u, p)p \quad ((x, p) \in T^*M, u \in \mathbf{R}), \\ \dot{u} = \dfrac{\partial H}{\partial p}(x, u, p) - H(x, u, p). \end{cases}$$

In the research of weak KAM theory for contact Hamiltonian systems, we generalize the fundamental results of Aubry-Mather theory and weak KAM theory to the case of the contact Hsmiltonian systems.

Weak KAM theory for contact Hamiltonian systems contains following themes:

• Variational principle for contact Hamiltonian systems and its applications (see [Wang-Wang-Yan 2017], [Wang-Yan 2019], b[Wang-Wang-Yan 2019]).

• Aubry-Mather and weak KAM theories for contact Hamiltonian systems (see a[Wang-Wang-Yan 2018], b[Wang-Wang-Yan 2018], a[Wang-Wang-Yan 2019]).

• Herglotz' generalized variational principle and contact type Hamilton-Jacobi equations (see a[Cannarsa, 2019]).

• On the vanishing contact structure for viscosity solutions of contact type Hamilton-Jacobi equations (see [Zhao-Cheng 2018]).

Ⅱ.5. Weak KAM theory for conformally symplectic systems

We define the concept of conformally symplectic system first. Let M is n-dimensional C^∞ manidold. We let $C^\infty(M)$ denote the linear space of all C^∞ functions on M and $\mathsf{X}(M)$ denote the linear space of all C^∞ vector fields on M.

[Definition] Let $X \in \mathsf{X}(M)$. We define C^∞ function $L_X f$ on M by
$(L_X f)(x) = (Tf)X(x)$.
We call C^∞ function $L_X f$ the Lie-derivative of f with vector field X.

(Remark) Since

$$\left.\frac{d}{dt} f(\varphi_t(x))\right|_{t=0} = Tf\left(\left.\frac{d}{dt}\varphi_t(x)\right|_{t=0}\right) = Tf(X(x)),$$

we have

$$L_X f(x) = \left.\frac{d}{dt} f(\varphi_t(x))\right|_{t=0}. \qquad (*)$$

[Proposition 1] For any $f_1, f_2 \in C^\infty(M)$ and any $\lambda_1, \lambda_2 \in \mathbf{R}$, we have
$L_X(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 L_X f_1 + \lambda_2 L_X f_2$.

(Example) Let $M \overset{\circ}{\subset} \mathbf{R}^n$ and $x_1, \cdots, x_n$ is a coordinate of $\mathbf{R}^n$. Then we have

$$(L_X f)(x) = Df(x)X(x) = \nabla f(x) \cdot X(x) = \left(\sum_{i=1}^n X_i(x)\frac{\partial f}{\partial x_i}\right) = \left(\sum_{i=1}^n X_i(x)\frac{\partial}{\partial x_i}\right)f.$$

Therefore we can consider map $L_X : C^\infty(M) \to C^\infty(M)$ as a first-order differential operator. If we use chart of manifold, then we can also consider the map $L_X : C^\infty(M) \to C^\infty(M)$ as a first-order differential operatpr for general n-dimensional C^∞ manidold M.

[Definition] $\mathcal{L}_1(C^\infty(M), C^\infty(M))$ denote the linear space of all first-order differential operators from $C^\infty(M)$ to $C^\infty(M)$.

[Proposition 2] Map $X \mapsto L_X : \mathsf{X}(M) \to \mathcal{L}_1(C^\infty(M), C^\infty(M))$ is bijective.

(Proof) See [Arnold 1989].

[Proposition 3] If $X, Y \in \mathsf{X}(M)$, then $L_Y L_X - L_X L_Y$ is a first-order differential operator from $C^\infty(M)$ to $C^\infty(M)$.

(Proof) We prove the proposition in the case of $M \overset{\circ}{\subset} \mathbf{R}^n$. If consider (*), then for any $f \in C^\infty(\mathbf{R}^n)$, we have

$$L_Y L_X f = \sum_{i=1}^{n} Y_i(x)\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} X_i(x)\frac{\partial}{\partial x_j}\right)f = \sum_{i,j=1}^{n}\left[Y_i\frac{\partial X_j}{\partial x_i}\frac{\partial f}{\partial x_j} + Y_i X_j\frac{\partial^2 f}{\partial x_i \partial x_j}\right].$$

Therefore we obtain

$$(L_Y L_X - L_X L_Y)f = \sum_{i,j=1}^{n}\left[Y_i\frac{\partial X_j}{\partial x_i} - X_i\frac{\partial Y_j}{\partial x_i}\right]\frac{\partial f}{\partial x_j}. \square$$

[**Definition**] Suppose that $X, Y, Z \in \mathrm{X}(M)$. If $L_Z = L_Y L_X - L_X L_Y$, then the vector field Z is called the Poisson bracket of X and Y and is denoted by $[X, Y]$. So

$$L_{[X,Y]} = L_Y L_X - L_X L_Y.$$

[Definition] Let $(x_1, \cdots, x_n)$ is a local coordinate of n-dimensional C^∞ manifold M and $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ is a local coordinate of cotangent bundle T^*M of M. Then $\omega = \sum_{i=1}^{n} dx_i \wedge dp_i$ define a sympletic form of T^*M. ω is called the standard symplectic structure on T^*M.

[Definition] Suppose that ω is differneltial 2-form on n-dimensional C^∞ manifold M and X is C^∞ vector field on M. Then $L_X\omega$ differneltial 2-form on M by

$$(L_X\omega)(X_1, X_2) = X(\omega(X_1, X_2)) - \omega([X, X_1], X_2) - \omega(X_1, [X, X_2]), (\forall X_1, X_2 \in \mathrm{X}(M)).$$

The differneltial 2-form $L_X\omega$ is called Lie erivative of ω with vector field X.

[Definition] Suppose that ω is standard symplectic structure of T^*M and X is C^∞ vector field on T^*M. If there exists constant $\lambda \in \mathbf{R} \setminus \{0\}$ such that

$$\mathcal{L}_X\omega = \lambda\omega,$$

then X is called the conformally symplectic vector field.

Obviously the case of symplectic corresponds to the case of $\lambda = 0$ that is limit case. □

Conformally symplectic system emerges in interesting situation such as physics, geometry, clelstial mechanics, economy, and transporting model et al. In the weak KAM theory for conformally symplectic systems, we research the analogy of weak KAM theory to the class of dissipertive system that is conformally symplectic system. Especially we research the analogy of Aubry-Mather theory. In thie research, we existence of Aubry-Mather set in conformally symplectic systems, the structure of Aubry-Mather set and analyze the dynamical meaning of Aubry-Mather set and research the role of Aubry-Mather set in driving asymptotic dynamics of the system and action-minimizing property, attractiveness, and repelling property of Aubry-Mather set.

This theory can be considered as a generalization of the previous results on Aubry-Mather sets in the dissipative context to conformally symplectic flows on any compact manifold. And this theory is mostly focused in understanding what happens after these invariant Lagrangian submanifolds stop to exist or, more generally, what can be said about the dynamics and the invariant sets of a dissipative system(the above are quoted from [Marò-Sorrentino 2016]).

Weak KAM theory for conformally symplectic systems contains following themes:

• Periodic orbits for dissipative twist maps (see [Casdagli 1987]).

• Dynamical properties of diffeomorphisms of the annulus and of the torus (see [Calvez 2000]).

• KAM theory for conformally symplectic systems (see a[Calleja, 2013]).

• Local behavior near quasi-periodic solutions of conformally symplectic systems (see b[Calleja, 2013]).

• Aubry-Mather theory for conformally symplectic systems (see [Marò-Sorrentino 2016]).

Ⅱ.6. Weak KAM theory for Hamiltonian systems depending on unknown functions.

In this theory, we extend Fathi's weak KAM theory to the case of the Hamiltonian H depends on unknown function like $H = H(x, u, p)$. We establish a variation principle for the initial value problems of evolutionary Hamilton-Jacobi equations in which the Hamiltonian depends on unknown function explicitly and clarify the internal connection between the viscosity solutions and the minimizers. Moreover we clarify the convergence of viscosity solutions of evolutionary Hamilton-Jacobi equations that satisfy the initial condition to the weak KAM solutions of stationary Hamilton-Jacobi equations.

Weak KAM theory for Hamiltonian systems depending on unknown functions contains following themes:

• Solution semi-group under proper conditions (see [Su-Wang-Yan 2014]).

• Fundamental solution under Lipschitz conditions (see b[Wang-Yan 2014]).

• Variational principle under Osgood conditions (see c[Wang-Yan 2014]).

• Convergence of Lax-Oleinik semi-group for Hamiltonian systems depending on unknown functions (see ([Su-Wang-Yan 2016]).

• Weak KAM solutions of Hamilton-Jacobi equations with decreasing dependence on unknown functions (see [Wang, 2021]).

Ⅱ.7. Infinite-dimensional weak KAM theory

In this theory, for a class of Hamiltonians defined on the cotangent bundle of $L^2(0,1)$-infinite dimensional torus T we confirm existence of viscosity solutions for Hanioton-Jacobi equations on T. Thus we prove weak KAM theorem on $L^2(0,1)$-infinite dimensional torus T.

Infinite dimensional weak KAM theory contains following themes:

• Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities (see [Gozzi 1996]).

• Gradient flows in metric spaces and the Wasserstein spaces of probability (see [Ambrosio, 2005]).

• Hamilton-Jacobi equations in the Wasserstein space (see [Gangbo, 2008]).

• Weak KAM theorem for the nonlinear Vlasov equation (see [Gangbo-Tudorascu 2009]).

• Weak KAM theory for infinite-dimensional Lagrangians systems (see a[Gangbo-Tudorascu 2010], b[Gangbo-Tudorascu 2010] , [Shi-Yang 2016]).

• Extension of the weak KAM theory to the Wasserstein torus (see a[Gangbo-Tudorascu 2012], [Gangbo-Tudorascu 2014]).

• Minimizers of calculus of variations problems in Hilbert spaces (see [Gomes-Nurbekyan 2015]).

• Infinite-dimensional weak KAM theory via random variables (see [Gomes-Nurbekyan 2016]).

• Differentiability in the Wasserstein space and well-posedness for Hamilton-Jacobi equations (see [Gangbo-Tudorascu 2018]).

Ⅱ.8. Stochasitic weak KAM theory

In this theory, we research stochastic analogy of Aubry-Mather theory for the case that deterministic control problems are replaced to diffusion control problems. And we research the existence and uniqueniss of invariant mesure for Burgers equation with random forcing oscillating term, and asymptotic properties of the invariant measure. On the other hand, we research exit points problem from certain domain of stochastic process that is given by perturbation of dynamical system according to addition of small noise by the qualitative analysis of viscosity solutions of Hamilton-Jacobi equations with Neumann boundary condition.

Stochasitic weak KAM theory contains following themes:

• Invariant measures for Burgers equation with stochastic forcing (see [E-Sinai, 2000]).

• Stochastic Aubry-Mather theory (see [Iturriaga, 2005]).

• Weak KAM theory in the stationary ergodic setting (see [Davini-Siconolfi 2012], [Davini-Siconolfi 2014]).

• Randomly perturbed dynamical systems and Aubry-Mather theory (see [Siconolfi, 2009]).

• Stochastic homogenization of a nonconvex Hamilton-Jacobi equation (see [Armstrong-Tran -Yu 2013]).

Ⅱ.9. Discrete weak KAM theory

In the discrete weak KAM theory, we consider continuous function $c: X \times X \to \mathbf{R}$ that is called as cost function defiend on a distance space X instead of Lagrangian in weak KAM theory. We say that function $u: X \to \mathbf{R}$ is the α –subsolution to cost function c if there exists $\alpha \in \mathbf{R}$ such that

$$\forall x, y \in X, u(y) - u(x) \le c(x, y) + \alpha .$$

Then we call u is dominated by $c + \alpha$, which we denote by $u \prec c + \alpha$. We say that the smallest constant α that there exists α-subslution is the critical cinstant, which we denote by $\alpha[0]$. And we define discrete Lax-Oleinik semi-group for cost function c $T_c^- u$ and $T_c^+ u$ by

$$T_c^+ u(x) = \sup_{y \in X} [u(y) - c(y, x)] .$$

A continuous function $u: X \to \mathbf{R}$ that satisfies $T_c^- u + \alpha[0] = u$ is called a negative weak KAM solution for cost function c. On the other hand a continuous function $u: X \to \mathbf{R}$ that satisfies $T_c^+ u + \alpha[0] = u$ is called a positive weak KAM solution for cost function c.

In discrete weak KAM theory, we perform the research to transfer the various results concerned with subsolution, critical subsolution, weak KAM solution, Lax-Oleinik semi-group, Aubry set, α function of Mather, Mañé potential, and barrier function etc in weak KAM theory that was developed for Lagrangian continuous dynamical systems to the case of discrete setting in which we consider continuous function $c: X \times X \to \mathbf{R}$ called the cost function defined on suitable distance space X. In the discrete weak KAM theory, we prove existence of negative weak KAM solution and positive weak KAM solution for cost function under the assumtions of uniform superlinarity and uniform boundedness and we research existence of $C^{1,1}$ critical subsolution and approximate problem of effective Hamiltonian by discrete weak KAM solution etc.

Discrete weak KAM theory contains following themes:

• Weak KAM pairs and Monge-Kantorovich duality (see a[Bernard-Buffoni 2008]).

• Minimizing orbits in the discrete Aubry-Mather model. [Garibaldi 2009]

• Existence of $C^{1,1}$ critical subsolutions in discrete weak KAM theory (see [Zavidovique 2010]).

• Strict sub-solutions and Mañé potential in discrete weak KAM theory (see [Zavidovique 2012]).

• [Bernard-Zavidovique 2013]: Regularization of subsolutions in discrete weak KAM theory (see [Bernard-Zavidovique 2013]).

• Convergence of discrete Aubry-Mather model in the continuous limit (see [Su-Thieullen 2015]).

Ⅱ.10. Quantum weak KAM theory

In this theory, we find quantum analogy of the minimizing principle of Mather for Lagrangian dynamial systems that means understanding the connection between solutions of Shrödinger equations when we send Plank constant to zero, thus in the semi-classical limit $h \to 0$ (here h denotes Plank constant). And we research connection between spectrum of Schrödinger operator and effective Hamiltonian in weak KAM theory.

Quantum weak KAM theory contains following themes:

• Effective Hamiltonians and quantum states (see [Evans 2001]).

• Quantum analog of Weak KAM theory (see b[Evans 2004]).

• Wigner measures and the semi-classical limit to the Aubry-Mather measure (see [Gomes, 2011]).

• Coherent states and quantum asymptotic features by weak KAM theory (see [Cardin, 2014]).

• Schrödinger spectra and the effective Hamiltonian of weak KAM theory on the flat torus (see [Zanelli 2016]).

• Weak KAM approach to the periodic stationary Hartree equation (see [Zanelli, 2021]).

Ⅱ.11. Weak KAM theory for weakely coupled systems

We call

$$\frac{\partial u_i}{\partial t} + H_i(x, D_x u_i) + \sum_{j=1}^{m} b_{ij}(x) u_j(t, x) = 0, \ ((t, x) \in (0, +\infty) \times \mathbf{T}^N, i = 1, \cdots, m) \qquad \text{(EHJ)}$$

$$u_i(0, x) = u_i^0(x), \ (i = 1, \cdots, m) \quad (u_i^0 \in Lip(\mathbf{T}^N))$$

the evolutionary type of weakely coupled system of Hamilton-Jacobi equations. The other hand we call

$$H_i(x, D_x u_i) + \sum_{j=1}^{m} b_{ij}(x) u_j(x) = c, \ (x \in \mathbf{T}^N, i = 1, \cdots, m, c \in \mathbf{R}) \qquad \text{(SHJ)}$$

the stationary type of weakely coupled system of Hamilton-Jacobi equations. Here the assumptions for functions and coefficients are as follows:

1) Assumptions for Hamiltonian. For $i = 1, \cdots, m$

(H1) $H_i : \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ is continuous.

(H2) for any $x \in M$, $p \mapsto H_i(x, p)$ is strong convex on $\mathbf{R}^N$.

(H3) There are exist functions $\alpha, \beta : \mathbf{R}_+ \to \mathbf{R}$ such that

$\alpha(|p|) \le H_i(x, p) \le \beta(|p|), (\forall (x, p) \in \mathbf{T}^N)$.

2) Assumption for combination matrix

① $x \mapsto B(x)$ is continuous.

② $b_{ii} \geq 0,\ b_{ij} \leq 0, (j \neq i),\ \sum_{j=1}^{m} b_{ij} \geq 0, (\forall i \in \{1, \cdots, m\})$.

③ $B(x)$:non-degenerate $\left(\overset{\textbf{def}}{\Leftrightarrow} \sum_{j=1}^{m} b_{ij} = 0, (\forall i = 1, \cdots, m) \right)$.

④ $B(x)$: irreducible $\left(\overset{\textbf{def}}{\Leftrightarrow} \forall \mathcal{I} \subset \{1, \cdots, m\} : \mathcal{I} \neq \{1, \cdots, m\}, \exists i \in \mathcal{I}, \exists j \notin \mathcal{I} : b_{ij} \neq 0 \right)$.

In the Weak KAM theory for weakely coupled systems, we research for generalized weak KAM type theorem for solutions of weakely coupled system of Hamilton-Jacobi equations and Aubry set of weakely coupled system of Hamilton-Jacobi equations.

Moreover we research large time behavior of viscosity solutions of weakely coupled system of Hamilton-Jacobi equations.

Weak KAM theory for weakely coupled system contains following themes:

• Comparison results for weakly coupled systems of eikonal equations (see [Camilli, 2008]).

• Degenerate equations and weakly coupled systems (see [Cagnetti, 2012]).

• Large time behavior of weakly coupled systems of Hamilton-Jacobi equations (see [Cagnetti, 2012], [Camilli, 2012] , [Mitake-Tran 2012], [Mitake-Tran 2014]).

• Aubry-Mather theory for weakly coupled systems of Hamilton-Jacobi equations (see [Davini-Zavidovique 2012], [Zavidovique, 2016]).

• Adjoint methods for obstacle problems and weakly coupled systems of PDE (see [Cagnetti-Gomes-Tran 2013]).

• Weak KAM theory for a weakly coupled system of Hamilton-Jacobi equations (see [Figalli-Goems, 2016]).

• Lagrangian appraoch to weakely coupled Hamilton-Jacobi systems (see [Mitake, 2016]).

Ⅲ. Applications of weak KAM theory

Ⅲ.1. Large time behavior of evolutionary Hamilton-Jacobii equations.

In the theory, we research asymtotic behavior of viscosity solutions of the initial value problem for evolutionary Hamilton-Jacobi equation

$u_t + H(x, Du) = 0$,

asymtotic behavior of viscosity solutions of periodic first-order Hamilton-Javobi equation with space dependence, for asymptotic behavior for a class of weakly coupled system of Hamilton-Jacobi equations, asymptotic behavior of mean field game equation, asymptotic behavior of solutions of evolutionary Hamilton-Jacobi equations related to optimal switching problems, aymtotic behavior of weakely coupled systems of fully nonlinear parabolic linear equations, asymptotic behavior of solutions of obstruction problems for degenerate viscosity Hamilton-Jacobi equations, asymptotic behavior of solutions of time-periodic Hamilton-Jacobi equations that satisfy Dirichlet boundary conditions, and asymptotic behavior of viscosity solutions of evolutionary Hamilton-Jacobi equations in which Hamiltonian depends on inknown function etc.

Theory on large time behavior of Hamilton-Jacobi equations contains following themes:

• Regularity and large time behavior of solutions of a conservation law without convexity (see [Dafermos 1985]).

• Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations (see [Barles-Souganidis 2001]).

• Convergence to steady states or periodic solutions in Hamilton-Jacobi equations (see [Roquejoffre 2001]).

• A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations (see [Davini-Siconolfi 2006]).

• Asymptotic solutions of Hamilton-Jacobi equations in Euclidean n space (see a[Fujita-Ishii-Loreti 2006] , [Fujita 2007] , [Fujita-Uchiyama 2007], [Ishii 2008]).

• Asymtotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck oprator (see b[Fujita-Ishii-Loreti 2006]).

• Large time behavior of solutions of initial-boundary value problems for Hamilton-Jacobi equations (see [Mitake 2008], [Mitake 2009]).

• Gradient bounds for nonlinear degenerate parabolic equations and large time behavior of systems (see [Leya-Nguyen 2016]).

• Large-time behavior for obstacle problems for degenerate viscous Hamilton-Jacobi equations (see [Mitake-Tran 2013]).

• Uniqueness sets of additive eigenvalue problems and applications (see [Mitake-Tran 2018]).

• Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians (see [Ichihara-Ishii 2009]).

Ⅲ.2. Homogenization of Hamilton-Jacobi equations

In this theory, we research the convergence problems of solutions in Hamilton-Jacobi equations involving parameter when parameter goes to certain value. And we also research the calculation method of effective Hamiltonian and the qualitative properties of effective Hamiltonian. Let us consider classical example of the research related to homogenization by [Lions-Crrandall 1982]. The fundamental result is that for $T>0$ viscosity solution $u^{\varepsilon}(x,t)$ of initial problem of evolutionary Hamilton-Jacobi equation with Hamiltonian $H:\mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R}; (x,p)\mapsto H(x,p)$ which is continuous and 1-periodic with x and superliniar with p

$$\frac{\partial u^{\varepsilon}}{\partial t}+H(x/\varepsilon, Du^{\varepsilon})=0, ((x,t)\in\mathbf{R}^N\times(0,+\infty)) \tag{1}$$

$$u^{\varepsilon}(x,0)=u_0(x),\ (x\in\mathbf{R}^N);\ u_0\in BUC(\mathbf{R}^N) \tag{2}$$

converges to solution which satisfy initial condition (2) $u(x,t)$ of evolutionary Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t}+\overline{H}(Du)=0,\ \ ((x,t)\in\mathbf{R}^N\times(0,+\infty)) \tag{3}$$

on $\mathbf{R}^N\times[0,T]$ in function splace $BUC(\mathbf{R}^N\times[0,T])$ when $\varepsilon\to 0$. Here function $\overline{H}(p)$ is so-colled effective Hamiltonian and is determined as follows: For any $p\in\mathbf{R}^N$, there exists $\overline{H}(p)\in\mathbf{R}$ such that there exists periodic continuous solution $\mathsf{v}(y), (y\in\mathbf{R}^N)$ of Hamilton-Jacobi equation

$$H(y, p+D_y\mathsf{v})=\overline{H}(p). \tag{4}$$

After the above classical research the homogenization problems have been researched in various situations and for various types of Hamilton-Jacobi equations.

The theory on homogenization of Hamilton-Jacobi equations contains following themes:

• Fundamental of homogenizations of Hamilton-Jacobi equations (see a[Lions, 1987]).
• Homogenization of Hamilton-Jacobi equations in the Heisenberg group (see [Birindelli- Wigniolle 2003]).
• Perturbation problems in homogenization of Hamilton-Jacobi equations (see [Cardaliaguet, 2018]).
• Periodic homogenisation of nonlinear PDEs (see [Evans 1992], [Concordel 1996], [Concordel 1997], [Lions, 2009]).
• Computing the effective Hamiltonian using a varational approach (see [Gomes, 2004]).
• Multiscale problems and homogenization for second-order Hamilton-Jacobi equations (see [Alvarez, 2007]).
• Homogenization of metric Hamilton-Jacobi equations (see [Oberman, 2009]).
• Inverse problems in periodic homogenization of Hamilton-Jacobi equations (see [Luo-Tran-Yu 2016]).
• Exact and approximate correctors for stochastic Hamiltonians (see b[Davini-Siconolfi 2009]).
• Approximation for effective Hamiltonians for homogenization of Hamilton-Jacobi equations (see [Luo, 2011]).
• Stochastic homogenization of Hamilton-Jacobi equations (see [Souganidis, 1999], Rezakhanlou, 2000], [Lions, 2003], [Lions-Souganidis 2010], a[Armstrong, 2012], b[Armstrong, 2012], [Armstrong-Tran-Yu 2014], [Armstrong-Cardaliaguet 2016] , [Gao 2016]).
• Homogenization and non-homogenization of nonconvex Hamilton-Jacobi equations (see [Feldman, 2016]).

Ⅲ.3. Weak KAM theory related to optimal transportation problems

Let us explain the concept of optima transportation based on [Villani 2003], pp. 1-2, first.

Assume that we are given a pile of sand (say), and a hole that we have to completely fill up with the sand.

Obviously, the pile and the hole must have the same volume. Let us normalize the mass of the pile to 1. We shall model both the pile and the hole by probability measures μ, ν, difined respectively on some measure spaces X and Y. Whenever A and B are measurable subset of X and Y respectively, $\mu(A)$ gives a measure of how much sand is located inside A; and $\nu(B)$ of how much sand can be piled in B.

Moving the sand around needs some effort, which is modeled by a measurable cost function defined on $X\times Y$. Informally, $c(x,y)$ tells how much it costs to transport one unit of mass from lacation x to location y. It is natural to assume at least that c is measurable and nonnegative. One should not a priori exclude thd possibility that c thakes infinite values, and so c should be a mesurable map from $X\times Y$ to $\mathbf{R}\cup\{+\infty\}$.

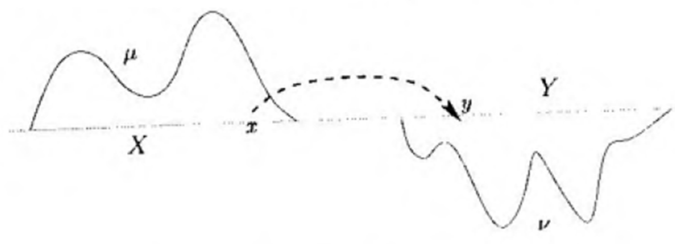

Figure 1. The mass transportation problem

The central question in optimal transportation problem is the following

Basic problem: How to realize the transportation at minimal cost ?

Before studying this question, we have to make clear what a way of transportation, or a transference plan, is. We shall model transference plans by proberbility measures π on the product space $X \times Y$. Informally, $d\pi(x, y)$ measures the amount of mass transferred from location x to location y. We do not a priori exclude the possibility that some mass located at point x may be split into several parts (several possible destination y's). For a transference plan $\pi \in P(X \times Y)$ to be admissible, it is of course necessary that all the mass taken from point x coincide with $d\mu(x)$, and that all the mass transferred to y coincide with $d\nu(y)$. This means

$$\int_Y d\pi(x, y) = d\mu(x), \quad \int_X d\pi(x, y) = d\nu(y).$$

More rigorously, we require that

$$\pi(A \times Y) = \mu(A), \quad \pi(X \times B) = \nu(B) \tag{1}$$

For all measurable subsets A of X and B of Y.

Those probability measures π satisfy (1) will be the admissible transference plans. We shall denote the set of all such probability measures by

$\Pi(\mu, \nu) = \{\pi \in P(X \times Y)$; (1) holds for all measurable A, B $\}$.

We now have a clear mathematical definition of our basic problem. In this form, it is known as

Kantorovich's optimal transportations problems:

Minimize $I[\pi] = \int_{X \times Y} c(x, y) d\pi(x, y)$ for $\pi \in \Pi(\mu, \nu)$.□

In weak KAM theory related to optimal transportation problems, we research the connection between optimal transportation problem and Mather theory and between optimal transportation problem and weak KAM theory.

The theory related to optimal transportation problems contains following themes:

- Fundamental theory of the optimal transportation problems (see [Gangbo 1997], [Villani 2003], [Ambrosio 2003]).
- Weak KAM pairs and Monge-Kantorovich duality (see a[Bernard-Buffoni 2007]).
- Optimal mass transportation and Mather theory (see b[Bernard-Buffoni 2007]).
- Continuity of optimal control costs and weak KAM theory (see [Agrachev, 2010]).
- Optimal transportation on non-compact manifolds (see [Fathi-Figalli 2010]).
- Logarithmic divergences from optimal transport and Rényi geometry (see [Wong 2018]).

Ⅲ.4. Weak KAM theory related to optimal switching problems

Optimal switching problems are the problems finding the orbits minimizing action functional as the orbits of system in which the dynamics change by the switching between different setting or different mode. In the theory on weak KAM theory related to optimal switching problems, we extend many concdpts of weak KAM theory to the case that optimal switching mode is considered and consider the variation problems for the solutions of the weakely coupled system of Hamilton-Jacobi equations and extend weak KAM theory and Aubry-Mather theory to optimal switching problems. And we research existence and reguraliry of action minimizer, large time behavior of solutions of non-stationary systems, and asymptotic limit of genelarized Lax-Oleinik semi-gorup.

The theory on optimal switching problems contains following themes:

- Optimal stochastic switching and the Dirichlet problem for the Bellman equation (see [Evans-Friedman 1979]).
- Optimal switching control of diffusion processes and associated implicit variational problems (see [Belbas 1981]).
- System of first order quasi-variational inequalities connected with optimal switching problem (see [Dolcetta, 1983]).
- Optimal switching for ordinary differential equations (see [Dolcetta-Evans 1984]).
- Weak KAM and Aubry-Mather theories in an optimal switching setting (see [Farias 2013]).

Ⅲ.5. Weak KAM theory related to optimal control problems

In this theory, we prove the version of weak KAM theorem concerned with optimal control of control systems and research Aubry-Mather problems corresponding to the systems. And we research the asymptotic properties of the corresponding value functions in singular perturbation optimal control problems with variables depending to parameter ε when ε goes to 0. Moreover we research the convergence of appropriate limit equation involving effective Hamiltonian to subsolution and supersolution in the sense of weak semilimit. This research is performed using some tools of weak KAM theory, especially the concept of Aubry set.

Weak KAM theory related to the optimal control problem contains following themes:

• Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations (see [Bardi, 1997])
• Viscosity solutions and analysis in L^∞ (see [Barron 1999]).
• Pontryagin approximations for optimal design (see [Carlsson 2006]).
• Stochastic optimal control under state constraints (see [Rutquist 2017]).
• Optimal control for nonlinear descriptor systems (see [Sjöberg 2006]).
• Continuity of optimal control costs and weak KAM theory (see [Agrachev, 2010]).

Ⅲ.6. Mean field game

Mean field game system denotes equilibrium arrangement in the game which has infinite number of players. In mean field game thoery we consider N-player Nash equilibrium for stochastic problem and establish “mean field” nonlinear partial differential equation when N goes to infinity and moreover show that this nonlinear problem is well-posed essentially, thus that has unique solution.

In mean field game thoery we develop an equivalent of weak KAM theory to mean field game. This allow for us to describe large time behavior of time-depending mean field game systems. Main resut is the existence of limit of value function when time goes to infinity. And we clarify the existence of mean field limit for ergodic constant related to corresponding Hamilton-Jacobi equation to mean field game.

Mean field game theory contains following themes:

• Fundamental theory of mean field games (see a[Lasry-Lions, 2006], b[Lasry-Lions, 2006], [Lasry-Lions 2007], [Lions 2008], [Cardaliaguet 2010], [Lions 2010], [Guéant-Lasry-Lions 2011]).
• Discrete time, finite state space mean field games (see [Gomes, 2010]).
• Mean field games and numerical methods for the planning problem (see [Achdou, 2012]).
• A semi-discrete in time approximation for a model first order-finite horizon mean field game problem (see [Camilli-Silva 2012]).
• Long time average of mean field games and weak KAM theory (see [Cardaliaguet, 2012], [Cardaliaguet 2013], [Cardaliaguet, 2013]).
• Convergence of finite state mean-field games through γ-convergence (see [Ferreira-Gomes 2014]).
• Potential mean field game (see [Briani-Cardaliaguet 2018], b[Cardaliaguet, 2019], [Masoero 2019]).
• Mean field games on networks (see [Camilli, 2015], [Cacace, 2016], [Camilli, 2016]).
• Second order mean field games with degenerate diffusion and local coupling (see [Cardaliaguet, 2015]).
• One-dimensional, forward-forward mean-field games with congestion (see [Gomes-Sedjro 2017]).
• Probabilistic theory of mean field games (see [Carmona-Delarue 2018]).
• Mean field games systems (see [Cardaliaguet-Graber 2015], [Cirant-Nurbekyan 2018], [Mészáros-Silva 2018]).
• Existence of a solution to an equation arising from the theory of mean field games (see [Gangbo-Święch 2015]).
• Existence of oscillating solutions in non-monotone mean-field games (see [Cirant 2019]).
• Long time behavior of master equation in mean field game theory (see a[Cardaliaguet, 2019], c[Cardaliaguet, 2019]).

Ⅲ.7. Research on construction of smooth time functions on Lorentzian manifolds

In this theory, we are concerned with the existence of smooth time functions on connected time-oriented Lorentzian manifolds. The problem is tackled in a more general abstract setting, namely in a manifold M where is just defined a field of tangent convex cones $(C_X)_{X\in M}$ enjoying mild continuity properties.

Under some conditions on its integral curves, we will construct a time function. The approach is based on the definition of an intrinsic length for curves indicating how a curve is far from being an integral trajectory of C_X. We find connections with topics pertaining to Hamilton-Jacobi equations, and make use of tools and results issued from weak KAM theory.

See [Fathi-Siconolfi 2012] for the detailed contents related to construction of smooth time functions on Lorentzian manifolds.

Ⅲ.8. Inverse Lyapunov theorems

See [Siconolfi-Terrone 2007] for inverse Lyapunov theorems.

3. Open problems in weak KAM theory

We denote the conjectures, open problems, and the point at issues in each detailed research fields of weak KAM theory. They contain 17 of the points at the issue that are newly proposed in this paper. The 17 points we are newly proposed expressed as [Jong, 2023].

Ⅰ. Open problems related to fundamental problems of weak KAM theory

[Problem Ⅰ.1-1]([Jong, 2023])

Generalize the theory related to Lax-Oleinik type operators of [Wang-Yan 2012] that was developed on compact manifold to the case of the time-periodic Lagrangian

$L: TM \times \mathbf{R} \to \mathbf{R}; (x, v, t) \mapsto L(x, v, t),\ L(x, v, t+1) = L(x, v, t)$

defined on finite-dimensional C^∞ manifold M.

[Problem Ⅰ.1-2]([Jong, 2023])

In [Bernard 2012] Bernard proved the existence of weak KAM solution and $C^{1,1}$ subsolution in the case that underlying manifold is $\mathbf{R}^n$.

Problem: Prove the existence of weak KAM solution and $C^{1,1}$ subsolution in the case that $\exists$ underlying manifold is general finite-dimensional manifold.

[Problem Ⅰ.2-1](a[Arnaud 2010], pp. 1669, Question 2)

Does an example of an invariant curve with an irrational rotation number that is not C^1 exist?

[Problem Ⅰ.2-2](a[Arnaud 2010], pp. 1671, Question 3)

Are there examples of Tonelli Hamiltonians or twist maps that are C^0-integrable but not C^1-integrable?

[Problem Ⅰ.2-3](a[Arnaud 2010], pp. 1672, Question 4)

Do two twist maps f and g and two minimizing measures μ_f for f and μ_g for g exist, so that μ_f and μ_g have the same support but are not equivalent (i.e. not mutually absolutely continuous)?

[Problem Ⅰ.2-4](a[Arnaud 2010], pp. 1672, Question 5)

Do there exist any minimizing measures with non zero Lyapunov exponents that are not uniformly hyperbolic?

[Problem Ⅰ.2-5](a[Arnaud 2010], pp. 1672, Question 6)

Do any examples of minimizing measures with zero Lyapunov exponents that are not supported in a C^1 curve exist?

[**Open Problem Ⅰ**.3-**1]**([Soga 2016], pp. 31)

The function H is assumed to satisfy the following:

(A1) $H(x, t, p): \mathbf{T}^2 \times \mathbf{R} \to \mathbf{R}, C^2$

(A2) $H_{pp} > 0$

(A3) $\lim_{|p| \to +\infty} \dfrac{H(x, t, p)}{|p|} = +\infty$

From (A1)-(A3) we obtain the Legendre transform $L(x, t, \xi)$ of $H(x, t, \cdot)$, which is given by

$L(x, t, \xi) = \sup_{p \in \mathbf{R}} \{\xi p - H(x, t, p)\}$.

(A4) There exists $\alpha > 0$ such that $|L_x| \le \alpha(|L| + 1)$.

The regularity criterion of solutions to

$$u_t + H(x, t, c + u)_x = 0 \qquad (1.1)$$

and

$$v_t + H(x, t, c + v_x) = h(c) \qquad (1.2)$$

under (A1)-(A4) remains an important open problem. Here $c, h(c)$ are constants.

[**Problem Ⅰ**.3-**2]**([Jong, 2023])

In a[Bernard 2007], Bernard proved the $C^{1,1}$ critical subsolution of Hamilton-Jacobi equations in the case that underlying manifold is compact manifold.

Problem: Prove the existence of $C^{1,1}$ critical subsolution of Hamilton-Jacobi equations in the case that underlying manifold is finite-dimensional manifold.

[Conjecture Ⅰ.4-1]([Rifford 2013], pp. 16, Conhecture 8.1. Mañé’s Conjecture)

(M, g) will be a smooth connected compact Riemannian manifold without boundary of dimension $n \geq 2$.

Following Mañé’s paper [Mañé 1996], given a Tonelli Hamiltonian $H: T^*M \to \mathbf{R}$ of C^k (with $k \geq 2$) and an potential $V: M \to \mathbf{R}$ of C^k (with $k \geq 2$), we define the Hamiltonian $H_V: T^*M \to \mathbf{R}$ by

$H_V(x, p) := H(x, p) + V(x) \quad \forall (x, p) \in T^*M$.

Denote by $C^k(M)$ the set of C^k potentials on M equipped with the C^k topology.

The Mañé’s conjecture in C^k topology (with $k \geq 2$) can be stated as follows:

For every Tonelli Hamiltonian $H: T^*M \to \mathbf{R}$ of C^k (with $k \geq 2$), there is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set $\widetilde{\mathcal{A}}(H_V)$ of the Hamiltonian H_V is either an equilibrium point or a periodic orbit. □

A natural way to attack the Mañé Conjecture in any dimension would be to prove first a density result, then a stability result. Namely, given an Tonelli Hamiltonian of class C^k, first one could show that the set of potentials $V \in C^k(M)$ such that $\widetilde{\mathcal{A}}(H_V)$ is either a hyperbolic equilibrium point or a hyperbolic periodic orbit is dense, and then prove that the latter property is open in C^k topology. The stability part is indeed contained in results obtained by Contreras and Iturriaga in [Contreras-Iturriaga 1999], so we can consider that the Mañé Conjecture reduces to the density part(the above was quoted from [Rifford 2013]).

[Conjecture Ⅰ.4-2]([Rifford 2013], pp. 16, Conhecture 8.2. Mañé’s density Conjecture)

For every Tonelli Hamiltonian $H: T^*M \to \mathbf{R}$ of class (with $k \geq 2$) there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of H_V is either an equilibrium point or a periodic orbit. □

In a series of papers in a[Figalli-Reffoed 2010] and b[Figalli-Reffoed 2010], Figalli and Refford made progress toward a proof of the Mañé Conjecture in C^2 topology. Their approach is based on a combination of thechniques combing from finite dimensional control theory and Hamilton-Jacobi theory, together with some of the ideas which were used to prove C^1-clossing lemmas for dynamical systems. The following result is a weak form of some of the results that they obtained in a[Figalli-Reffoed 2010] and b[Figalli-Reffoed 2010].

[Theorem 8.3]([Rifford 2013], pp. 17)

Let $H: T^*M \to \mathbf{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 4$, and fix $\varepsilon > 0$. Assume that there is a critical subsolution which is of class C^{k+1}. Then there exists a potential $V: M \to \mathbf{R}$ of class C^{k-1}, with $\|V\|_{C^2} < \varepsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an equilibrium point or a periodic orbit. □

This result together with stability results by Contreas and Iturriaga [Contreras-Iturriaga 1999] shows that we can more or less consider that the Mañé Conjecture for Hamiltonian of class at least C^4 is equivalent to the:

[Conjecture Ⅰ.4-3]([Rifford 2013], pp. 17, Conjecture 8.4. Mañé’s regularity Conjecture)

For every Tonelli Hamiltonian $H: T^*M \to \mathbf{R}$ of class C^k, with $k \geq 4$ there is a set $\mathcal{D} \subset C^4(M)$ which is dense in $C^2(M)$ (with respect to the C^2 topology) such that the following holds: For every $V \in \mathcal{D}$, the Hamiltonian H_V admits a critical subsolution of class C^5.

[Problem Ⅰ.4-4](a[Arnaud 2010], pp. 1654, Question 1)

Is there a means of distinguishing between the hyperbolic and the non hyperbolic Aubry or Aubry-Mather sets? Is there a means of seeing the Lyapunov exponents of a minimizing measure when knowing only the measure and not the dynamic?

[Problem Ⅰ.4-5]([Jong, 2023])

In b[Cannarsa, 2019], Cannarsa et al researched dynamical properties of generalized characteristic semi-flows related to Hamilton-Jacobi equation for mechanical Hamiltonian systems in the case that underlying manifold is n-dimensional torus and they constructed connection between generalized characteristic semi-flow and ω-limit set of projected Aubry set.

Problem: Genralize the results of b[Cannarsa, 2019] to the case that underlying manifold is compact manifold or finite-dimensional manifold or to the case of general Hamiltonian systems.

[Open Problem Ⅰ.5-1]([Evans 2005], pp. 150)

Concordel in [Concordel 1996], [Concordel 1997] initiated the systematic study of the geometric properties of the effective Hamiltonian $\overline{H}$, but many questions are still open.

Consider, say, the basic example $H(p,x)=\frac{|p|^2}{2}+W(x)$ and ask how the geometric properties of the periodic potential W influence the geometric properties of $\overline{H}$, and vice versa. For example, if we know that $\overline{H}$ has a "flat spot" at its minimum, what does this imply about W?

It would be interesting to have some more careful numerical studies here, as for instance in [Gomes-Oberman 2004].

[Open Problem Ⅰ.5-2]([Evans 2005], pp. 151)

It is, I think, very significant that the theory a[Lions, 1998] of Lions, Papanicolaou and Varadhan leads to the existence of solutions to the generalized eikonal equation

$$H(D_x u, x)=\overline{H}(P) \tag{E}$$

even if the Hamiltonian H is nonconvex in the momenta p: all that is really needed is the coercivity condition that $\lim_{|p|\to\infty} H(p,x)=\infty$, uniformly for $x\in\mathbf{T}^n$. In this case it remains a major problem to interpret $\overline{H}$ in terms of dynamics.

[Open Problem Ⅰ.5-3]([Evans 2005], pp. 151)

Fathi and Siconolfi [Fathi-Siconolfi 2005] have made great progress here, constructing much of the previously discussed theory under the hypothesis that $p\mapsto H(p,x)$ be geometrically quasiconvex, meaning that for each real number λ and $x\in\mathbf{T}^n$, the sublevel set $\{p \mid H(p,x)\le\lambda\}$ is convex.

The case of Hamiltonians which are coercive, but nonconvex and nonquasiconvex in p, is completely open.

[Problem Ⅰ.5-4]([Jong, 2023])

Evans in [Evans 2017] discovered various new integral identities and extended prior researches for the variational approximation in the case of the mechanical Hamiltonian $H(p,x)=\frac{1}{2}|p|^2+W(x)$.

Problem: Generalize the results of [Evans 2017] to the case of more general Hamiltonian.

[Problem Ⅰ.5-5]([Jong, 2023])

Evans in [Evans 2017] discovered various new integral identities for mechanical Hamiltonian $H(p,x)=\frac{1}{2}|p|^2+W(x)$ and extended prior researches for the variational approximation in the case that underlying manifold is n-dimensional torus $\mathbf{T}^n$.

Problem: Generalize the results of [Evans 2017] to the case that underlying manifold is compact manifold or finite-dimensional manifold.

[Problem Ⅰ.6-1]([Jong, 2023])

Generalize the results in b[Gomes 2002] that proved stability of viscosity solution and Mather set under the small perturbation of Hamiltonian on n-dimensional torus $\mathbf{T}^n$ to the case of the Hamiltonian on compact manifold or finite-dimensional manifold.

[Problem Ⅰ.6-2]([Jong, 2023])

Generalize the results in [Chen-Zhou 2017] that gave perturbation estimation of weak KAM solution u_ε with parameter ε for nealy integrable Tonelli Hamiltonian

$$H_\varepsilon=H_0(p)+\varepsilon H_1(q,p,t),\ (q,p,t)\in\mathbf{T}^n\times\mathbf{R}^n\times\mathbf{T}$$

on n-dimensional torus $\mathbf{T}^n$ and proved stability of Mather set $\widetilde{\mathcal{M}}_\varepsilon$, Aubry set $\widetilde{\mathcal{A}}_\varepsilon$, Mañé set $\widetilde{\mathcal{N}}_\varepsilon$ and backword calibrated curve to the case of Hamiltonian on compact manifold or finite-dimensional manifold.

[Conjecture Ⅰ.7.1]([Luo, 2011], pp. 13)

Consider mechanical Hamiltonian

$$H:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R};(p,x)\mapsto H(p,x),\quad H(p,x)=\sum_{1\le i,j\le n}a_{ij}(x)p_ip_j+V(x).$$

Here $a_{ij}:\mathbf{R}^n\to\mathbf{R}$, $V:\mathbf{R}^n\to\mathbf{R}$ is continuous and $\mathbf{T}^n$-periodic. And assume that $a_{ij}(x)$ satisfies the uniform strong convex condition

$$\Lambda|\xi|^2\ge\sum_{1\le i,j\le n}a_{ij}(x)\xi_i\xi_j\ge\lambda|\xi|^2,(\forall\xi\in\mathbf{R}^n;\Lambda>\lambda>0).$$

Let us $\overline{H}:\mathbf{R}^n\to\mathbf{R};\,p\mapsto\overline{H}(p)$ is effective Hamiltonian with Hamiltonian H. We define function $f:\mathbf{R}^n\to\mathbf{R}$ by

$$f(x)=4\Lambda|x|^2+\max_{x\in\mathbf{R}^n}V(x).$$

Suppose that $\Omega\subset\mathbf{R}^n$ is an open bounded set and consider Hamilton-Jacobi equation

$$(HJa)^\varepsilon\quad H(Du^\varepsilon,x/\varepsilon)=f(x),\ (x\in\Omega\setminus\{0\}\subset\mathbf{R}^n);\ u^\varepsilon(0)=0.$$

Conjecture; Then following holds:

Suppose that $n>1$, $\overline{H}(p)>\max_{x\in\mathbf{T}^1}V(x)$, and u^ε is viscosity solution of Hamilton-Jacobi equation $(HJa)^\varepsilon$. If there exists $x_0\in\mathbf{R}$ such that

$$u^\varepsilon(x_0)-p\cdot x_0=\min_{x\in\mathbf{R}}\{u^\varepsilon(x)-p\cdot x\},$$

then

$$|f(x_0)-\overline{H}(p)|\le O(\varepsilon)$$

holds.

[**Open Problem Ⅰ.7-2**]([Soga 2016], pp. 31-32)

Let $\overline{u}^{(c)}$ is $\mathbf{Z}^2$-periodic solution of Hamilton-Jacobi equation and $\overline{u}_\Delta^{(c)}$ is the solution of space-time-periodic scheme. Then an estimate of the error between $\overline{u}_\Delta^{(c)}$ and $\overline{u}^{(c)}$ without the Diophantine condition or without the condition $\overline{u}_\Delta^{(c)}\in C^1$ also remains open. The latter is particularly interesting in the context of a rigorous treatment of numerical approximations of Aubry-Mather sets.

[Conjecture Ⅰ-11-1]([Farias 2013], pp. 96, Conjecture 96)

Motivated by a[Bernard 2007]'s results, we present the following conjecture.

Given a subsolution u of

$$\max_{x\in M}\{H_i(x,du_i(x))-c_0,\ \max_{j\ne i}\{u_i(x)-u_j(x)-\psi(x,i,j)\}\}=0,\ \forall i\in\mathcal{I}\tag{1.9}$$

, there exists a subsolution v such that, for every mode $i\in\mathcal{I}$, $\mathsf{v}(\cdot,i)$ is in $C^{1,1}(M)$, at least when $\mathcal{I}=\{1,2\}$.

[Open Problem Ⅰ-11-2]([Zavidovique, 2016], pp 13)

Does there exist a C^1 subsolution of weakly coupled system of stationary Hamilton-Jacobi equations ?

[Open Problem Ⅰ-11-3]([Zavidovique, 2016], pp. 14**)**

Are subsolutions of weakly coupled system of stationary Hamilton-Jacobi equations differential on Aubry set ?

Ⅱ. Open problems related variation of weak KAM theory

[Problem Ⅱ.1-1]([Qian-Tran-Yu 2017], pp. 13, Question 1)

Let $H\in C(\mathbf{R}^n)$ be a coercive and even Hamiltonian, and $V\in C(\mathbf{T}^n)$ be a given potential. Let $\overline{H}$ be the effective Hamiltonian associated with $H(p)-V(p)$. Is it true that $\overline{H}$ is also even? In general, we may ask what properties of the original Hamiltonian will be preserved under homogenization.

[Problem Ⅱ.1-2]([Jong, 2023])

Fathi in [Fathi 2008] derived the results related to Lax-Oleinik semi-group, minimizing measure, regurality of weak KAM solution, and action minimizing invariant set under the assumption of superlinearity.

Problem: Derive the results in [Fathi 2008] under the assumption of coercivity.

[Open Problem Ⅱ.3-1]([Siconolfi 2006], pp. 1305)

Something similar to the Aubry–Mather sets exists for nonconvex Hamiltonian this is still an important open problem.

[Conjecture Ⅱ.3-2]([Qian-Tran-Yu 2017], pp. 14, Conjecture 1)

Assume that

(H7) " $\varphi \in C([0,\infty),\mathbf{R})$ satisfying that $\lim_{s\to\infty}\varphi(s)=+\infty$ and there exist $m\in\mathbf{N}$ and $0=s_0<s_1<\cdots<s_m<\infty=s_{2m+1}$ such that φ is strictly increasing in (s_{2i},s_{2i+1}), and is strictly decreasing in (s_{2i+1},s_{2i+2})."

holds. Assume further that $\varphi(0)=\min\varphi=0$. Let $H(p)=\varphi(p)$ for all $p\in\mathbf{R}^n$, and $V\in C(\mathbf{T}^n)$ be a given potential function. Let $\overline{H}$ be the effective Hamiltonian corresponding to $H(p)-V(p)$. If

$$osc_{\mathbf{T}^n}V=\max_{\mathbf{T}^n}V-\min_{\mathbf{T}^n}V\geq\max_{i,j}(M_i-m_j)$$

then the effective Hamiltonian $\overline{H}$ is quasiconvex.

[Conjecture Ⅱ.3-3]([Qian-Tran-Yu 2017], pp. 22, Conjecture 2)

Assume that $\varphi:[0,\infty)\to\mathbf{R}$ is continuous and coercivw. Set $H(p,x,w)=\varphi(|p|)-V(x,w)$ for $(p,x,w)\in\mathbf{R}^n\times\mathbf{R}^n\times\Omega$. Then H is regularly homogenizable.

[Problem Ⅱ.3-4]([Qian-Tran-Yu 2017], pp. 20, Question 2)

Assume that

$$\begin{cases} H(p)=\min\{|p-c_1|,|p+c_1|\},\ (p=(p_1,p_2)\in\mathbf{R}^2) \\ V(x)=S*(\sin^2(2\pi x_1)+\sin^2(2\pi x_2)),\ (x=(x_1,x_2)\in\mathbf{R}^2) \end{cases}$$

holds. Does there exist L such that when $S>L$, $\overline{H}$ is quasiconve?

[Problem Ⅱ.3-5]([Qian-Tran-Yu 2017], pp. 25, Question 3)

Let w be a periodic seni-concave (or semi-convex) function. Denote $\mathcal{D}$ as the collection of all regular gradients, that is,

$$\mathcal{D}=\{Dw(x);\ \ w \text{ is differentiable at } x\}.$$

Is $\mathcal{D}$ a connected set ?

[Problem Ⅱ.3-6]([Qian-Tran-Yu 2017], pp 25, Question 4)

Assume that $n=2$ and $H(p)=\min\{|p-e_1|,|p+e_1|\},\ (p\in\mathbf{R}^2)$ for all $p\in\mathbf{R}^2$, where $e_1=(1,0)$. Does there exist $L>0$ such that, if

$$osc_{\mathbf{R}^2\times\Omega}V=ess\sup_{\Omega}V(0,\omega)-ess\inf_{\Omega}V(0,\omega)>L$$

then $H-V$ is regulaly homogenizable ?

[Problem Ⅱ.4-1]([Jong, 2023])

Generalize the results related to Lax-Oleinik semi-group, minimizing measure, regularity of weak KAM solution, action-minimizing invariant set in [Fathi 2008] that was not treated in a[Wang-Wang-Yan 2019] to the case of contact Hamiltonian systesms.

[Problem Ⅱ.4-2]([Jong, 2023])

Obtain the results on Aubry-Mather theory and weak KAM theory that was not obtained in [Fathi 2008] and also was not treated in a[Wang-Wang-Yan 2019] to the case of contact Hamiltonian systesms.

[Problem Ⅱ.4-3]([Jong, 2023])

Establish the moderate increasing condition on the contact variable in [Wang-Wang-Yan 2019] more weakly and develop the theory in a[Wang-Wang-Yan 2019].

[Problem Ⅱ.4-4]([Jong, 2023])

In a[Wang-Wang-Yan 2019] Wang et al generalized the fundamental results of Aubry-Mather theory and weak KAM theory for Hamiltonian sestems defined on compact manifold to the case of contact Hamiltonian systems defined on compact manifold.

Problem: Generalize the results in a[Wang-Wang-Yan 2019] in the case of contact Hamiltonian system on finite-dimensional manifold.

[Problem Ⅱ.5-1]([Jong, 2023])
a[Calleja, 2013] developped KAM theory for conformally symplectic systems.
Problem: Develop Aubry-Mather theory for conformally symplectic systems.

[Problem Ⅱ.5-2]([Jong, 2023])
a[Calleja, 2013] developped KAM theory for conformally symplectic systems.
Problem: Develop weak KAM theory for conformally symplectic systems.

[Problem Ⅱ.7-1]([Jong, 2023])
Let us consider Lagrangian on Hilbert space $L^2(I)$ (where $I=(0,1)$)

$$L(M,N)=\frac{1}{2}\|N\|^2_{L^2(I)}-W(N),\quad (M,N)\in L^2(I)\times L^2(I).$$

Here

$$W\in C^2(\mathbf{T}^1)\ \text{ and }\ W(M):=\int_{I^2}W(Mz-M\bar{z})dzd\bar{z},\ (M\in L^2(I)).$$

Define Aubry set of the Lagrangian system with above Lagrangian and clarify the connection between the Aubry set and the corresponding Hamilton-Jacobi equation.

Refer to [Fathi 2008] for the connections between Aubry set of Lagrangian system and viscosity solution of accociated Hamilton-Jacobi equation.

[Problem Ⅱ.7-2]([Jong, 2023])
a[Gangbo-Tudorascu 2010] proved weak KAM theorem for the Lagrangian system given by Lagrangian on Hilbert space $L^2(I)$ (here $I=(0,1)$)

$$L(M,N)=\frac{1}{2}\|N\|^2_{L^2(I)}-W(N),\quad (M,N)\in L^2(I)\times L^2(I)$$

$$\text{(here } W\in C^2(\mathbf{T}^1)\ \text{ and }\ W(M):=\int_{I^2}W(Mz-M\bar{z})dzd\bar{z},\ (M\in L^2(I))\text{).}$$

Problem: Prove weak KAM theorem for the Lagrangian system given by Lagrangian on general separable Hilbert space $\mathbf{H}$

$$L(M,N)=\frac{1}{2}\|N\|^2-W(N),\quad (M,N)\in\mathbf{H}\times\mathbf{H}.$$

[Open Problem Ⅱ.8-1]([Davini-Siconolfi 2012], pp. 23, Open Question (1))
This is the third of a series of papers we have devoted to the analysis of critical equations for stationary ergodic Hamiltonians, see a[Davini-Siconolfi 2008], b[Davini-Siconolfi 2008], by using the metric approach combined with some tools from Random Set Theory. This method has allowed to get a complete picture of the setup when the state variable space is 1–dimensional, as specified in the introduction, and, we think, has revealed to be effective also in the multidimensional setting, highlighting some interesting analogies with the compact case. However many crucial problems are still to be clarified. The more striking is:
(1) In case of existence of an exact corrector, is the random Aubry set almost surely nonempty ?

[Open Problem Ⅱ.8-2]([Davini-Siconolfi 2012], pp. 23, Open Question (1′))
(1′) Is it impossible the simultaneous existence of an exact corrector and a global weakly strict admissible critical subsolution ?

[Open Problem Ⅱ.8-3]([Davini-Siconolfi 2012], pp. 23, Open Question (2))
In this respect, it should be helpful to strengthen
“ **[Theorem 5.3]** Assume that $c>c_f$ or $c=c_f$ and $\mathcal{A}_f(\omega)=\varnothing$ a.s. in ω. Then there exists a critical admissible subsolution which is wekly strict in $\mathbf{R}^N\setminus\mathcal{A}_f(\omega)$ a.s. in ω.”
, as in periodic case. So we would also like to know:
(2) If the Aubry set is a.s. empty, there exist strict global critical subsolutions?
Can we find one of such subsolution which is, in addition, smooth?

[Open Problem Ⅱ.8-4]([Davini-Siconolfi 2012], pp. 23-24, Open Question (3))

If the answer to (1), (1′) is positive, another question urges itself upon us:

(3) Is any exact corrector the Lax extension from the Aubry set of an admissible trace ?

Or, in other terms, is the Aubry set a uniqueness set for the critical equation, as in the deterministic compact case? Notice that both questions (1) and (3) have positive answer when $N = 1$, see b[Davini-Siconolfi 2009], and in any space dimension when $c = c_f = \sup_x \min_p H(x, p, \omega)$ and the critical stable norm in nondegenerate, see b[Davini-Siconolfi 2008].

[Open Problem Ⅱ.8-5]([Davini-Siconolfi 2012], pp. 24, Open Question (4))

Another subject of interest is about approximate correctors. So far we don't have any counterexamples to their existence when exact correctors do not exist. Hence the main question is:

(4) Do approximate correctors always exist?

This issue is also strongly related to homogenization problems and a positive answer would be an important step towards generalizations of the results proved in [Rezakhanlou-Tarver 2000], [Rockafellar 1970] to more general Hamiltonians.

[Open Problem Ⅱ.8-6]([Davini-Siconolfi 2012], pp. 25, Open Question (5))

Note that the existence results of [Ishii 2000] for approximate correctors in the almost-periodic case are based on an ergodic approximation of the Hamilton–Jacobi equation, and so are not constructive. A final question, which stems from the previous discussion, then is

(5) At least in the almost-periodic case, are the approximate correctors representable through Lax formulae?

[Open Problem Ⅱ.10.-1](b[Evans 2004], pp. 311)

Evans in Theorem 7.1 of b[Evans 2004] proved the minimizer of $\int_{\mathbf{T}^n} \frac{h^2}{2} |D\psi|^2 - |\psi|^2 \, dx$ satisfy

$$-\frac{h^2}{2}\Delta\psi + W\psi - E\psi = O(h).$$

Evans wrote in pp. 311 of b[Evans 2004] as follows: Regarding our state ψ as a quasimode, we furthermore derive some error estimates, although it remains an open problem to improve these bounds.

4. Conclusion

In this paper, we performed the comprehensive systemization of weak KAM theory for one to review the whole of fields and detailed fields of weak KAM theory. We systemized weak KAM theory as fundamentals of weak KAM theory, variations of weak KAM theory, and applications of weak KAM theory largely and then we systemized these three of the fields to 26 of detailed fields in the total. Moreover we systemized each of the detailed fields to various themes.

We systemized comprehensively the conjectures, the open problems, and points at the issue that proposed in weak KAM theory as well. They contain 17 problems that are proposed newly in this paper.

You would be able to survey the whole features of weak KAM theory that is being studied widely in various fields by this paper and obtain a compass in research of weak KAM theory.

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