Embedding the Einstein tensor in the Klein-Gordon Equation using Geometric Algebra Cl3,0

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Abstract

In this paper we will use Geometric Algebra to be able to embed the Klein-Gordon equation for a particle in a non-Euclidean field (gravitational field) arriving to the following equation:

$$e^{\beta}\nabla_{\beta}(\nabla_{\alpha}(\psi^{\dagger}\psi)e^{\alpha}) = \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^{\dagger}\psi$$

Where $\psi^{\dagger}\psi$ is the wavefunction collapsed (multiplied by its reverse), this way:

$$\psi^{\dagger}\psi = (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) = \rho + \vec{j}$$

Being ρ and $\vec{j}$ the probability density and the fermionic current respectively.
The equation above can be factored to be simplified into:

$$\nabla_{\alpha}\psi = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}\,\psi e_{\alpha}$$

Meaning that the energy of a particle is somehow decreased by a term that depends on the Ricci scalar (the curvature of the space where it lies in):

$$E_{particle} = mc^2 - \frac{\hbar^2}{m}R$$

Anyhow, this reduction is completely negligible in the general case, being several orders of magnitude below the normal energy.

Following other path, we will find another equation:

$$\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}(\nabla_{\alpha}(\psi^{\dagger}\psi)e^{\alpha})\right) + \frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R + mc^2\right)\psi^{\dagger}\psi - \frac{c^4}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right) = 0$$

This equation (that are in fact 8 embedded equations) have 14 or 15 unknown variables: 8 coefficients of the wavefunction $\psi^0 \; to \; \psi^7$ and 6 metric elements g_{ij} (i,j from 1 to 3) with a possible added g_{00}.

The rest of the needed equations (8 equations more) come from the continuity equation:

$$e^{\lambda}\nabla_{\lambda}T = 0$$

With T defined as:

$$T = g^{\mu\nu}T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right)\psi^{\dagger}\psi + \frac{1}{2}\frac{\hbar^2}{m}\left(e^{\beta}\nabla_{\beta}(\nabla_{\alpha}\psi^{\dagger}\psi)e^{\alpha}\right)$$

So, the equation is in fact, solvable.

Keywords

Geometric Algebra, Einstein Tensor, Klein-Gordon Equation, Bra-ket product, non-Euclidean metric

1. Introduction

In this paper we will embed the Klein-Gordon equation for a particle in a non-Euclidean field (gravitational field) using Geometric Algebra and the Einstein equations. This will lead to new equations that we will show in the paper.

2. Geometric Algebra $Cl_{3,0}$. Basis vectors

There is a discipline in mathematics that is called Geometric Algebra [1][3] also known as Clifford Algebras.

In the specific Geometric Algebra $Cl_{3,0}$, it is considered a three-dimensional space, so we need three independent vectors to define a basis. The classical definition of a basis is as follows:

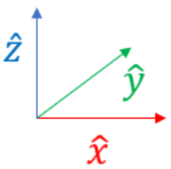
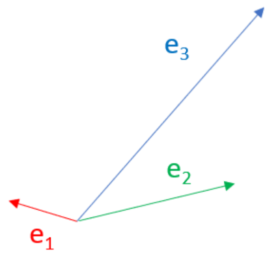


Fig. 1 Basis vectors in three-dimensional space.

In this paper we will use the nomenclature e_i (without any hat or vector sign) to name these three vectors instead the classical $\hat{x}\; \hat{y}\; \hat{z}$. Above, I have considered an orthonormal basis as an example.

But in the general case, this is not even necessary. The only necessary constraint to form a basis is that the three vectors are linearly independent (this is, they do not lie on the same plane). An example below:



In geometric algebra, it is defined an operation called the geometric product. The geometric product is not represented by any symbol. It is the implicit operation when two vectors are represented one after the other.

Its definition is:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j$$

Being:

$$e_i \cdot e_j = \|e_i\| \|e_j\| \cos(\alpha_{ij})$$

The classical definition of the scalar product. The product of the two norms (the length) of the vectors by the cosine of the angle formed by them (we have called it α_{ij} in this case).

The result of the scalar product is a number, a scalar. An important property of the scalar product is that it is commutative:

$$e_i \cdot e_j = e_j \cdot e_i = \|e_i\| \|e_j\| \cos(\alpha_{ij})$$

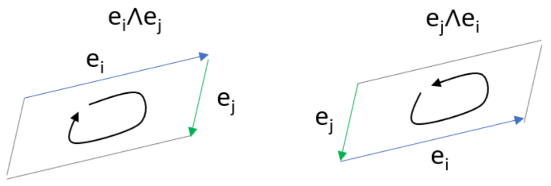
As the cosine of the angle is included in the product, you can check that when e_i and e_j are perpendicular (right angle), the scalar product is zero. And the vectors are colinear (the angle is zero), the scalar product is just the product of the modules of the vectors.

The other element of the geometric product above is:

$$e_i \wedge e_j$$

What it is called the outer, exterior or wedge product of the two vectors.

The result of this operation is not a number. It is another entity that is not a number and not a vector. It is called a bivector. The bivector is an entity that represents an oriented surface area (in a same way that a vector "represents" an oriented line segment).



It can be checked above that the module (area of the surface) when reversing the order of the exterior product is the same. But the orientation (its sign) changes. So, the exterior product is anticommutative:

$$e_i \wedge e_j = -e_j \wedge e_i$$

The module (area of the surface) of the exterior product is:

$$\|e_i \wedge e_j\| = \|e_j \wedge e_i\| = \|e_i\|\|e_j\| \sin(\alpha_{ij})$$

You can see that when the vectors are colinear (the angle is zero), the exterior product result is zero. And when the vectors are perpendicular, the module of the exterior product is the product of the modules of the vectors.

Coming back to the definition of the geometric product:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j$$

We can see that when we perform the square of a vector, this is, the product of a vector by itself (the vector is colinear with itself, its angle is zero) the result is:

$$(e_i)^2 = e_i e_i = e_i \cdot e_i + e_i \wedge e_i = \|e_i\|\|e_i\| \cdot 1 + 0 = \|e_i\|\|e_i\| = \|e_i\|^2$$

So, the square of a vector is its norm squared. The important thing here, is that the result is just a number. It is not a vector, it is not a bivector, it is just a number. We have converted a vector to a number just multiplying it by itself.

If now, we multiply (geometric product) two perpendicular vectors (the angle between them is a right angle):

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = 0 + e_i \wedge e_j = e_i \wedge e_j$$

So, you can see that the result is a pure bivector. It does not include vectors or scalars, just a bivector.

If we reverse the angle, we have:

$$e_j e_i = e_j \cdot e_i + e_j \wedge e_i = 0 + e_j \wedge e_i = e_j \wedge e_i = -e_i \wedge e_j = -e_i e_j$$

So, when two vectors are perpendicular, not only the exterior product, but also the geometric product is anticommutative.

From the equations above we can obtain the following equations.

$$e_i \cdot e_j = \frac{1}{2}\left(e_i e_j + e_j e_i\right)$$

$$e_i \wedge e_j = \frac{1}{2}\left(e_i e_j - e_j e_i\right)$$

The demonstration comes directly from the definition of the geometric product. If we sum a geometric product by its reverse, we put the definition of geometric product, we take into account that the scalar product is commutative and the exterior product anticommutative:

$$\begin{aligned} e_i e_j + e_j e_i &= e_i \cdot e_j + e_i \wedge e_j + e_j \cdot e_i + e_j \wedge e_i = e_i \cdot e_j + e_i \wedge e_j + e_i \cdot e_j - e_i \wedge e_j \\ &= 2\left(e_i \cdot e_j\right) \end{aligned}$$

$$e_i \cdot e_j = \frac{1}{2}\left(e_i e_j + e_j e_i\right)$$

If instead of summing, we subtract:

$$e_ie_j - e_je_i = e_i \cdot e_j + e_i \wedge e_j - e_j \cdot e_i - e_j \wedge e_i = e_i \cdot e_j + e_i \wedge e_j - e_i \cdot e_j + e_i \wedge e_j$$
$$= 2(e_i \wedge e_j)$$

$$e_i \wedge e_j = \frac{1}{2}(e_ie_j - e_je_i)$$

We will see in next chapters that when we apply the exterior product instead of the geometric product of two vectors, this means that we want only the result that appears in the plane they form (in the bivector they form). And we "remove" from the result the scalars (that will appear with the scalar product of the vectors) and also, we remove the possible result in vectors (in more complicated products that we will see in next chapters).

Another point to comment is that the exterior product of bivectors (instead of vectors) is defined in the opposite way (summing instead of subtracting). I am not going to enter into details, you can check it in [3].

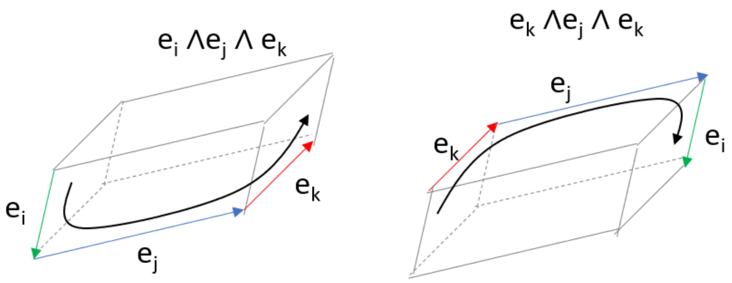
$$(e_ie_j) \wedge (e_re_s) = \frac{1}{2}(e_ie_je_re_s + e_re_se_je_i)$$

The same way, the scalar product of bivectors is also defined as the opposite of vectors. See [3].

$$(e_ie_j) \cdot (e_re_s) = \frac{1}{2}(e_ie_je_re_s - e_re_se_je_i)$$

Also, to remark that the geometric product is always associative and distributive as you can see in [3]. But in general, is not commutative or anticommutative as commented (it depends on the specific product) We will see more examples in the following chapters.

To conclude this chapter about geometric algebra, we will define the trivector. When two vectors are exterior multiplied, they form a bivector as seen above. The same way, when three vectors are exterior multiplied, they create an oriented volume, called the trivector:



You can see again, that when we reverse the vectors, we get the same volume (module of the trivector) but with different orientation (sign):

$$e_i \wedge e_j \wedge e_k = -e_k \wedge e_j \wedge e_i$$

We will check more thing regarding reversion and change of signs in the next chapter.

3. Geometric Algebra $Cl_{3,0}$. Different types of bases

3.1 Orthonormal basis

In an orthonormal basis, the norm of the basis vectors is equal to one. And the basis vectors are perpendicular to each other.

So, from the properties commented in chapter 2, we can get obtain the following equations (for orthonormal basis):

$$(e_i)^2 = e_ie_i = e_i \cdot e_i = 1$$
$$e_ie_j = e_i \wedge e_j = -e_j \wedge e_i = -e_je_i \quad (when\ i \neq j)$$
$$e_i \cdot e_j = e_j \cdot e_i = 0 \quad (when\ i \neq j)$$

Making the equations explicit for three dimensions:

$$(e_1)^2 = e_1e_1 = 1$$
$$(e_2)^2 = e_2e_2 = 1$$
$$(e_3)^2 = e_3e_3 = 1$$
$$e_1e_2 = -e_2e_1$$
$$e_2e_3 = -e_3e_2$$
$$e_3e_1 = -e_1e_1$$

We can define the inverse of a vector and name it e^i , as the vector that fulfills (Einstein summation is not implied here):

$$(e_i)^{-1}e_i \equiv e^ie_i = 1 = e_i(e_i)^{-1} \equiv e_ie^i$$

To calculate e^i we can post multiply by e_i:

$$(e_i)^{-1}e_ie_i \equiv e^ie_ie_i = 1 \cdot e_i$$
$$e^i(e_i)^2 = e_i$$
$$e^i \cdot 1 = e_i$$
$$e^i = e_i = (e_i)^{-1}$$

So, in orthonormal metric the inverse of a basis vector is itself. It is important to remark here that in Geometric Algebra there are no covectors (or 1-forms). There are only scalars, bivectors, trivectors… We will see that the concept of covector in Geometric Algebra is just a vector that is the inverse of another vector.

In traditional algebra you cannot define the inverse of a vector, so it is used a different type of element. In Geometric Algebra, the covectors are also vectors. And in fact, the product of inverse vectors by vectors outputs scalars as it would be expected by the product of a covector by a vector.

3.2. Geometric Algebra $Cl_{3,0}$. Orthogonal but not orthonormal basis

In an orthogonal basis, the vectors are perpendicular to each other. But in general, the norm of the vectors is not one. In Geometric Algebra $Cl_{3,0}$, the norm of the basis vectors is always positive and different from zero. The 0 in the name $Cl_{3,0}$, makes reference that there are no basis vectors with negative norm. and the absence of a third number makes reference that also, there no basis vectors with zero norm.

From the properties commented in chapter 2, we can get obtain the following equations (for orthogonal, not orthonormal basis):

$$(e_i)^2 = e_ie_i = e_i \cdot e_i = \|e_i\|^2 = g_{ii}$$
$$e_ie_j = e_i \wedge e_j = -e_j \wedge e_i = -e_je_i \quad (when\ i \neq j)$$
$$e_i \cdot e_j = e_j \cdot e_i = 0 \quad (when\ i \neq j)$$

Making the equations explicit for three dimensions:

$$(e_1)^2 = e_1e_1 = \|e_1\|^2 = g_{11}$$

$$(e_2)^2 = e_2 e_2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = e_3 e_3 = \|e_3\|^2 = g_{33}$$
$$e_1 e_2 = -e_2 e_1$$
$$e_2 e_3 = -e_3 e_2$$
$$e_3 e_1 = -e_1 e_1$$

Where the g_{ii} makes reference to the metric tensor components. See paper [2]. Take into account that when you multiply two colinear vectors (and a vector is colinear with itself), its geometric product is equal to the scalar product. And this is exactly the definition of g_{ii} (the scalar product of e_i with itself).

The definition of the inverse of a vector, and naming it e^i , is the vector that fulfills (not Einstein summation is implied here):

$$(e_i)^{-1} e_i \equiv e^i e_i = 1 = e_i (e_i)^{-1} \equiv e_i e^i$$

To calculate e^i we can post multiply by e_i:

$$(e_i)^{-1} e_i e_i \equiv e^i e_i e_i = 1 \cdot e_i$$
$$e^i (e_i)^2 = e_i$$
$$e^i \|e_i\|^2 = e_i$$
$$e^i g_{ii} = e_i$$
$$e^i = \frac{e_i}{g_{ii}} = \frac{e_i}{\|e_i\|^2} = (e_i)^{-1}$$

So, in orthogonal metric the inverse of a basis vector is itself divided by its norm squared (by g_{ii}). Everything commented regarding covectors in 3.1 applies also here.

One important consequence of this, is that if the basis vectors are orthogonal (as in this chapter), all the basis vectors and all the inverse of the basis vectors are also orthogonal among them (when i≠j). this is:

$$e^i \cdot e_j = \frac{e_i}{g_{ii}} \cdot e_j = \frac{1}{g_{ii}} (e_i \cdot e_j) = \frac{1}{2g_{ii}} (e_i e_j + e_j e_i) = 0$$
$$e^i \cdot e^j = \frac{e_i}{g_{ii}} \cdot \frac{e_j}{g_{jj}} = \frac{1}{2g_{ii} g_{jj}} (e_i \cdot e_j) = \frac{1}{2g_{ii} g_{jj}} (e_i e_j + e_j e_i) = 0$$

In the last equation (but when i=j) we get:

$$e^i \cdot e^i = (e^i)^2 = \frac{e_i}{g_{ii}} \cdot \frac{e_i}{g_{ii}} = \frac{1}{g_{ii} g_{ii}} (e_i \cdot e_i) = \frac{1}{g_{ii} g_{ii}} (e_i e_i) = \frac{1}{(g_{ii})^2} \cdot 1 = \frac{1}{(g_{ii})^2}$$

These last properties apply also to chapter 3.1 (orthonormal basis) but in that case the elements g_{ii} or g_{jj} are always 1.

3.3. Geometric Algebra $Cl_{3,0}$. Non-Orthogonal (and therefore not orthonormal) basis

In a non-orthogonal basis, the vectors are not perpendicular from each other. And in general, the norm of the vectors is not one. As commented in 3.2, in Geometric Algebra $Cl_{3,0}$, the norm of the basis vectors is always positive and different from zero.

From the properties commented in chapter 2 and also in [2], we can get obtain the following equations (for orthogonal, not orthonormal basis):

$$(e_i)^2 = e_i e_i = \|e_i\|^2 = g_{ii}$$
$$e_i e_j = 2g_{ij} - e_j e_i = 2g_{ji} - e_j e_i$$
$$e_i \cdot e_j = e_j \cdot e_i = g_{ij} = g_{ji}$$
$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = g_{ij} + e_i \wedge e_j$$

Making the equations explicit for three dimensions:

$$(e_1)^2 = e_1 e_1 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = e_2 e_2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = e_3 e_3 = \|e_3\|^2 = g_{33}$$
$$e_1 e_2 = 2g_{12} - e_2 e_1 = 2g_{21} - e_2 e_1$$

$$e_2e_3 = 2g_{23} - e_3e_2 = 2g_{32} - e_3e_2$$
$$e_3e_1 = 2g_{31} - e_1e_3 = 2g_{13} - e_1e_3$$

Where the g_{ij} makes reference again to the metric tensor components (the scalar products of the basis vectors). See paper [2] for more information. You can obtain the above equations from the definition of scalar product in geometric algebra as commented in chapter 2.

$$e_i \cdot e_j = g_{ij} = \frac{1}{2}\left(e_ie_j + e_je_i\right)$$

Multiplying by 2:

$$2g_{ij} = e_ie_j + e_je_i$$

Rearranging terms (and knowing that the metric tensor is symmetric):

$$e_ie_j = 2g_{ij} - e_je_i = 2g_{ji} - e_je_i$$

Now, we will define again the inverse of the basis vectors and name them e^i. To obtain the inverse of the basis vectors is this case, you have to get the inverse of the metric tensor, so you are able to define a vector e^i that fulfills for every i and every j the following (Einstein summation does not apply):

$$(e_i)^{-1}e_i \equiv e^ie_i = 1 = e_i(e_i)^{-1} \equiv e_ie^i$$

$$e^i \cdot e_j = e_i \cdot e^j = \frac{1}{2}(e_ie^j + e^je_i) = 0 \quad for\ i \neq j$$

In general, this is written as:

$$e^i \cdot e_j = \delta^i_j$$

Where δ^i_j is the Kronecker Delta, that is equal to 1 when i=j and 0 when i≠j.

If we multiply two inverse vectors between them, in non-orthogonal metric, we do not obtain zero as a general case. See below:

$$e^i \cdot e^j = \frac{1}{2}(e^ie^j + e^je^i) = g^{ij} = g^{ji}$$

So:

$$e^ie^j = 2g^{ij} - e^je^i$$

And:

$$e^ie^i = (e^i)^2 = e^i \cdot e^i = g^{ii}$$

In this paper, we will work mainly with orthogonal (or orthonormal basis), so do not worry about these above points. For more info regarding how to invert the metric you have a lot of literature [58][59][60][61][62][64].

What we will do in general, is to make all the calculations with orthogonal metrics and then try to generalize to the case of non-orthogonal metric applying the above relations.

3.4. Geometric Algebra $Cl_{3,0}$. Sum of geometric products of basis vectors

We will calculate the following sum. Take into account that the product inside the sum is geometric (not scalar) and that we have not imposed anything regarding the basis (it can be not orthogonal).

$$S = \sum_{i=1}^{3} \sum_{j=1}^{3} e_i e_j$$

If we operate, we get:

$$S = e_1 e_1 + e_1 e_2 + e_1 e_3 +$$

$$+ e_2 e_1 + e_2 e_2 + e_2 e_3 +$$

$$+ e_3 e_1 + e_3 e_2 + e_3 e_3 =$$

$$e_1 e_1 + e_2 e_2 + e_3 e_3 +$$

$$+ (e_1 e_2 + e_2 e_1) +$$

$$+ (e_2 e_3 + e_3 e_2) +$$

$$+ (e_3 e_1 + e_1 e_3) =$$

$$e_1 \cdot e_1 + e_2 \cdot e_2 + e_3 \cdot e_3 +$$

$$+ 2(e_1 \cdot e_2) +$$

$$+ 2(e_2 \cdot e_3) +$$

$$+ 2(e_3 \cdot e_1)$$

As the scalar product is always symmetric (independently if the basis is orthogonal or not) we can convert the elements that are multiplied by 2, in the sum of two scalar products reversed (with the same result).

$$S = e_1 \cdot e_1 + e_2 \cdot e_2 + e_3 \cdot e_3 +$$

$$+ e_1 \cdot e_2 + e_2 \cdot e_1 +$$

$$+ e_2 \cdot e_3 + e_3 \cdot e_2$$

$$+ e_3 \cdot e_1 + e_1 \cdot e_3 =$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} e_i \cdot e_j = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}$$

So:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} e_i e_j = \sum_{i=1}^{3} \sum_{j=1}^{3} e_i \cdot e_j = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}$$

As commented, this holds, independently of the type of metric. And in fact, it holds even for more than three dimensions, but I have preferred to do it explicitly for three dimensions to avoid any doubt and avoid getting lost in the subindices.

Now, consider a symmetric tensor (or a symmetric matrix if you want) that have the components a^{ij}:

$$a^{ij} = a^{ji}$$

And now want to perform the sum (don't worry, I will explain the reason of all this later):

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a^{ij} e_i e_j$$

Making the same calculation as above (and only if a^{ij} is symmetric) we will obtain a similar result:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a^{ij} e_i e_j = \sum_{i=1}^{3} \sum_{j=1}^{3} a^{ij} (e_i \cdot e_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} a^{ij} g_{ij}$$

Or using the Einstein notation to simplify:

$$a^{ij} e_i e_j = a^{ij}(e_i \cdot e_j) = a^{ij} g_{ij} \qquad only\ if\ a^{ij} = a^{ji}$$

Similarly, we can obtain:

$$a_{ij} e^i e^j = a_{ij}(e^i \cdot e^j) = a_{ij} g^{ij} \qquad only\ if\ a_{ij} = a_{ji}$$

But if:

$$a_i^j e^i e_j = a_i^j (e^i \cdot e_j) = a_i^j \delta_j^i = a_i^i \qquad only\ if\ a_i^j = a_j^i$$

$$a_i^j e_j e^i = a_i^j (e_j \cdot e^i) = a_i^j \delta_j^i = a_i^i \qquad only\ if\ a_i^j = a_j^i$$

Where the last move of above equations is a property of the Kronecker Delta that you can check in [59][60][61][62].

3.5. Geometric Algebra $Cl_{3,0}$. Expanding the basis

One of the properties of the Geometric Algebra is that the number of elements that conform the algebra of a certain realm are more than the number of dimensions of that realm. In three dimensions we have three basis vectors as commented, but we have 8 different elements that conform that algebra, that are:

- The scalars
- The three vectors
- The three bivectors
- One trivector

We will call these elements with these names:

$$\begin{gathered} e_0 \rightarrow scalars \\ e_1 \\ e_2 \\ e_3 \\ e_4 = e_2 e_3 \\ e_5 = e_3 e_1 \\ e_6 = e_1 e_2 \\ e_7 = e_1 e_2 e_3 \end{gathered}$$

Regarding e_0 I will comment later. In Geometric Algebra probably you would expect $e_0=1$. And this is the natural move, but I will come back to this later, as commented.

The elements e_4, e_5, e_6 are bivectors whose square is negative, as we will see now. And e_7 is the trivector whose square is also negative, as we will see.

In general, we will work with orthogonal (not necessarily orthonormal) basis. About the non-orthogonal case, we will talk explicitly in certain points of the paper. If nothing is said, along the paper we will work with orthogonal metric that fulfills the following, already commented, relations:

$$(e_i)^2 = e_i e_i = e_i \cdot e_i = \|e_i\|^2 = g_{ii}$$
$$e_i e_j = e_i \wedge e_j = -e_j \wedge e_i = -e_j e_i$$
$$e_i \cdot e_j = e_j \cdot e_i = 0 \qquad (when\ i \neq j)$$

This is, in 3 dimensions:

$$(e_1)^2 = e_1 e_1 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = e_2 e_2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = e_3 e_3 = \|e_3\|^2 = g_{33}$$
$$e_1 e_2 = -e_2 e_1$$
$$e_2 e_3 = -e_3 e_2$$
$$e_3 e_1 = -e_1 e_1$$

The last three equations are key in orthogonal metric and are the ones that will make working with bivectors or the trivector much easier. Because they permit us to swap the order of the vectors in any geometric product, just adding a minus sign for each swap. These means that the result will be the same if we make an even number of swaps. And will be the negative of the original if we make an odd number of swaps.

An example. We have the following trivectors and we want to sum them:

$$7e_1e_2e_3 + 2e_2e_1e_3$$

We swap e_2 and e_1 in the second element and we add a minus sign. This is the same as using one of the equations above.

$$7e_1e_2e_3 - 2e_1e_2e_3 = 5e_1e_2e_3$$

But, take into account that when a basis vector is squared, it is converted to a number, so it does not count as a vector anymore. It is just a number that can be moved in the product not changing signs. For example:

$$7e_1e_2e_3e_2 + 2e_1e_3$$

We swap e_3 and the last e_2 in the first element, adding a minus sign.

$$-7e_1e_2e_2e_3 + 2e_3e_1$$

Now, we perform the square of e_2, getting its norm and converting it into a number.

$$-7e_1(e_2)^2e_3 + 2e_3e_1 = -7e_1\|e_2\|^2e_3 + 2e_3e_1 = -7e_1g_{22}e_3 + 2e_3e_1$$

Now, g_{22} is just a number, so I can move to the beginning of the element (not changing the sign), we are moving a number, a scalar, not a vector:

$$-7e_1g_{22}e_3 + 2e_3e_1 = -7g_{22}e_1e_3 + 2e_3e_1$$

And now, we exchange e_1 and e_3 in the first element and yes now, we have to add a minus sign (multiply by -1).

$$-7g_{22}e_1e_3 + 2e_3e_1 = 7g_{22}e_3e_1 + 2e_3e_1 = (7g_{22} + 2)e_3e_1$$

If instead, we swap the e_1 and e_3 in the second element we get:

$$-7g_{22}e_1e_3 + 2e_3e_1 = -7g_{22}e_1e_3 - 2e_1e_3 = (-7g_{22} - 2)e_1e_3 = -(7g_{22} + 2)e_1e_3$$

This is the negative as the first result, but take into account that the vectors that multiply are reversed, so in fact, it is the same result. I could swap them and change the sign again and both results will be the same.

Another way to see it is using the nomenclature we have defined in the beginning of the chapter:

$$(7g_{22} + 2)e_3e_1 = (7g_{22} + 2)e_5$$

But in the second case, we have to reverse to be able to use that nomenclature. Swapping the vectors and adding a minus sign (changing the sign):

$$-(7g_{22} + 2)e_1e_3 = -\big(-(7g_{22} + 2)\big)e_3e_1 = (7g_{22} + 2)e_3e_1 = (7g_{22} + 2)e_5$$

For more info regrading this type of operations you can check [1][2][3][4][5][6].

As commented, all these swapping's with changing of sign can only be applied in orthogonal bases. In non-orthogonal bases you should apply the equations in the beginning of chapter. 3.3.

Knowing this rule, I would just show the squares of the bivectors and the trivector to check that they are in fact negative:

$$(e_4)^2 = (e_2e_3)^2 = e_2e_3e_2e_3 = -e_2e_3e_3e_2 = -e_2g_{33}e_2 = -g_{33}e_2e_2 = -g_{33}g_{22}$$
$$(e_5)^2 = (e_3e_1)^2 = e_3e_1e_3e_1 = -e_3e_1e_1e_3 = -e_3g_{11}e_3 = -g_{11}e_3e_3 = -g_{11}g_{33}$$
$$(e_6)^2 = (e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_2e_2e_1 = -e_1g_{22}e_1 = -g_{22}e_1e_1 = -g_{22}g_{11}$$
$$(e_7)^2 = (e_1e_2e_3)^2 = e_1e_2e_3e_1e_2e_3 = +e_1e_2e_3e_3e_1e_2 = g_{33}e_1e_2e_1e_2 = -g_{33}e_1e_1e_2e_2 = -g_{33}g_{11}g_{22}$$

Remind that the g_{ij} are just numbers, so you can move them as you want along the product. I keep the order obtained in the operations to facilitate the understanding, but you can swap them as you want not changing the sign or the result.

Just to close the chapter, I will comment that an entity that is composed by the sum of scalars, vectors, bivectors etc… is called a multivector. As an example:

$$A = 3 + 2e_1 - 3e_1 + 7e_3e_1$$

This entity A is called a multivector. We will see that in Geometric Algebra any object can be defined by a multivector expression.

The most important comment of this section is the following. In Geometric Algebra, once you have defined the number of dimensions (in this case 3) and the consequent degrees of freedom (or different basis vectors and their combinations, in this case 8, from e_0 to e_7), it does not matter how many operations (sums, geometric products, even exponentials etc…) you do, the number of basis vectors and their combinations are always the same (8 in this case). You can multiply the times you want any multivector by another one, you will only finish with 8 coefficients that multiply 8 basis vectors from e_0 to e_7 (considering also basis vectors their product combinations). Nothing else. This is key in Geometric Algebra and its power.

If you are familiarized with matrices, tensors or tensors products, you know that in those cases the number of elements could grow to infinite (the number of dimensions also). In Geometric Algebra, there is a limit. And this KEY as we will see.

3.6. Geometric Algebra $Cl_{3,0}$. Comments about e_0 and e_7

Before, I have commented that the natural move is that:

$$e_0 = 1$$

And in general, this is what I would have written in any of my previous papers. But in this case, as we will see later, it is possible that we need a "degree of freedom more" or the possibility that e_0 is a scalar function that depends on certain parameters that we will see later.

So, instead of defining e_0 equal to 1, we will define it as a scalar (this is important, it is a scalar or a function whose output is a scalar, not vectors, not bivectors etc…):

$$e_0 = \sqrt{g_{00}}$$

So:

$$(e_0)^2 = \|e_0\|^2 = g_{00}$$

As commented g_{00}, is a scalar or a function that outputs a scalar (positive-definite). The problem is the conceptual meaning of e_0 and g_{00}. Normally g_{00} would mean the scalar product of vectors. In this case, it is not that. It is a function that appear only at certain operations that we will see later.

Regarding the possible values of g_{00} are (we will comment later):

$$g_{00} = 1$$

$$g_{00} = \|e_1\|^2\|e_2\|^2\|e_3\|^2$$

$$g_{00} = \frac{1}{\|e_1\|^2\|e_2\|^2\|e_3\|^2}$$

$$g_{00} = independent\ scalar\ function\ (positive\ definite)$$

As commented, we will keep this nomenclature of g_{00} as in the end it is discovered that it is equal to 1 or to whatever other result we will substitute in the equations. If we put directly that it is equal to 1, it will be more difficult to modify the equations.

Anyhow, for the shake of simplicity for orthonormal metric, we will consider e_0=1 as it should be, except exceptional situations. For other metrics, we will keep it indicated as e_0.

Regarding e_7 the important property as commented is this:

$$(e_7)^2 = (e_1e_2e_3)^2 = e_1e_2e_3e_1e_2e_3 = -g_{33}g_{11}g_{22}$$

This means, its square is negative, and it is a "neutral" vector. Meaning "neutral" that it does not have any "preferred" direction or orientation. The bivectors e_4, e_5, e_6 have also negative square but with "preferred" directions.

$$(e_4)^2 = (e_2e_3)^2 = e_2e_3e_2e_3 = -g_{33}g_{22}$$
$$(e_5)^2 = (e_3e_1)^2 = e_3e_1e_3e_1 = -g_{11}g_{33}$$

$$(e_6)^2 = (e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -g_{22} g_{11}$$

But e_7 has a negative square and does not point anywhere specific. It applies to the volume in general (not a surface or a line). If you have read the papers [4][5][6] probably you have already seen the possibility that the time vector can be associated with e_7 (the trivector). The reason is that the square of e_7 is negative and that taking this consideration is completely coherent with Dirac Equation, Maxwell equations and Gell-Mann matrices [5][6][26][63].

When we come to general relativity, the thing gets more complicated. We will see that depending on the context, the scalars e_0 (as considered in APS[43]) or the trivector e_7 can represent time depending on the context. We will see later, but first we need to understand the spinor in Geometric Algebra to understand the different possible contexts.

What we will keep from previous papers [4][5][6][26][63]is that as the square of e_7 is negative and does not have any preferred direction, when the imaginary unit *i* is used in traditional algebra, we will substitute it in Geometric Algebra by the trivector e_7. The reason is that in Geometric Algebra there are already elements as e_7 (appearing in a natural way) whose square is negative.

And the imaginary unit *i* is used in traditional algebra as an "unknown or generic" element whose square is negative. In Geometric Algebra, what you have to do is, depending on the context, to use the corresponding already exiting element in the Algebra (of all the ones whose square is negative) instead of using *i*. As commented, we will used e_7 for the reasons commented above.

4. The reverse of a multivector and the reverse product

If we have multivector, the reverse of it can be defined as a multivector with the same coefficients but where all the products of basis vectors are reversed. An example:

$$A = 3 + 2e_1 - 3e_1 + 7e_3 e_1 + 2e_2 e_3 - 5e_1 e_2 e_3$$

Its reverse will be:

$$A^\dagger = 3 + 2e_1 - 3e_1 + 7e_1 e_3 + 2e_2 e_3 - 5e_3 e_2 e_1$$

This in orthogonal metric (not in general) can be converted using chapter 3.2 equations into:

$$A^\dagger = 3 + 2e_1 - 3e_1 - 7e_3 e_1 - 2e_2 e_3 + 5e_1 e_2 e_3 = A^*$$

Being A^* the conjugate multivector. This means, in orthogonal metric the reverse of a multivector is the same as a conjugate of the multivector. The conjugate means changing the sign of the elements whose square is negative (this means: bivectors and trivector) and keeping the same sign for scalars and vectors (whose square is positive)

In a non-orthogonal metric, you should use equations in chapter 3.3 instead of those in chapter 3.2, so in a general case, reverse and conjugate will not be the same.

Anyhow, as commented, in this paper we will focus on orthogonal basis, so here reverse and conjugate will be the same in most cases (but this is not true for a general case).

Calculating the reverse for the different basis vectors, we have:

$$e_0^\dagger = e_0$$
$$e_1^\dagger = e_1$$
$$e_2^\dagger = e_2$$

$$e_3^\dagger = e_3$$
$$e_4^\dagger = (e_2e_3)^\dagger = e_3e_2$$
$$e_5^\dagger = (e_3e_1)^\dagger = e_1e_3$$
$$e_6^\dagger = (e_1e_2)^\dagger = e_2e_1$$
$$e_7^\dagger = (e_1e_2e_3)^\dagger = e_3e_2e_1$$

One important property is that a product of basis vectors multiplied by its reverse is always positive definite (also in non-orthogonal metrics):

$$e_0e_0^\dagger = e_0e_0 = \|e_0\|^2 = g_{00}$$
$$e_1e_1^\dagger = e_1e_1 = \|e_1\|^2 = g_{11}$$
$$e_2e_2^\dagger = e_2e_2 = \|e_2\|^2 = g_{22}$$
$$e_3e_3^\dagger = e_3e_3 = \|e_3\|^2 = g_{33}$$
$$e_4e_4^\dagger = e_2e_3(e_2e_3)^\dagger = e_2e_3e_3e_2 = e_2g_{33}e_2 = g_{33}e_2e_2 = g_{33}g_{22} \equiv g_{44}$$
$$e_5e_5^\dagger = e_3e_1(e_3e_1)^\dagger = e_3e_1e_1e_3 = e_3g_{11}e_3 = g_{11}e_3e_3 = g_{11}g_{33} \equiv g_{55}$$
$$e_6e_6^\dagger = e_1e_2(e_1e_2)^\dagger = e_1e_2e_2e_1 = e_1g_{22}e_1 = g_{22}e_1e_1 = g_{22}g_{11} \equiv g_{66}$$
$$e_7e_7^\dagger = e_1e_2e_3(e_1e_2e_3)^\dagger = e_1e_2e_3e_3e_2e_1 = g_{33}e_1e_2e_2e_1 = g_{33}g_{22}e_1e_1 = g_{33}g_{22}g_{11} \equiv g_{77}$$

Where I have defined the g_{ii} as the result of these products also for basis vectors with i>3. And also, as commented it is defined a g_{00} as the square for e_0 to have one degree of freedom more (even that very probably defining it as 1, should be ok, meaning just a that pre-normalization has been de-facto done).

As you can guess, the reverse product is just defined as multivector by the reverse of other (or the same) multivector following the rules commented above.

An important thing to comment, is that the reverse should not be mixed up with the inverse.

The inverse of a product of basis vectors is defined as the inverse of each basis vector in reverse order. This is, for example:

$$(e_7)^{-1} = (e_1e_2e_3)^{-1} = (e_3)^{-1}(e_2)^{-1}(e_1)^{-1} = e^3e^2e^1 = e^7$$

Where in the last steps above, I have used the definition of the superscripts as defined in chapters 3.1, 3.2 and 3.3, as the inverse of the basis vectors. We can check that this hold:

$$e_7e^7 = e_1e_2e_3e^3e^2e^1 = e_1e_2 \cdot 1 \cdot e^2e^1 = e_1 \cdot 1 \cdot e^1 = 1$$

So, in fact, it corresponds to the inverse of e_7. The same applies, to the rest of vectors:

$$(e_1)^{-1} = e^1$$

$$(e_2)^{-1} = e^2$$

$$(e_3)^{-1} = e^3$$

$$(e_4)^{-1} = (e_2e_3)^{-1} = (e_3)^{-1}(e_2)^{-1} = e^3e^2 = e^4$$

$$(e_5)^{-1} = (e_3e_1)^{-1} = (e_1)^{-1}(e_3)^{-1} = e^1e^3 = e^5$$

$$(e_6)^{-1} = (e_1e_2)^{-1} = (e_2)^{-1}(e_1)^{-1} = e^2e^1 = e^6$$

$$(e_7)^{-1} = (e_1e_2e_3)^{-1} = (e_3)^{-1}(e_2)^{-1}(e_1)^{-1} = e^3e^2e^1 = e^7$$

So, you can see that the inverse, also reverse the order, but besides that, it inverses the basis vectors (converts the subscripts in superscripts and vice-versa).

5. Spinor in Geometric Algebra $Cl_{3,0}$

A spinor in matrix notation has this form:

$$\psi = \begin{pmatrix} \psi_{1r} + \psi_{1i} i \\ \psi_{2r} + \psi_{2i} i \\ \psi_{3r} + \psi_{3i} i \\ \psi_{4r} + \psi_{4i} i \end{pmatrix}$$

As you can see, it has eight parameters:

$$\psi_{1r} \ \ \psi_{1i} \ \ \psi_{2r} \ \ \psi_{2i} \ \ \psi_{3r} \ \ \psi_{3i} \ \ \psi_{4r} \ and \ \psi_{4i}$$

In Geometric Algebra, the spinor has this form:

$$\psi = \psi^{\mu} e_{\mu} = \psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7$$

Where the e_i are the elements (scalars, vectors, bivectors and trivector) as defined in chapter 3.5.

The ψ^i are the coefficients of the spinor or wavefunction. You can see that they are also eight as in the matrix notation. You can find a relation between both in [5] [31]and [63]. There you can find that that relation is coherent with Dirac Equation and Strong Force Interaction (Gell-Mann matrices).

For this paper we will just stick to that these 8 coefficients are sufficient to define a spinor or wavefunction. And calculating them is what we need to define the state of a particle or a related filed.

6. Probability density and probability current

As we saw in [63] we can calculate probability density and probability current multiplying the reverse of the wavefunction by itself, this way:

$$\psi^{\dagger}\psi = \left(\psi^0 e_0^{\dagger} + \psi^1 e_1^{\dagger} + \psi^2 e_2^{\dagger} + \psi^3 e_3^{\dagger} + \psi^4 e_4^{\dagger} + \psi^5 e_5^{\dagger} + \psi^6 e_6^{\dagger} + \psi^7 e_7^{\dagger}\right)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7)$$

Where all the vectors, bivectors and the trivector and their reverses, are as defined in chapter 4 and previous ones.

Only in the case of orthogonal metric (not in the general case), this can be simplified as (the reverse is the same as the conjugate):

$$\psi^{\dagger}\psi = \psi^{*}\psi = (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7)$$

As you can see in Annex A2, the result of this multiplication is for the orthogonal case is:

$$\psi^{\dagger}\psi = \rho + \vec{j}$$

Being:

$$\rho = (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22} g_{33} + (\psi^5)^2 g_{33} g_{11} + (\psi^6)^2 g_{11} g_{22} + (\psi^7)^2 g_{11} g_{22} g_{33}$$

$$\vec{j} = 2(\psi^0\psi^1 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22}g_{33})e_1 + 2(+\psi^0\psi^2 + \psi^1\psi^6 g_{11} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33}g_{11})e_2 + 2(+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^6\psi^7 g_{11}g_{22})e_3$$

Being ρ the probability and $\vec{j}$ the fermionic current.

But we can say that even in the general case where the basis is not orthogonal or even if the product above is defined another way, the result will have for sure have this form:

$$\psi^\dagger\psi = j^\mu e_\mu$$

In Annexes A1, A2, A3 and A4, you can find that in whatever metric you are or however this product is defined, the result will always have this form:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Where μ and ν go from 0 to 7 in the most general case. This means, independently of the metric, independently if the product is correctly defined or are some elements pending (see Annexes A1, A2, A3 and A4 for details), what it is true is that the result, will have the form above.

Even if we calculate wrongly the coefficients of j^μ, we can continue with our study as these coefficients will represent a general case. In case they change the value, we will change the operations done, but the study following will be perfectly correct as the meaning of the coefficients j^μ is general. This is the power of geometric algebra. We know the form of the results even if we have calculated them wrong. We know that the result will have 8 components j^μ (very important, scalar coefficients or functions that output a scalar) multiplying 8 basis vectors (considering their product combinations also, this means, considering them from e_o to e_7).

Last comment to make are the measuring units of this $j^\mu e_\nu$. For the j^0 component the units are density of probability in 3D space, this means probability/cubic length. Probability does not have units, so it is L^{-3}.

The components j^1 to j^3 are called the probability current and its units are density of probability multiplied by velocity. As probability does not have units, the density has L^{-3} and the speed has LT^{-1}, the total units are $L^{-2}T^{-1}$. To make these units coherent with j^0, we have to multiply j^0 by c (the speed of light) or the opposite, to divide the components of j^1 to j^3 by it.

As commented, for orthogonal basis, j^μ only has components from 0 to 3. For the general case, it would have components from 0 to 7 and the measuring units should be harmonized with the units that have the components from 0 to 3. But we will not care about that now, we will just consider that we can find a coherent following expression with coherent units:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Just to finalize, I will comment that to be consequent with certain papers in the literature, sometimes I will use the following nomenclature, but you can check that the concept is the same, just changing the name of j to V, and the dummy index form μ to ρ:

$$\psi^\dagger\psi = j^\mu e_\mu = V^\rho e_\rho$$

7. Definition of Covariant Operator in Geometric Algebra

We will define the following operator:

$$e^{\mu}\nabla_{\mu}$$

Where ∇_{μ} is the covariant derivative. This means, if it is applied to a scalar function, it will be just the partial derivative with respect to μ of it. If f is a scalar function:

$$e^{\mu}\nabla_{\mu}f = e^{\mu}\frac{\partial f}{\partial e^{\mu}}$$

Where the partial derivative is taken respect to the coordinate variable that corresponds to the vector e_{μ}. This means, that ∂e^{1} would mean derivative to the coordinate variable associated to e_1 (typically x in cartesian coordinates, or r in polar coordinates or called e^1 in the general case). It is important to recall that in this paper, the coefficients that multiply the vectors ar scalars (not "covectors"), so the rule above, apply to them (to the coefficients). It does not apply to the vectors as you can see below.

If the function includes vectors, apart from the partial derivative of the coefficients that multiply these vectors, we will have to apply the covariant derivative to the vectors.

The covariant derivative of the basis vectors (you can check this in different literature of Relativity or Riemann geometries [58]-[62]) are the Christoffel symbols. So, in a general case:

$$e^{\mu}\nabla_{\mu}(f^{\nu}e_{\nu}) = e^{\mu}\left(\nabla_{\mu}f^{\nu}\right)e_{\nu} + e^{\mu}f^{\nu}\left(\nabla_{\mu}e_{\nu}\right)$$

Where we have used the product rule of the covariant derivative of a product. And we are keeping the same order of the elements (mainly vectors). Remember they are nor commutative in the general case.

Now, for the scalar coefficients f^{ν} we can use the same equation shown before (partial derivative equation). For the other term (the covariant derivative of a basis vector) we will use the Christoffel symbols as they are defined [58]-[62].

$$e^{\mu}\nabla_{\mu}(f^{\nu}e_{\nu}) = e^{\mu}\left(\nabla_{\mu}f^{\nu}\right)e_{\nu} + e^{\mu}f^{\nu}\left(\nabla_{\mu}e_{\nu}\right) = e^{\mu}\frac{\partial f^{\nu}}{\partial e^{\mu}}e_{\nu} + e^{\mu}f^{\nu}\Gamma^{\lambda}_{\mu\nu}e_{\lambda}$$

As the partial derivative of the coefficients of f and the Christoffel symbols are just scalars (yes, in this context, Christoffel symbols are just scalars that multiply vectors) we can move the vectors as follows:

$$e^{\mu}\nabla_{\mu}(f^{\nu}e_{\nu}) = e^{\mu}\frac{\partial f^{\nu}}{\partial e^{\mu}}e_{\nu} + e^{\mu}f^{\nu}\Gamma^{\lambda}_{\mu\nu}e_{\lambda} = e^{\mu}e_{\nu}\frac{\partial f^{\nu}}{\partial e^{\mu}} + e^{\mu}e_{\lambda}f^{\nu}\Gamma^{\lambda}_{\mu\nu}$$

Another thing to comment is that we can calculate also the covariant derivative of the inverse of a vector this way:

$$\nabla_{\beta}\left(e_{\mu}(e_{\alpha})^{-1}\right) = \nabla_{\beta}\left(e_{\mu}e^{\alpha}\right) = \nabla_{\beta}\left(\delta^{\alpha}_{\mu}\right) = 0$$
$$\nabla_{\beta}\left(e_{\mu}\right)e^{\alpha} + e_{\mu}\nabla_{\beta}(e^{\alpha}) = \Gamma^{\lambda}_{\beta\mu}e_{\lambda}e^{\alpha} + e_{\mu}\nabla_{\beta}(e^{\alpha}) = 0$$
$$e_{\mu}\nabla_{\beta}(e^{\alpha}) = -\Gamma^{\lambda}_{\beta\mu}e_{\lambda}e^{\alpha}$$
$$e_{\mu}\nabla_{\beta}(e^{\alpha}) = -\Gamma^{\lambda}_{\beta\mu}\delta^{\alpha}_{\lambda}$$

$$e_{\mu}\nabla_{\beta}(e^{\alpha}) = -\Gamma^{\alpha}_{\beta\mu}$$
$$e^{\mu}e_{\mu}\nabla_{\beta}(e^{\alpha}) = -\Gamma^{\alpha}_{\beta\mu}e^{\mu}$$
$$\nabla_{\beta}(e^{\alpha}) = -\Gamma^{\alpha}_{\beta\mu}e^{\mu}$$

So, this above, added with classical definition covariant derivative of basis vector:

$$\nabla_{\beta}(e_{\alpha}) = \Gamma^{\mu}_{\beta\alpha}e_{\mu}$$

They are the ones we will need in following chapters. Also, to comment something that we will need in some steps. The geometric product is not commutative in general. But sometimes we will have to commute the vectors. To do so, we have to consider one of these three scenarios:

- The metric is orthogonal. So, the geometric product is the same as scalar product, and therefore commutative.
- We are in a situation as in chapter 3.4. This is, the symmetry of the sums in certain situations, "convert" the geometric products in scalar products. So, the same as commented above applies.
- The other option is directly that we are forced to change the definition of the operators, using scalar products instead of geometric products. As an example, in certain situations, we can say, instead of using the operator:

$$e^{\mu}\nabla_{\mu}$$

  We could decide to use:

$$e^{\mu} \cdot \nabla_{\mu}$$

  Loosing generality (all the non-commutative elements will be lost), rigor and probably some solutions, but as a way to move forward.

Just to finish we will define the reverse (the reverse not the inverse) of the covariant operator to a function f as:

$$\left(e^{\mu}\nabla_{\mu}f\right)^{\dagger} = f\nabla_{\mu}^{\dagger}e^{\mu} = \left(f\nabla_{\mu}^{\dagger}\right)e^{\mu} = \left(\nabla_{\mu}f\right)e^{\mu}$$

This means, when we see the reverse operator, we have to take into account these things:

- The operator applies to the function on the **left** of it (not on the right as it is usual).
- The vector that accompanies it, it is located on the right of the operator, not on the left as defined from the non-reverse operator.

Probably you are asking why the vector that accompanies the function is not reversed as well. In general, I would say that the logic thing would be to reverse it. Creating sometimes changes on signs (or even real changes in result in non-orthogonal metric). In this paper I will keep it as not reversed to facilitate the things and the message, but it could be that in the future, the definition, changes to reversed.

Also, you can ask why the f is not reversed as well. The answer is that to keep the symmetry, it should be reversed. But to simplify the nomenclature, we will keep f not reversed, and just indicate it directly if this is the case.

Another thing we could think about is that if the operator is reversed, we should add a minus sign to the derivative as we are deriving in the opposite direction to the one represented by the variable. This is true in fact. But as we will always make double derivatives (in the left and in the right, see later), in the end, this will only lead to a change of sign in the final results, not affecting the implicit meaning. Anyhow, this is something that probably has to be taken into account in the future (and also if it is needed or not to reverse the vectors that accompany the derivative/del operator).

The last comment is that in Geometric Algebra everything is done keeping symmetries. When a double operator has to be applied (like a Laplacian) it is not generally done as a double operator on the left. Instead, it is done like a simple operator in the left and another simple operator on the right (that is applying to the elements on the left).

The reason for this is that in geometric algebra the order of the vectors matters. As it is not the same pre-multiplying than post-multiplying. Because the products are not in general commutative or anticommutative, it depends on the product itself (the number of vectors and its grade). So, the only way to keep the symmetries is to keep the balance of operators on the left and in the right as much as possible.

When this happens, we will have the convention that we will start applying the reverse del operator (the one in the right, and afterwards the non-reverse del operator, the one in the left). This is just by convention. Taking into account that normally we work with commutators in our calculations, a change of this will only lead to a change of signs in the final results.

Apart from this, this will let us also facilitate the factorization of the equations that will be key to simplify them in following chapters.

8. Ricci tensor in Geometric Algebra

As we can see in different papers, the Ricci Tensor can be considered as the Laplacian of the basis vectors. Taking into account what we have commented about the covariant derivative in the previous chapter, we can calculate the Laplacian as a covariant derivative on the left and another covariant derivative on the right. And to be in the most general case as possible, instead of applying to the basis vectors, I will apply to a complete field that includes coefficients and vectors:

$$V^\rho e_\rho$$

If you want to apply only to basis vectors just consider:

$$V^\rho = 1 \, for \, every \, \rho$$

And:

$$V^\rho_{,\mu} = 0$$

Where the comma represents partial derivative with respect to e^μ.

Ok, so let's apply the operator defined in chapter 7 to $V^\rho e_\rho$ to the left and the reverse of it, to the right. We will start operating the one of the right (the reverse operator). This is just by convention as commented in chapter 7. If we do the opposite, we will obtain a different result. But we will see that it does not even really matters, as we will perform also this operation later.

$$e^\mu \nabla_\mu V^\rho e_\rho \nabla^\dagger_\nu e^\nu$$

$$e^\mu \nabla_\mu \left(\left(V^\rho e_\rho \nabla^\dagger_\nu \right) e^\nu \right)$$

$$e^\mu \nabla_\mu \left(\nabla_\nu V^\rho e_\rho \right) e^\nu$$

$$e^\mu \nabla_\mu \left(\left(\nabla_\nu V^\rho e_\rho \right) e^\nu \right)$$

Very important to remark the coefficients V^ρ are just scalars. Their covariant derivative is just the partial derivative.

And for the vectors, we will apply the equations shown in chapter seven:

$$\nabla_\beta (e_\alpha) = \Gamma^\mu_{\beta\alpha} e_\mu$$

$$\nabla_\beta (e^\alpha) = -\Gamma^\alpha_{\beta\mu} e^\mu$$

And to remark that in this context, the Christoffel symbols are just scalar coefficients, that multiply vectors. So, the covariant derivative of the Christoffel symbol itself is the partial derivative. The covariant of the vectors that accompany them will be done naturally following the derivative product rule.

We start calculating, the expression inside the brackets:

$$\nabla_\nu V^\rho e_\rho = V^\rho_{,\nu} e_\rho + V^\rho \Gamma^\sigma_{\nu\rho} e_\sigma$$

I change the name of the dummy coefficients for convenience and to follow [57]:

$$\nabla_\nu V^\rho e_\rho = V^\rho_{,\nu} e_\rho + V^\sigma \Gamma^\rho_{\nu\sigma} e_\rho$$

Now I just post-multiply by the vector that appeared in the original equation at the beginning of the paper:

$$\left(\nabla_\nu V^\rho e_\rho\right) e^\nu = V^\rho_{,\nu} e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma} e_\rho e^\nu$$

Now, I proceed with the covariant derivative that was in the left (that applies to all the expression above, including the two vectors):

$$\begin{gathered}\nabla_\mu \left(\left(\nabla_\nu V^\rho e_\rho\right) e^\nu \right) = \nabla_\mu \left(V^\rho_{,\nu} e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma} e_\rho e^\nu \right) = \\ V^\rho_{,\nu\mu} e_\rho e^\nu + V^\rho_{,\nu} \Gamma^\sigma_{\rho\mu} e_\sigma e^\nu - V^\rho_{,\nu} e_\rho \Gamma^\nu_{\mu\sigma} e^\sigma + \\ + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma} \Gamma^\lambda_{\rho\mu} e_\lambda e^\nu - V^\sigma \Gamma^\rho_{\nu\sigma} e_\rho \Gamma^\nu_{\mu\lambda} e^\lambda =\end{gathered}$$

I change again the name of dummy variables to follow [57] nomenclature:

$$\begin{gathered}V^\rho_{,\nu\mu} e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e_\rho e^\nu - V^\rho_{,\lambda} e_\rho \Gamma^\lambda_{\mu\nu} e^\nu + \\ + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e_\rho e^\nu + V^\sigma \Gamma^\lambda_{\nu\sigma} \Gamma^\rho_{\lambda\mu} e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} e_\rho \Gamma^\lambda_{\mu\nu} e^\nu\end{gathered}$$

$$\begin{gathered}V^\rho_{,\nu\mu} e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e_\rho e^\nu - V^\rho_{,\lambda} \Gamma^\lambda_{\mu\nu} e_\rho e^\nu + \\ + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e_\rho e^\nu + V^\sigma \Gamma^\lambda_{\nu\sigma} \Gamma^\rho_{\lambda\mu} e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\mu\nu} e_\rho e^\nu\end{gathered}$$

Now, we pre-multiply by the vector as it was stated in original equation in the beginning of the chapter:

$$\begin{gathered}e^\mu \nabla_\mu V^\rho e_\rho \nabla^\dagger_\nu e^\nu = \\ e^\mu \nabla_\mu \left(\left(\nabla_\nu V^\rho e_\rho\right) e^\nu \right) = \\ V^\rho_{,\nu\mu} e^\mu e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\rho_{,\lambda} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu + \\ + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\lambda_{\nu\sigma} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu\end{gathered}$$

Now, we calculate the result with the operations reversed. This is, the operator on the left with respect to ν and the reverse operator in the right with respect to μ:

$$\begin{gathered}e^\nu \nabla_\nu V^\rho e_\rho \nabla^\dagger_\mu e^\mu = \\ e^\nu \nabla_\nu \left(\left(\nabla_\mu V^\rho e_\rho\right) e^\mu \right) = \\ V^\rho_{,\mu\nu} e^\nu e_\rho e^\mu + V^\lambda_{,\mu} \Gamma^\rho_{\lambda\nu} e^\nu e_\rho e^\mu - V^\rho_{,\lambda} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu + \\ + V^\sigma_{,\nu} \Gamma^\rho_{\mu\sigma} e^\nu e_\rho e^\mu + V^\sigma \Gamma^\rho_{\mu\sigma,\nu} e^\nu e_\rho e^\mu + V^\sigma \Gamma^\lambda_{\mu\sigma} \Gamma^\rho_{\lambda\nu} e^\nu e_\rho e^\mu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu\end{gathered}$$

Noe, let’s calculate the subtraction of one to another (let’s say the commutator of this operation):

$$\begin{gathered}e^\mu \nabla_\mu V^\rho e_\rho \nabla^\dagger_\nu e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla^\dagger_\mu e^\mu = \\ V^\rho_{,\nu\mu} e^\mu e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\rho_{,\lambda} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu + \\ + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\lambda_{\nu\sigma} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu - \\ - V^\rho_{,\mu\nu} e^\nu e_\rho e^\mu - V^\lambda_{,\mu} \Gamma^\rho_{\lambda\nu} e^\nu e_\rho e^\mu + V^\rho_{,\lambda} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu + \\ - V^\sigma_{,\nu} \Gamma^\rho_{\mu\sigma} e^\nu e_\rho e^\mu - V^\sigma \Gamma^\rho_{\mu\sigma,\nu} e^\nu e_\rho e^\mu - V^\sigma \Gamma^\lambda_{\mu\sigma} \Gamma^\rho_{\lambda\nu} e^\nu e_\rho e^\mu + V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu =\end{gathered}$$

To be able to perform, this operation we have to be able to “move” vectors inside the products. This can only be done if we are in one of three cases commented in chapter 7.

So, we will consider that we are in one of these three cases (the most typical, we are in orthogonal metric) and let's move the position of the vectors inside the products at our convenience:

$$e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu}-e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}=$$
$$V^{\rho}_{,\nu\mu}e^{\mu}e_{\rho}e^{\nu}+V^{\lambda}_{,\nu}\Gamma^{\rho}_{\lambda\mu}e^{\mu}e_{\rho}e^{\nu}-V^{\rho}_{,\lambda}\Gamma^{\lambda}_{\mu\nu}e^{\mu}e_{\rho}e^{\nu}+$$
$$+V^{\sigma}_{,\mu}\Gamma^{\rho}_{\nu\sigma}e^{\mu}e_{\rho}e^{\nu}+V^{\sigma}\Gamma^{\rho}_{\nu\sigma,\mu}e^{\mu}e_{\rho}e^{\nu}+V^{\sigma}\Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho}_{\lambda\mu}e^{\mu}e_{\rho}e^{\nu}-V^{\sigma}\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\nu}e^{\mu}e_{\rho}e^{\nu}-$$
$$-V^{\rho}_{,\mu\nu}e^{\mu}e_{\rho}e^{\nu}-V^{\lambda}_{,\mu}\Gamma^{\rho}_{\lambda\nu}e^{\mu}e_{\rho}e^{\nu}+V^{\rho}_{,\lambda}\Gamma^{\lambda}_{\nu\mu}e^{\mu}e_{\rho}e^{\nu}+$$
$$-V^{\sigma}_{,\nu}\Gamma^{\rho}_{\mu\sigma}e^{\mu}e_{\rho}e^{\nu}-V^{\sigma}\Gamma^{\rho}_{\mu\sigma,\nu}e^{\mu}e_{\rho}e^{\nu}-V^{\sigma}\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho}_{\lambda\nu}e^{\mu}e_{\rho}e^{\nu}+V^{\sigma}\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\mu}e^{\mu}e_{\rho}e^{\nu}=$$

We see that the only elements left (the ones that do not cancel) are the ones in bold. See [57] for more info.

$$V^{\rho}_{,\nu\mu}e^{\mu}e_{\rho}e^{\nu}+V^{\lambda}_{,\nu}\Gamma^{\rho}_{\lambda\mu}e^{\mu}e_{\rho}e^{\nu}-V^{\rho}_{,\lambda}\Gamma^{\lambda}_{\mu\nu}e^{\mu}e_{\rho}e^{\nu}+$$
$$+V^{\sigma}_{,\mu}\Gamma^{\rho}_{\nu\sigma}e^{\mu}e_{\rho}e^{\nu}+\boldsymbol{V^{\sigma}\Gamma^{\rho}_{\nu\sigma,\mu}e^{\mu}e_{\rho}e^{\nu}}+\boldsymbol{V^{\sigma}\Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho}_{\lambda\mu}e^{\mu}e_{\rho}e^{\nu}}-V^{\sigma}\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\nu}e^{\mu}e_{\rho}e^{\nu}-$$
$$-V^{\rho}_{,\mu\nu}e^{\mu}e_{\rho}e^{\nu}-V^{\lambda}_{,\mu}\Gamma^{\rho}_{\lambda\nu}e^{\mu}e_{\rho}e^{\nu}+V^{\rho}_{,\lambda}\Gamma^{\lambda}_{\nu\mu}e^{\mu}e_{\rho}e^{\nu}+$$
$$-V^{\sigma}_{,\nu}\Gamma^{\rho}_{\mu\sigma}e^{\mu}e_{\rho}e^{\nu}-\boldsymbol{V^{\sigma}\Gamma^{\rho}_{\mu\sigma,\nu}e^{\mu}e_{\rho}e^{\nu}}-\boldsymbol{V^{\sigma}\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho}_{\lambda\nu}e^{\mu}e_{\rho}e^{\nu}}+V^{\sigma}\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\mu}e^{\mu}e_{\rho}e^{\nu}=$$

This is:

$$=V^{\sigma}\left(\Gamma^{\rho}_{\nu\sigma,\mu}+\Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho}_{\lambda\mu}-\Gamma^{\rho}_{\mu\sigma,\nu}-\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho}_{\lambda\nu}\right)e^{\mu}e_{\rho}e^{\nu}=$$
$$=V^{\sigma}\left(\Gamma^{\rho}_{\nu\sigma,\mu}-\Gamma^{\rho}_{\mu\sigma,\nu}+\Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho}_{\lambda\mu}-\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho}_{\lambda\nu}\right)e^{\mu}e_{\rho}e^{\nu}=$$

As V^{σ} and the Christoffel symbols are just scalars in this context I can move it freely inside the product.

$$=\left(\Gamma^{\rho}_{\nu\sigma,\mu}-\Gamma^{\rho}_{\mu\sigma,\nu}+\Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\nu\sigma}-\Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\mu\sigma}\right)V^{\sigma}e^{\mu}e_{\rho}e^{\nu}=$$
$$=R^{\rho}_{\sigma\mu\nu}V^{\sigma}e^{\mu}e_{\rho}e^{\nu}$$

Where $R^{\rho}_{\sigma\mu\nu}$ is the Riemann tensor, as commented in [57].

Now, if we consider that we are within one of the three cases commented in chapter 7, we can consider that this product is scalar and therefore:

$$e^{\mu}e_{\rho}=e^{\mu}\cdot e_{\rho}=\delta^{\mu}_{\rho}$$

So:

$$R^{\rho}_{\sigma\mu\nu}V^{\sigma}e^{\mu}e_{\rho}e^{\nu}=R^{\rho}_{\sigma\mu\nu}V^{\sigma}\delta^{\mu}_{\rho}e^{\nu}=R^{\mu}_{\sigma\mu\nu}V^{\sigma}e^{\nu}$$

Now checking [57] we can see that the last element is the Ricci tensor.

$$R^{\mu}_{\sigma\mu\nu}V^{\sigma}e^{\nu}=R_{\sigma\nu}V^{\sigma}e^{\nu}$$

So summing up we can say that (in the last step, I have just used the property that dummy indices can be renamed as convenience:

$$e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu}-e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}=R_{\sigma\nu}V^{\sigma}e^{\nu}=R_{\mu\nu}V^{\mu}e^{\nu}$$

If we want to isolate the Ricci tensor, we could do:

$$(R_{\sigma\nu}V^{\sigma}e^{\nu})e_{\nu}V_{\sigma}=R_{\sigma\nu}V^{\sigma}e^{\nu}e_{\nu}V_{\sigma}=R_{\sigma\nu}V^{\sigma}\cdot 1\cdot V_{\sigma}=R_{\sigma\nu}V^{\sigma}V_{\sigma}=R_{\sigma\nu}$$

$$\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu}-e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right)e_{\nu}V_{\sigma}=(R_{\sigma\nu}V^{\sigma}e^{\nu})e_{\nu}V_{\sigma}=R_{\sigma\nu}$$

$$R_{\sigma\nu}=\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu}-e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right)e_{\nu}V_{\sigma}$$

If we want to calculate the Ricci scalar[57]-[62], we can do:

$$\begin{aligned} R = g^{\sigma\nu}R_{\sigma\nu} &= g^{\sigma\nu}\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right)e_{\nu}V_{\sigma} \\ &= \left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right)g^{\sigma\nu}e_{\nu}V_{\sigma} \end{aligned}$$

Another way to obtain it (but not isolating it):

$$\begin{gathered} \left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right) = R_{\sigma\nu}V^{\sigma}e^{\nu} \\ g^{\sigma\lambda}g^{\nu\theta}\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right) = g^{\sigma\lambda}g^{\nu\theta}R_{\sigma\nu}V^{\sigma}e^{\nu} \\ g^{\sigma\lambda}g^{\nu\theta}\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right) = R^{\lambda\theta}V^{\sigma}e^{\nu} \\ g_{\lambda\theta}g^{\sigma\lambda}g^{\nu\theta}\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right) = g_{\lambda\theta}R^{\lambda\theta}V^{\sigma}e^{\nu} \\ g_{\lambda\theta}g^{\sigma\lambda}g^{\nu\theta}\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right) = RV^{\sigma}e^{\nu} \\ g^{\sigma\nu}\left(e^{\mu}\nabla_{\mu}V^{\rho}e_{\rho}\nabla_{\nu}^{\dagger}e^{\nu} - e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla_{\mu}^{\dagger}e^{\mu}\right) = RV^{\sigma}e^{\nu} \end{gathered}$$

9. Klein-Gordon equation of a field

We consider the definition of stress-energy tensor of a scalar field [65]-[67]. We will not use natural units. It is better to use real units with factors so we can control that the measuring units of the variables are coherent:

$$G_{\mu\nu} = T_{\mu\nu} = 2\hbar^2\partial_{\mu}\phi\partial_{\nu}\phi - \hbar^2 g_{\mu\nu}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi + g_{\mu\nu}m^2c^2\phi^2$$

We divide by 2m:

$$T_{\mu\nu} = \frac{\hbar^2}{m}\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi + \frac{1}{2}g_{\mu\nu}mc^2\phi^2$$

It is important to check that the measuring units are coherent. $\frac{\hbar^2}{m}$ units are Energy·L^2. But there are always two derivatives with respect two spatial coordinates that creates a L^{-2}. So, the units of the first two elements are energy. The last element mc^2 is energy also. So, in principle ok. But the stress energy tensor should have units that are Energy·L^{-3}. Do not worry, we will solve this later, as the field that only appears in the right-hand side elements will have L^{-3} units, leaving everything ok.

The first, thing we will do is to apply the operator we defined in chapter 7. But as there are some vectors missing to be able to do that, we will just multiply and divide by them, leaving everything ok.

$$T_{\mu\nu} = \frac{\hbar^2}{m}e_{\mu}e^{\mu}\partial_{\mu}\phi\partial_{\nu}\phi e^{\nu}e_{\nu} - \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_{\alpha}e^{\alpha}\partial_{\alpha}\phi\partial_{\beta}\phi e^{\beta}e_{\beta} + \frac{1}{2}g_{\mu\nu}mc^2\phi^2$$

$$T_{\mu\nu} = \frac{\hbar^2}{m}e_{\mu}\left(e^{\mu}\partial_{\mu}\phi\partial_{\nu}\phi e^{\nu}\right)e_{\nu} - \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_{\alpha}\left(e^{\alpha}\partial_{\alpha}\phi\partial_{\beta}\phi e^{\beta}\right)e_{\beta} + \frac{1}{2}g_{\mu\nu}mc^2\phi^2$$

And here’s the drill. Instead of applying this to a scalar filed as it was original conceived by the equation, we will apply it to a vector field. We have the tools commented in chapters 7 and 8 to make all the operation so we can do it. We will apply to a general field that is:

$$V^{\rho}e_{\rho}$$

And the double derivatives, will be left and reverse right derivatives (keeping the symmetries as always in geometric algebra), instead of two left derivatives.

$$T_{\mu\nu}=\frac{\hbar^2}{m}e_\mu\left(e^\mu\nabla_\mu V^\rho e_\rho\nabla_\nu^\dagger e^\nu\right)e_\nu-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\alpha\nabla_\alpha V^\rho e_\rho\nabla_\beta^\dagger e^\beta\right)e_\beta+\frac{1}{2}g_{\mu\nu}mc^2V^\rho e_\rho$$

I add the following elements to the equation. I can do it, because its sum is zero:

$$-\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)e_\nu+\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta$$
$$+\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\overline{\nabla}_\mu e^\mu\right)e_\nu-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta$$

Once added, we have:

$$\begin{aligned}T_{\mu\nu}=&\frac{\hbar^2}{m}e_\mu\left(e^\mu\nabla_\mu V^\rho e_\rho\nabla_\nu^\dagger e^\nu\right)e_\nu-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\alpha\nabla_\alpha V^\rho e_\rho\nabla_\beta^\dagger e^\beta\right)e_\beta+\frac{1}{2}g_{\mu\nu}mc^2V^\rho e_\rho\\&-\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)e_\nu+\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta\\&+\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)e_\nu-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta\end{aligned}$$

Reordering:

$$\begin{aligned}T_{\mu\nu}=&\frac{\hbar^2}{m}e_\mu\left(e^\mu\nabla_\mu V^\rho e_\rho\nabla_\nu^\dagger e^\nu\right)e_\nu-\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\overline{\nabla}_\mu e^\mu\right)e_\nu\\&-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\alpha\nabla_\alpha V^\rho e_\rho\nabla_\beta^\dagger e^\beta\right)e_\beta\\&+\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta+\frac{1}{2}g_{\mu\nu}mc^2V^\rho e_\rho\\&+\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)e_\nu-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta\end{aligned}$$

Factorizing as possible:

$$\begin{aligned}T_{\mu\nu}=&\frac{\hbar^2}{m}e_\mu\left(e^\mu\nabla_\mu V^\rho e_\rho\nabla_\nu^\dagger e^\nu-e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)e_\nu\\&-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\alpha\nabla_\alpha V^\rho e_\rho\nabla_\beta^\dagger e^\beta-e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta\\&+\frac{1}{2}g_{\mu\nu}mc^2V^\rho e_\rho+\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)e_\nu\\&-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta\end{aligned}$$

Applying the relation to the Ricci tensor commented in 8:

$$\begin{aligned}T_{\mu\nu}=&\frac{\hbar^2}{m}e_\mu\left(R_{\sigma\lambda}V^\sigma e^\lambda\right)e_\nu-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(R_{\sigma\lambda}V^\sigma e^\lambda\right)e_\beta+\frac{1}{2}g_{\mu\nu}mc^2V^\rho e_\rho\\&+\frac{\hbar^2}{m}e_\mu\left(e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)e_\nu-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)e_\beta\end{aligned}$$

Now, again we will suppose that the vectors can be moved inside the product, following one of the three possible cases commented in 7 (orthogonal metric, sum over symmetric elements or defining from the beginning that the products are scalar instead of geometric, losing solutions and rigor).

$$\begin{aligned}T_{\mu\nu}=&\frac{\hbar^2}{m}e_\mu e_\nu\left(R_{\sigma\lambda}V^\sigma e^\lambda\right)-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha e_\beta\left(R_{\sigma\lambda}V^\sigma e^\lambda\right)+\frac{1}{2}g_{\mu\nu}mc^2V^\rho e_\rho\\&+\frac{\hbar^2}{m}e_\mu e_\nu\left(e^\nu\nabla_\nu V^\rho e_\rho\nabla_\mu^\dagger e^\mu\right)-\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}e_\alpha e_\beta\left(e^\beta\nabla_\beta V^\rho e_\rho\nabla_\alpha^\dagger e^\alpha\right)\end{aligned}$$

If the products are scalars (following the three cases in chapter 7) the geometric product of two vectors is the metric (or delta if they are inverse).

$$T_{\mu\nu} = \frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) - \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}g^{\alpha\beta}g_{\alpha\beta}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla^{\dagger}_{\mu}e^{\mu}\right) - \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}g^{\alpha\beta}g_{\alpha\beta}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

Operating:

$$T_{\mu\nu} = \frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) - \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\nu}\nabla_{\nu}V^{\rho}e_{\rho}\nabla^{\dagger}_{\mu}e^{\mu}\right) - \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

Changing the dummy variables names:

$$T_{\mu\nu} = \frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) - \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\beta}\nabla^{\dagger}_{\alpha}e^{\alpha}\right) - \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

Operating:

$$T_{\mu\nu} = \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

$$T_{\mu\nu} = \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

Now I multiply by $e_{\sigma}e^{\sigma}$ to simplify the operations and get to the Ricci scalar. I could obtain the same result, multiplying by $g^{\lambda\sigma}g_{\lambda\sigma}$:

$$T_{\mu\nu} = \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e^{\lambda}e_{\sigma}e^{\sigma}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

Here, I can move the vectors inside the product considering the 3 cases of cahpater 7 (this is not even necessary if I use $g^{\lambda\sigma}g_{\lambda\sigma}$ instead of $e_{\sigma}e^{\sigma}$:

$$T_{\mu\nu} = \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e_{\sigma}e^{\lambda}e^{\sigma}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

$$T_{\mu\nu} = \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(R_{\sigma\lambda}V^{\sigma}e_{\sigma}g^{\lambda\sigma}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

Now, I just change nomenclature of dummy indices:

$$T_{\mu\nu} = \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(g^{\lambda\rho}R_{\rho\lambda}V^{\rho}e_{\rho}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

The following move, I am not sure if it can be done or not. If it cannot be done. Just substitute R by $g^{\lambda\rho}R_{\rho\lambda}$ in the following equations.

$$T_{\mu\nu} = \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(RV^{\rho}e_{\rho}\right) + \frac{1}{2}g_{\mu\nu}mc^2V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

$$T_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R + mc^2\right)V^{\rho}e_{\rho} + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}V^{\rho}e_{\rho}\nabla^{\dagger}_{\alpha}e^{\alpha}\right)$$

Here, it comes another drill. We have seen that the solution to:

$$\psi^{\dagger}\psi = j^{\mu}e_{\mu}$$

And just changing nomenclature, we can consider that it has the form:

$$\psi^\dagger\psi = j^\mu e_\mu = V^\rho e_\rho$$

So why not applying the above equations to $\psi^\dagger\psi$ when appears $V^\rho e_\rho$? This is to apply the equation to collapsed waveform of a particle. This is to its probability and fermionic current. As you know the units of $\psi^\dagger\psi$ is L^{-3}. This is because the probability does not have units, but $\psi^\dagger\psi$ represents the density of probability. This is probability divided by volume (L^{-3}). So here, we solve the issue of the measuring units. They are Energy·L^{-3} in all the elements.

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu}\left(\frac{\hbar^2}{m}R + mc^2\right)\psi^\dagger\psi + \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)$$

$$T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) e_\mu \psi^\dagger\psi e_\nu + \frac{1}{2}\frac{\hbar^2}{m} e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right) e_\nu$$

Oner thing we could do to simplify even more, considering we can move the vectors freely inside the products and that they are scalar multiplied (3 cases of chapter 7) is:

$$T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) e_\mu e_\nu \psi^\dagger\psi + \frac{1}{2}\frac{\hbar^2}{m} e_\mu e_\nu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)$$

$$T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) g_{\mu\nu} \psi^\dagger\psi + \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)$$

Now, we can define a multivector (not even tensor):

$$T = g^{\mu\nu} T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) g^{\mu\nu} g_{\mu\nu} \psi^\dagger\psi + \frac{1}{2}\frac{\hbar^2}{m} g^{\mu\nu} g_{\mu\nu}\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)$$

$$T = g^{\mu\nu} T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) \psi^\dagger\psi + \frac{1}{2}\frac{\hbar^2}{m}\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)$$

Which result is not a scalar. It is a multivector with elements in the eight vectors (scalars, 3 vectors, 3 bivectors and trivector).

Above, the stress-energy tensor is treated as independent of the particle, or the field we are considering. Below, we will see two examples of using this equation, taking into account possible relations between the particle and this tensor.

9. 1 Considering that the stress energy of the particle is the one of a point particle

If we follow [68][69], we can consider the stress energy tensor, just relates to the energy and momentum of the particle. Being coherent with the units, one option could be the energy density of the particle defined by its waveform collapse (squared by its reversed). The units are coherent Energy·L^{-3} and for the cross elements Force·L^{-2} (pressure) that has the same units as Energy·L^{-3}. So, a definition could be:

$$T_{\mu\nu} = mc^2 e_\mu \psi^\dagger \psi e_\nu$$

I remind you that:

$$\psi^\dagger\psi = \psi^\mu e_\mu^\dagger \psi^\nu e_\nu = \left(\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger\right)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) =$$
$$(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_3 e_2 + \psi^5 e_1 e_3 + \psi^6 e_2 e_1 + \psi^7 e_3 e_2 e_1)(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3)$$

So, this is in fact a complicate operation, not a trivial one, with one scalar as result. It has result in all 8 vectors (scalars, 3 vectors, 3 bivectors and the trivector).

You can see in Annexes A1, A2, A3, A4 different examples of the calculation. For example, the most simple on (orthonormal metric) A1, gives:

$$\psi^\dagger\psi = \rho + \vec{j} \qquad (29.1)$$

With:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

And:

$$\vec{j} = 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 + 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2 + 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3$$

So considering the definition of the Stress Energy tensor, as commented above:

$$T_{\mu\nu} = mc^2 e_\mu \psi^\dagger\psi e_\nu$$

And introducing the equation found in the end of chapter 9:

$$T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) g_{\mu\nu}\psi^\dagger\psi + \frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)$$

$$mc^2 e_\mu \psi^\dagger\psi e_\nu = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) e_\mu\psi^\dagger\psi e_\nu + \frac{1}{2}\frac{\hbar^2}{m} e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)e_\nu$$

$$mc^2 e_\mu \psi^\dagger\psi e_\nu - \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right) e_\mu\psi^\dagger\psi e_\nu = \frac{1}{2}\frac{\hbar^2}{m} e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)e_\nu$$

$$-\frac{1}{2}\left(\frac{\hbar^2}{m}R - mc^2\right) e_\mu\psi^\dagger\psi e_\nu = \frac{1}{2}\frac{\hbar^2}{m} e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)e_\nu$$
$$\frac{1}{2}\left(mc^2 - \frac{\hbar^2}{m}R\right) e_\mu\psi^\dagger\psi e_\nu = \frac{1}{2}\frac{\hbar^2}{m} e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)e_\nu$$

$$\left(mc^2 - \frac{\hbar^2}{m}R\right) e_\mu\psi^\dagger\psi e_\nu = \frac{\hbar^2}{m} e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)e_\nu$$
$$\frac{\hbar^2}{m} e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)e_\nu = \left(mc^2 - \frac{\hbar^2}{m}R\right) e_\mu\psi^\dagger\psi e_\nu$$
$$e_\mu\left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha\right)e_\nu = \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right) e_\mu\psi^\dagger\psi e_\nu$$
$$e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha = \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^\dagger\psi$$

$$e^\beta \nabla_\beta (\nabla_\alpha(\psi^\dagger\psi) e^\alpha) = \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^\dagger\psi$$

We can see that equation obtained, takes into account to calculate waveform not only the energy of the particle but also curvature conditions of the the space-time in its position (scalar curvature R). This is, is like the energy to be taken into account is not mc^2 alone but also, we have to subtract the other element:

$$E_{particle} = mc^2 - \frac{\hbar^2}{m}R$$

Another thing we can see is that the equation is so “simple” that it can be factored (a la Dirac way) easily:

$$e^{\beta}\nabla_{\beta}\psi^{\dagger}\psi\nabla_{\alpha}^{\dagger}e^{\alpha} = \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^{\dagger}\psi$$

$$e^{\beta}\nabla_{\beta}\psi^{\dagger}\psi\nabla_{\alpha}^{\dagger}e^{\alpha} = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}\psi^{\dagger}\psi\sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}$$

$$e^{\beta}\nabla_{\beta}\psi^{\dagger} = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}\psi^{\dagger}$$

$$\psi\nabla_{\alpha}^{\dagger}e^{\alpha} = \psi\sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}$$

$$(\nabla_{\alpha}\psi)e^{\alpha} = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}\ \psi$$

In the end the equations in alpha and beta are the same, just reversing sometimes or changing signs. We could simplify even more:

$$(\nabla_{\alpha}\psi)e^{\alpha}e_{\alpha} = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}\ \psi e_{\alpha}$$

$$\nabla_{\alpha}\psi = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}\ \psi e_{\alpha}$$

$$\nabla_{\beta}\psi^{\dagger} = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}e_{\beta}\psi^{\dagger}$$

9. 2 Introducing the Einstein Tensor in the Equation

Coming from the equation we got in the end of chapter 9:

$$T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right)e_{\mu}\psi^{\dagger}\psi e_{\nu} + \frac{1}{2}\frac{\hbar^2}{m}e_{\mu}\left(e^{\beta}\nabla_{\beta}\psi^{\dagger}\psi\nabla_{\alpha}^{\dagger}e^{\alpha}\right)e_{\nu}$$

And taking the Einstein General Relativity equation [58]-[62]:

$$\frac{8\pi G}{c^4}T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}$$

Operating this equation:

$$T_{\mu\nu} = \frac{c^4}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)$$

$$T_{\mu\nu} = \frac{c^4}{8\pi G}R_{\mu\nu} - \frac{1}{2}\frac{c^4}{8\pi G}g_{\mu\nu}R + \Lambda\frac{c^4}{8\pi G}g_{\mu\nu}$$

$$T_{\mu\nu} = \frac{c^4}{8\pi G}R_{\mu\nu} - \frac{c^4}{16\pi G}g_{\mu\nu}R + \Lambda\frac{c^4}{8\pi G}g_{\mu\nu}$$

And now, we introduce in the equation in the end of chapter 9:

$$T_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R + mc^2\right)\psi^{\dagger}\psi + \frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^{\beta}\nabla_{\beta}\psi^{\dagger}\psi\nabla_{\alpha}^{\dagger}e^{\alpha}\right)$$

$$\frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}\right)=\frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R+mc^2\right)\psi^\dagger\psi+\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)$$

$$\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)+\frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R+mc^2\right)\psi^\dagger\psi-\frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}\right)=0$$

$$\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^\beta\nabla_\beta\left(\nabla_\alpha(\psi^\dagger\psi)e^\alpha\right)\right)+\frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R+mc^2\right)\psi^\dagger\psi-\frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}\right)=0$$

This equation above seems (and it is) very complicated but it can be solvable.
The unknow variables are:

- $\psi^0\ \psi^1\ \psi^2\ \psi^3\psi^4\psi^5\psi^6\psi^7$
- $g_{11}\ g_{22}\ g_{33}g_{23}g_{31}g_{12}\ and\ it\ could\ be\ also\ g_{00}\ if\ it\ is\ not\ 1\ directly$

So, in total 14 (or 15) unknown variables. The equation above, only because it is a multi-vector equation, is converted into 8 equations (one per type of vector, bivector, scalar and trivector). So not even counting that it is also a tensor equation also (probably the equations obtained as a tensor equation are linearly dependent to the ones of the multivector), we will have 8 equations.

The rest of the equations we will get from the continuity equation[68]:

$$e^\lambda\nabla_\lambda T=0$$

With T defined as (end of chapter 9):

$$T=g^{\mu\nu}T_{\mu\nu}=\frac{1}{2}\left(\frac{\hbar^2}{m}R+mc^2\right)\psi^\dagger\psi+\frac{1}{2}\frac{\hbar^2}{m}\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)$$

These are another 8 equations. So, in total, we have 16 equations to solve 14 or 15 variables, so it should be ok. The system is over dimensioned. This means, we can take some of the unknowns as parameters, or even normalize the system as convenience (making those free parameters whatever value we want to make a normalization).

Coming back to this equation:

$$\frac{1}{2}\frac{\hbar^2}{m}g_{\mu\nu}\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)+\frac{1}{2}g_{\mu\nu}\left(\frac{\hbar^2}{m}R+mc^2\right)\psi^\dagger\psi-\frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}\right)=0$$

Putting it more symmetric (considering we are in one of the three cases of chapter 7):

$$\frac{1}{2}\frac{\hbar^2}{m}\mathrm{e}_\mu\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)\mathrm{e}_\nu+\frac{1}{2}\left(\frac{\hbar^2}{m}R+mc^2\right)\mathrm{e}_\mu\psi^\dagger\psi\mathrm{e}_\nu-\frac{c^4}{8\pi G}\left(R_{\mu\nu}-\frac{1}{2}\mathrm{e}_\mu R\mathrm{e}_\nu+\mathrm{e}_\mu\Lambda g_{\mu\nu}\right)=0$$

This equation, for sure can be factorized a la Dirac way somehow. But the quadratic equation solution has to be used, complicating the things. I will come back with this in next revisions of the paper.

10 Influence of Ricci scalar in the energy of a particle

We have seen in 9.1 the following equation:

$$E_{particle}=mc^2-\frac{\hbar^2}{m}R$$

But what is the influence of the second element? Let's check the influence in a proton at the surface of Earth

We know:

$$m_{proton} = 1.6726E - 27kg$$
$$\hbar = 1.05457E - 34J \cdot s$$
$$c = 299792458m/s$$

To calculate the Ricci scalar R is more complicated. If we use the Schwarzschild metric would be zero. What we can do is to calculate the Kretschmann scalar [70] considering Schwarzschild metric in the surface of Earth (related to the Ricci scalar curvature) and take its square root (its dimensions are L^{-4} and the Ricci scalar is L^{-2}. As commented, this is just a reference:

$$G = 6{,}6743E - \frac{11Nm^2}{kg}$$
$$M_{earth} = 5{,}9722E24\,kg$$
$$r = r_{earth} = 6{,}371E6m$$

$$\sqrt{Kretschmann\ scalar} = \sqrt{\frac{48G^2M^2}{c^4r^6}}\sqrt{\frac{48 \cdot (6{,}6743E - 11)^2(5{,}9722E24)^2}{299792458^4(6371E3)^6}}$$
$$= 1.18821E - 22m^{-2}$$

Coming back here, now considering a proton:

$$E_{particle} = mc^2 - \frac{\hbar^2}{m}R$$
$$= 1.6726E - 27 \cdot 299792458^2 - \frac{(1.05457E - 34)^2}{1.6726E - 27} \cdot 1.18821E$$
$$- 22 = 1.503257E - 10 - 7.9E - 64$$

We can see that the second element is several orders of magnitude lower than the original energy. Even if we consider R=1 (an example), we would be in a similar situation:

$$E_{particle} = mc^2 - \frac{\hbar^2}{m}R = 1.6726E - 27 \cdot 299792458^2 - \frac{(1.05457E - 34)^2}{1.6726E - 27} \cdot 1$$
$$= 1.503257E - 10 - 6.651E - 42$$

We can see that the second element is neglectable in general. And only in very big gravitational fields (with R very high), the second element could start having an effect.

11. Conclusions

In this paper we have used Geometric Algebra to be able to embed the Klein-Gordon equation for a particle in a non-Euclidean field (gravitational field) getting the following equation:

$$e^\beta \nabla_\beta(\nabla_\alpha(\psi^\dagger\psi)e^\alpha) = \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^\dagger\psi$$

Where $\psi^\dagger\psi$ is the wavefunction collapsed (multiplied by its reverse), this way:

$$\psi^\dagger\psi = (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) = \rho + \vec{j}$$

Being ρ and $\vec{j}$ the probability density and the fermionic current respectively.

The equation above can be factored to be simplified into:

$$\nabla_\alpha \psi = \sqrt{\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)}\ \psi e_\alpha$$

Meaning that the energy of a particle is somehow decreased by a term that depends on the Ricci scalar (the curvature of the space where it lies in)::

$$E_{particle} = mc^2 - \frac{\hbar^2}{m}R$$

Anyhow, this reduction is completely negligible in the general case being several orders of magnitude below the normal energy.

Following other path, we found another equation:

$$\frac{1}{2}\frac{\hbar^2}{m} g_{\mu\nu}\left(e^\beta \nabla_\beta\left(\nabla_\alpha(\psi^\dagger\psi)e^\alpha\right)\right) + \frac{1}{2} g_{\mu\nu}\left(\frac{\hbar^2}{m}R + mc^2\right)\psi^\dagger\psi - \frac{c^4}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right) = 0$$

This equation (that are in fact 8 embedded equations) have 14 or 15 unknown variables: 8 coefficients of the wavefunction $\psi^0\ to\ \psi^7$ and 6 metric elements g_{ij} (i,j from 1 to 3) with a possible added g_{00}.

The rest of the equations (8 equations more) come from the continuity equation:

$$e^\lambda \nabla_\lambda T = 0$$

With T defined as:

$$T = g^{\mu\nu}T_{\mu\nu} = \frac{1}{2}\left(\frac{\hbar^2}{m}R + mc^2\right)\psi^\dagger\psi + \frac{1}{2}\frac{\hbar^2}{m}\left(e^\beta \nabla_\beta(\nabla_\alpha(\psi^\dagger\psi)e^\alpha)\right)$$

So, the equation is in fact, solvable.

Bilbao, 8th December 2023 (viXra-v1).

12. Acknowledgements

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AAAAÁBCCCDEEIIILLLLLMMMOOOPSTU

If you consider this helpful, do not hesitate to drop your BTC here:

bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f

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A1. Annex A1. Bra-Ket product in Euclidean metric

The bra-ket product of a reversed spinor (in orthogonal metrics is the same as reverse) can be calculated as:

$$
\begin{gathered}
\psi^\dagger\psi = \psi^\mu e_\mu^\dagger \psi^\nu e_\nu = \left(\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger\right)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 \\
+ \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) = \psi^*\psi = \\
= (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 \\
+ \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) = \\
= (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_2 e_3 - \psi^5 e_3 e_1 - \psi^6 e_1 e_2 - \psi^7 e_1 e_2 e_3)(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 \\
+ \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) = \\
(\psi^0)^2 + \psi^0\psi^1 e_1 + \psi^0\psi^2 e_2 + \psi^0\psi^3 e_3 + \psi^0\psi^4 e_2 e_3 + \psi^0\psi^5 e_3 e_1 + \psi^0\psi^6 e_1 e_2 + \psi^0\psi^7 e_1 e_2 e_3 + \\
\psi^1\psi^0 e_1 + (\psi^1)^2 + \psi^1\psi^2 e_1 e_2 - \psi^1\psi^3 e_3 e_1 + \psi^1\psi^4 e_1 e_2 e_3 - \psi^1\psi^5 e_3 + \psi^1\psi^6 e_2 + \psi^1\psi^7 e_2 e_3 + \\
\psi^2\psi^0 e_2 - \psi^2\psi^1 e_1 e_2 + (\psi^2)^2 + \psi^2\psi^3 e_2 e_3 + \psi^2\psi^4 e_3 + \psi^2\psi^5 e_1 e_2 e_3 - \psi^2\psi^6 e_1 + \psi^2\psi^7 e_3 e_1 + \\
\psi^3\psi^0 e_3 + \psi^3\psi^1 e_3 e_1 - \psi^3\psi^2 e_2 e_3 + (\psi^3)^2 - \psi^3\psi^4 e_2 + \psi^3\psi^5 e_1 + \psi^3\psi^6 e_1 e_2 e_3 + \psi^3\psi^7 e_1 e_2 \\
-\psi^4\psi^0 e_2 e_3 - \psi^4\psi^1 e_1 e_2 e_3 + \psi^4\psi^2 e_3 - \psi^4\psi^3 e_2 + (\psi^4)^2 + \psi^4\psi^5 e_1 e_2 - \psi^4\psi^6 e_3 e_1 + \psi^4\psi^7 e_1 - \\
-\psi^5\psi^0 e_3 e_1 - \psi^5\psi^1 e_3 - \psi^5\psi^2 e_1 e_2 e_3 + \psi^5\psi^3 e_1 - \psi^5\psi^4 e_1 e_2 + (\psi^5)^2 + \psi^5\psi^6 e_2 e_3 + \psi^5\psi^7 e_2 - \\
-\psi^6\psi^0 e_1 e_2 + \psi^6\psi^1 e_2 - \psi^6\psi^2 e_1 - \psi^6\psi^3 e_1 e_2 e_3 + \psi^6\psi^4 e_3 e_1 - \psi^6\psi^5 e_2 e_3 + (\psi^6)^2 + \psi^6\psi^7 e_3 - \\
-\psi^7\psi^0 e_1 e_2 e_3 - \psi^7\psi^1 e_2 e_3 - \psi^7\psi^2 e_3 e_1 - \psi^7\psi^3 e_1 e_2 + \psi^7\psi^4 e_1 + \psi^7\psi^5 e_2 + \psi^7\psi^6 e_3 + (\psi^7)^2
\end{gathered}
$$

Please, take into account that for simplification I have considered directly $e_0 = 1$. If in the end, it has another value, it has just to be considered in the operations.

Continuing with the operation. If we separate from the result above only the scalars, we have:

$$(\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

We will call this sum ρ (probability density):

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

If we separate the components that multiply by e_1 we get:

$$\psi^0\psi^1 + \psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7 + \psi^5\psi^3 - \psi^6\psi^2 + \psi^7\psi^4 = 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)$$

In e_2 we get:

$$\psi^0\psi^2 + \psi^1\psi^6 + \psi^2\psi^0 - \psi^3\psi^4 - \psi^4\psi^3 + \psi^5\psi^7 + \psi^6\psi^1 + \psi^7\psi^5 = 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)$$

In e_3 we get:

$$\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^3\psi^0 + \psi^4\psi^2 - \psi^5\psi^1 + \psi^6\psi^7 + \psi^7\psi^6 = 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)$$

In e_2e_3 :

$$\psi^0\psi^4 + \psi^1\psi^7 + \psi^2\psi^3 - \psi^3\psi^2 - \psi^4\psi^0 + \psi^5\psi^6 - \psi^6\psi^5 - \psi^7\psi^1 = 0$$

In e_3e_1:

$$\psi^0\psi^5 - \psi^1\psi^3 + \psi^2\psi_{xyz} + \psi^3\psi^1 - \psi^4\psi^6 - \psi^5\psi^0 + \psi^6\psi^4 - \psi^7\psi^2 = 0$$

In e_1e_2:

$$\psi^0\psi^6 + \psi^1\psi^2 - \psi^2\psi^1 + \psi^3\psi^7 + \psi^4\psi^5 - \psi^5\psi^4 - \psi^6\psi^0 - \psi^7\psi^3 = 0$$

In $e_1e_2e_3$:

$$\psi^0\psi^7 + \psi^1\psi^4 + \psi^2\psi^5 + \psi^3\psi^6 - \psi^4\psi^1 - \psi^5\psi^2 - \psi^6\psi^3 - \psi^7\psi^0 = 0$$

If we call vector $\vec{j}$ (fermionic current) the sum in e_1, e_2 and e_3 , we get:

$$\vec{j} = 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 + 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2 + 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3$$

So, in total we have:

$$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j} \qquad (29.1)$$

With:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

And:

$$\vec{j} = 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 + 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2 + 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^\dagger\psi = j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

A2. Annex A2. Bra-Ket product in non-Euclidean metric (Orthogonal but not orthonormal)

We apply the following relations, when performing the multiplication:

$$(e_0)^2 = \|e_0\|^2 = g_{00}$$

$$(e_1)^2 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = \|e_3\|^2 = g_{33}$$

$$e_0 e_i = e_i e_0$$

$$e_2 e_3 = -e_3 e_2$$
$$e_3 e_1 = -e_1 e_3$$
$$e_1 e_2 = -e_2 e_1$$

For simplification we will consider directly $e_0 = 1$. If in the end, it has another value, it just will have to be considered in the operations.

$$\psi^\dagger \psi = \psi^\mu e_\mu^\dagger \psi^\nu e_\nu = \left(\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger\right)(\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) =$$
$$(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_3 e_2 + \psi^5 e_1 e_3 + \psi^6 e_2 e_1 + \psi^7 e_3 e_2 e_1)(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) =$$
$$\psi^{0^2} + \psi^0\psi^1 e_1 + \psi^0\psi^2 e_2 + \psi^0\psi^3 e_3 + \psi^0\psi^4 e_2 e_3 + \psi^0\psi^5 e_3 e_1 + \psi^0\psi^6 e_1 e_2 + \psi^0\psi^7 e_1 e_2 e_3 +$$
$$\psi^1\psi^0 e_1 + \psi^{1^2}\|e_1\|^2 + \psi^1\psi^2 e_1 e_2 - \psi^1\psi^3 e_3 e_1 + \psi^1\psi^4 e_1 e_2 e_3 - \psi^1\psi^5 \|e_1\|^2 e_3 + \psi^1\psi^6 \|e_1\|^2 e_2 + \psi^1\psi^7 \|e_1\|^2 e_2 e_3 +$$
$$\psi^2\psi^0 e_2 - \psi^2\psi^1 e_1 e_2 + \psi^{2^2}\|e_2\|^2 + \psi^2\psi^3 e_2 e_3 + \psi^2\psi^4 \|e_2\|^2 e_3 + \psi^2\psi^5 e_1 e_2 e_3 - \psi^2\psi^6 \psi_{xy} e_1 e_2 e_3 + \psi^3\psi^7 \|e_3\|^2 e_1 e_2$$
$$-\psi^4\psi^0 e_2 e_3 - \psi^4\psi^1 e_1 e_2 e_3 + \psi^4\psi^2 \|e_2\|^2 e_3 - \psi^4\psi^3 \|e_3\|^2 e_2 + \psi^{4^2}\|e_2\|^2\|e_3\|^2 + \psi^4\psi^5 \|e_3\|^2 e_1 e_2 - \psi^4\psi^6 \|e_2\|^2 e_3 e_1 + \psi^4\psi^7 \|e_2\|^2\|e_3\|^2 e_1 -$$
$$-\psi^5\psi^0 e_3 e_1 - \psi^5\psi^1 \|e_1\|^2 e_3 - \psi^5\psi^2 e_1 e_2 e_3 + \psi^5\psi^3 \|e_3\|^2 e_1 - \psi^5\psi^4 \|e_3\|^2 e_1 e_2 + \psi^{5^2}\|e_3\|^2\|e_1\|^2 + \psi^5\psi^6 \|e_1\|^2 e_2 e_3 + \psi^5\psi^7 \|e_3\|^2\|e_1\|^2 e_2 -$$
$$-\psi^6\psi^0 e_1 e_2 + \psi^6\psi^1 \|e_1\|^2 e_2 - \psi^6\psi^2 \|e_2\|^2 e_1 - \psi^6\psi^3 e_1 e_2 e_3 + \psi^6\psi^4 \|e_2\|^2 e_3 e_1 - \psi^6\psi^5 \|e_1\|^2 e_2 e_3 + \psi^{6^2}\|e_1\|^2\|e_2\|^2 + \psi^6\psi^7 \|e_1\|^2\|e_2\|^2 e_3 -$$
$$-\psi^7\psi^0 e_1 e_2 e_3 - \psi^7\psi^1 \|e_1\|^2 e_2 e_3 - \psi^7\psi^2 \|e_2\|^2 e_3 e_1 - \psi^7\psi^3 \|e_3\|^2 e_1 e_2 + \psi^7\psi^4 \|e_2\|^2\|e_3\|^2 e_1 + \psi^7\psi^5 \|e_1\|^2\|e_3\|^2 e_2 + \psi^7\psi^6 \|e_1\|^2\|e_2\|^2 e_3$$
$$+ \psi^{7^2}\|e_1\|^2\|e_2\|^2\|e_3\|^2$$

If we separate from the result above only the scalars, we have:

$$\rho = (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22} g_{33} + (\psi^5)^2 g_{33} g_{11} + (\psi^6)^2 g_{11} g_{22} + (\psi^7)^2 g_{11} g_{22} g_{33}$$

We will call above sum ρ (probability density).

Now, if we separate by e_1 :

$$\psi^0\psi^1 + \psi^1\psi^0 - \psi^2\psi^6\|e_2\|^2 + \psi^3\psi^5\|e_3\|^2 + \psi^4\psi^7\|e_2\|^2\|e_3\|^2 + \psi^5\psi^3\|e_3\|^2 - \psi^6\psi^2\|e_2\|^2$$
$$+ \psi^7\psi^4\|e_2\|^2\|e_3\|^2$$

$$2(\psi^0\psi^1 - \psi^2\psi^6\|e_2\|^2 + \psi^3\psi^5\|e_3\|^2 + \psi^4\psi^7\|e_2\|^2\|e_3\|^2)$$

$$\psi^0\psi^1 + \psi^1\psi^0 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22} g_{33} + \psi^5\psi^3 g_{33} - \psi^6\psi^2 g_{22} + \psi^7\psi^4 g_{22} g_{33}$$

$$2(\psi^0\psi^1 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22} g_{33})$$

By e_2 :

$$+\psi^0\psi^2 + \psi^1\psi^6\|e_1\|^2 + \psi^2\psi^0 - \psi^3\psi^4\|e_3\|^2 - \psi^4\psi^3\|e_3\|^2 + \psi^5\psi^7\|e_3\|^2\|e_1\|^2 + \psi^6\psi^1\|e_1\|^2$$
$$+ \psi^7\psi^5\|e_1\|^2\|e_3\|^2$$

$$2(+\psi^0\psi^2 + \psi^1\psi^6\|e_1\|^2 - \psi^3\psi^4\|e_3\|^2 + \psi^5\psi^7\|e_3\|^2\|e_1\|^2)$$

$$+\psi^0\psi^2 + \psi^1\psi^6 g_{11} + \psi^2\psi^0 - \psi^3\psi^4 g_{33} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33} g_{11} + \psi^6\psi^1 g_{11} + \psi^7\psi^5 g_{11} g_{33}$$
$$2(+\psi^0\psi^2 + \psi^1\psi^6 g_{11} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33} g_{11})$$

By e_3 :

$$+\psi^0\psi^3 - \psi^1\psi^5\|e_1\|^2 + \psi^2\psi^4\|e_2\|^2 + \psi^3\psi^0 + \psi^4\psi^2\|e_2\|^2 - \psi^5\psi^1\|e_1\|^2 + \psi^6\psi^7\|e_1\|^2\|e_2\|^2$$
$$+ \psi^7\psi^6\|e_1\|^2\|e_2\|^2$$

$$2(+\psi^0\psi^3 - \psi^1\psi^5\|e_1\|^2 + \psi^2\psi^4\|e_2\|^2 + \psi^6\psi^7\|e_1\|^2\|e_2\|^2)$$

$$+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^3\psi^0 + \psi^4\psi^2 g_{22} - \psi^5\psi^1 g_{11} + \psi^6\psi^7 g_{11} g_{22} + \psi^7\psi^6 g_{11} g_{22}$$

$$2(+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^6\psi^7 g_{11} g_{22})$$

In $e_2 e_3$ plane:

$$+\psi^0\psi^4 + \psi^1\psi^7\|e_1\|^2 + \psi^2\psi^3 - \psi^3\psi^2 - \psi^4\psi^0 + \psi^5\psi^6\|e_1\|^2 - \psi^6\psi^5\|e_1\|^2 - \psi^7\psi^1\|e_1\|^2 = 0$$

In $e_3 e_1$ plane:

$$+\psi^0\psi^5 - \psi^1\psi^3 + \psi^2\psi^7\|e_2\|^2 + \psi^3\psi^1 - \psi^4\psi^6\|e_2\|^2 - \psi^5\psi^0 + \psi^6\psi^4\|e_2\|^2 - \psi^7\psi^2\|e_2\|^2 = 0$$

In $e_1 e_2$ plane:

$$+\psi^0\psi^6+\psi^1\psi^2-\psi^2\psi^1+\psi^3\psi^7\|e_3\|^2+\psi^4\psi^5\|e_3\|^2-\psi^5\psi^4\|e_3\|^2-\psi^6\psi^0-\psi^7\psi^3\|e_3\|^2=0$$

In $e_1e_2e_3$ plane:

$$+\psi^0\psi^7+\psi^1\psi^4+\psi^2\psi^5+\psi^3\psi^6-\psi^4\psi^1-\psi^5\psi^2-\psi^6\psi^3-\psi^7\psi^0=0$$

So, in this case, we can sum up the result as:

$$\psi^\dagger\psi=\rho+\vec{j}$$

Being:

$$\rho=(\psi^0)^2+(\psi^1)^2g_{11}+(\psi^2)^2g_{22}+(\psi^3)^2g_{33}+(\psi^4)^2g_{22}g_{33}+(\psi^5)^2g_{33}g_{11}+(\psi^6)^2g_{11}g_{22}+(\psi^7)^2g_{11}g_{22}g_{33}$$

$$\begin{aligned}\vec{j}=2(\psi^0\psi^1-\psi^2\psi^6g_{22}+\psi^3\psi^5g_{33}+\psi^4\psi^7g_{22}g_{33})e_1\\+2(+\psi^0\psi^2+\psi^1\psi^6g_{11}-\psi^4\psi^3g_{33}+\psi^5\psi^7g_{33}g_{11})e_2\\+2(+\psi^0\psi^3-\psi^1\psi^5g_{11}+\psi^2\psi^4g_{22}+\psi^6\psi^7g_{11}g_{22})e_3\end{aligned}$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^\dagger\psi=j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

A3. Annex A3. Bra-Ket product between the reverse of a spinor and a spinor in non-Euclidean metric (Non orthogonal and non orthonormal). Debería llevar una capa forrada de armiño

We should do the following operation again:

$$\psi^\dagger\psi=\psi^\mu e_\mu^\dagger\psi^\nu e_\nu=(\psi^0e_0^\dagger+\psi^1e_1^\dagger+\psi^2e_2^\dagger+\psi^3e_3^\dagger+\psi^4e_4^\dagger+\psi^5e_5^\dagger+\psi^6e_6^\dagger+\psi^7e_7^\dagger)(\psi^0e_0+\psi^1e_1+\psi^2e_2+\psi^3e_3+\psi^4e_4+\psi^5e_5+\psi^6e_6+\psi^7e_7)=$$

$$(\psi^0+\psi^1e_1+\psi^2e_2+\psi^3e_3+\psi^4e_3e_2+\psi^5e_1e_3+\psi^6e_2e_1+\psi^7e_3e_2e_1)(\psi^0+\psi^1e_1+\psi^2e_2+\psi^3e_3+\psi^4e_2e_3+\psi^5e_3e_1+\psi^6e_1e_2+\psi^7e_1e_2e_3)=$$

But using the following rules commented in chapter 3.3.

$$\begin{gathered}(e_i)^2=e_ie_i=\|e_i\|^2=g_{ii}\\ e_ie_j=2g_{ij}-e_je_i=2g_{ji}-e_je_i\\ e_i\cdot e_j=e_j\cdot e_i=g_{ij}=g_{ji}\\ e_ie_j=e_i\cdot e_j+e_i\wedge e_j=g_{ij}+e_i\wedge e_j\end{gathered}$$

$$\begin{gathered}(e_1)^2=e_1e_1=\|e_1\|^2=g_{11}\\ (e_2)^2=e_2e_2=\|e_2\|^2=g_{22}\\ (e_3)^2=e_3e_3=\|e_3\|^2=g_{33}\\ e_1e_2=2g_{12}-e_2e_1=2g_{21}-e_2e_1\\ e_2e_3=2g_{23}-e_3e_2=2g_{32}-e_3e_2\\ e_3e_1=2g_{31}-e_1e_3=2g_{13}-e_1e_3\end{gathered}$$

I am not going to do it, but anyhow, you can understand that the result, whatever it is, will have this form:

$$\psi^\dagger\psi=j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

A4. Annex A4. Bra-Ket product between the inverse of a spinor and a spinor in non-Euclidean metric (Orthogonal but not orthonormal).

If instead of multiplying by the reverse, we multiply by the inverse (in orthogonal but not orthonormal metric), we should use the following rules from previous chapters:

$$(e_0)^2 = \|e_0\|^2 = g_{00}$$

$$(e_1)^2 = \|e_1\|^2 = g_{11}$$
$$(e_2)^2 = \|e_2\|^2 = g_{22}$$
$$(e_3)^2 = \|e_3\|^2 = g_{33}$$

$$e_0 e_i = e_i e_0$$

$$e_2 e_3 = -e_3 e_2$$
$$e_3 e_1 = -e_1 e_3$$
$$e_1 e_2 = -e_2 e_1$$

$$(e_i)^{-1} = e^i = \frac{e_i}{g_{ii}} = \frac{e_i}{\|e_i\|^2}$$

$$\left(e_i e_j\right)^{-1} = \frac{e_j e_i}{\|e_j\|^2 \|e_i\|^2} = \frac{e_j e_i}{g_{jj} g_{ii}}$$

Where all the above relation we have seen in previous chapters.
Operating:

$$\psi^{-1}\psi = \left(\psi^0 + \psi^1 \frac{e_1}{\|e_1\|^2} + \psi^2 \frac{e_2}{\|e_2\|^2} + \psi^3 \frac{e_3}{\|e_3\|^2} + \psi^4 \frac{e_3 e_2}{\|e_2\|^2\|e_3\|^2} + \psi^5 \frac{e_1 e_3}{\|e_3\|^2\|e_1\|^2} + \psi^6 \frac{e_2 e_1}{\|e_1\|^2\|e_2\|^2} + \psi^7 \frac{e_3 e_2 e_1}{\|e_1\|^2\|e_2\|^2\|e_3\|^2}\right)$$
$$(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3)$$
$$=$$

$$(\psi^0)^2 + \psi^1\psi^0 \frac{e_1}{\|e_1\|^2} + \psi^2\psi^0 \frac{e_2}{\|e_2\|^2} + \psi^3\psi^0 \frac{e_3}{\|e_3\|^2} - \psi^4\psi^0 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} - \psi^5\psi^0 \frac{e_3 e_1}{\|e_3\|^2\|e_1\|^2} - \psi^6\psi^0 \frac{e_1 e_2}{\|e_1\|^2\|e_2\|^2} - \psi^7\psi^0 \frac{e_1 e_2 e_3}{\|e_1\|^2\|e_2\|^2\|e_3\|^2} +$$

$$\psi^0\psi^1 e_1 + (\psi^1)^2 - \psi^2\psi^1 e_1 \frac{e_2}{\|e_2\|^2} + \psi^3\psi^1 \frac{e_3}{\|e_3\|^2} e_1 - \psi^4\psi^1 e_1 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} - \psi^5\psi^1 \frac{e_3}{\|e_3\|^2} + \psi^6\psi^1 \frac{e_2}{\|e_2\|^2} - \psi^7\psi^1 \frac{e_2 e_3}{\|e_2\|^2\|e_3\|^2} +$$

$$\psi^0\psi^2 e_2 + \psi^1\psi^2 \frac{e_1}{\|e_1\|^2} e_2 + (\psi^2)^2 - \psi^3\psi^2 e_2 \frac{e_3}{\|e_3\|^2} + \psi^4\psi^2 \frac{e_3}{\|e_3\|^2} - \psi^5\psi^2 e_1 e_2 \frac{e_3}{\|e_3\|^2} - \psi^6\psi^2 \frac{e_1}{\|e_1\|^2} - \psi^7\psi^2 \frac{e_3}{\|e_3\|^2} \frac{e_1}{\|e_1\|^2} +$$

$$\psi^0\psi^3 e_3 - \psi^1\psi^3 e_3 \frac{e_1}{\|e_1\|^2} + \psi^2\psi^3 \frac{e_2}{\|e_2\|^2} e_3 + (\psi^3)^2 - \psi^4\psi^3 \frac{e_2}{\|e_2\|^2} + \psi^5\psi^3 \frac{e_1}{\|e_1\|^2} - \psi^6\psi^3 \frac{e_1}{\|e_1\|^2} \frac{e_2}{\|e_2\|^2} e_3 - \psi^7\psi^3 \frac{e_1}{\|e_1\|^2} \frac{e_2}{\|e_2\|^2}$$

$$+\psi^0\psi^4 e_2 e_3 + \psi^1\psi^4 \frac{e_1}{\|e_1\|^2} e_2 e_3 + \psi^2\psi^4 e_3 - \psi^3\psi^4 e_2 + (\psi^4)^2 - \psi^5\psi^4 \frac{e_1}{\|e_1\|^2} e_2 + \psi^6\psi^4 e_3 \frac{e_1}{\|e_1\|^2} + \psi^7\psi^4 \frac{e_1}{\|e_1\|^2} +$$

$$+\psi^0\psi^5 e_3 e_1 - \psi^1\psi^5 e_3 + \psi^2\psi^5 e_1 \frac{e_2}{\|e_2\|^2} e_3 + \psi^3\psi^5 e_1 + \psi^4\psi^5 e_1 \frac{e_2}{\|e_2\|^2} + (\psi^5)^2 - \psi^6\psi^5 \frac{e_2}{\|e_2\|^2} e_3 + \psi^7\psi^6 \frac{e_2}{\|e_2\|^2} +$$

$$+\psi^0\psi^6 e_1 e_2 + \psi^1\psi^6 e_2 - \psi^2\psi^6 e_1 + \psi^3\psi^6 e_1 e_2 \frac{e_3}{\|e_3\|^2} - \psi^4\psi^6 \frac{e_3}{\|e_3\|^2} e_1 + \psi^5\psi^6 e_2 \frac{e_3}{\|e_3\|^2} + (\psi^6)^2 + \psi^7\psi^6 \frac{e_3}{\|e_3\|^2} +$$

$$+\psi^0\psi^7 e_1 e_2 e_3 + \psi^1\psi^7 e_2 e_3 + \psi^2\psi^7 e_3 e_1 + \psi^3\psi^7 e_1 e_2 + \psi^4\psi^7 e_1 + \psi^5\psi^7 e_2 + \psi^6\psi^7 e_3 + \left(\psi^7\right)^2$$

The scalar part is the same as the one multiplying by the reverse in a Euclidean orthonormal metric:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

This could be a hint, that probably this is the real operation that has to be done in general, instead of the reverse. The issue is that in orthonormal metric, the inverse and the reverse are the same operation. But this is not true in general, in non-orthonormal metrics.

If continuing with the operation, for example, we separate by e_1 we can see that the result is not as compact and in orthonormal or orthogonal solutions.

$$\psi^1\psi^0\frac{e_1}{\|e_1\|^2}+\psi^0\psi^1e_1-\psi^6\psi^2\frac{e_1}{\|e_1\|^2}+\psi^5\psi^3\frac{e_1}{\|e_1\|^2}+\psi^7\psi^4\frac{e_1}{\|e_1\|^2}+\psi^3\psi^5e_1-\psi^2\psi^6e_1+\psi^4\psi^7e_1$$

Even we can see that the result in the planes is not zero. Example e_2e_3:

$$-\psi^4\psi^0\frac{e_2e_3}{\|e_2\|^2\|e_3\|^2}-\psi^7\psi^1\frac{e_2e_3}{\|e_2\|^2\|e_3\|^2}-\psi^3\psi^2e_2\frac{e_3}{\|e_3\|^2}+\psi^2\psi^3\frac{e_2}{\|e_2\|^2}e_3+\psi^0\psi^4e_2e_3-\psi^6\psi^5\frac{e_2}{\|e_2\|^2}e_3+\psi^5\psi^6e_2\frac{e_3}{\|e_3\|^2}+\psi^1\psi^7e_2e_3$$

Or $e_1e_2e_3$, also different from zero:

$$\begin{aligned}-\psi^7\psi^0\frac{e_1e_2e_3}{\|e_1\|^2\|e_2\|^2\|e_3\|^2}-\psi^4\psi^1e_1\frac{e_2e_3}{\|e_2\|^2\|e_3\|^2}-\psi^5\psi^2e_1e_2\frac{e_3}{\|e_3\|^2}-\psi^6\psi^3\frac{e_1}{\|e_1\|^2}\frac{e_2}{\|e_2\|^2}e_3+\psi^1\psi^4\frac{e_1}{\|e_1\|^2}e_2e_3+\psi^2\psi^5e_1\frac{e_2}{\|e_2\|^2}e_3\\+\psi^3\psi^6e_1e_2\frac{e_3}{\|e_3\|^2}+\psi^0\psi^7e_1e_2e_3\end{aligned}$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^{-1}\psi=j^\mu e_\mu$$

Where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

In case that we perform this operation (multiplying by the inverse) in an orthonormal metric, we will get the same result as in Annex A1 (as the inverse is the same as the reverse in this case).

In case, that we perform this operation in a non-orthogonal (and therefore non-orthogonal case), we will have to follow the rules in chapter 3.3.

Anyhow, the result will always have this form:

$$\psi^{-1}\psi=j^\mu e_\mu$$