New equivalent of the Riemann hypothesis

Leonardo de Lima[1]

[1]Departamento de Ciências Exatas, Universidade Federal do Espírito Santo - UFES, Av. Fernando Ferrari, 514, Vitória, Espírito Santo, Brazil.

Contributing authors: leonardo.lima.88@edu.ufes.br;

Abstract

In this article, it is demonstrated that if the zeta function does not have a sequence of zeros whose real part converges to 1, then it cannot have any zeros in the critical strip, showing that the Riemann Hypothesis is false.

Keywords: Zeta function, Functional equation, Zeros of zeta function.

MSC Classification: 11-11 , 11Mxx

1 Introduction

The distribution of zeros of the Riemann zeta function is one of the great mysteries of modern mathematics. Its significance is related to the estimation of the error in the prime number theorem and a set of equivalent results (see [1],[2],[3],[4]).

One of the fundamental issues in the theory of the zeta function is the delimitation of a zero-free region in the critical strip. The first promising result in this direction was established by Vallée Poussin (see [5]), but this and recent findings have been unable to demarcate a zero-free region of the form $\Re(s) > 1 - \epsilon$ (see [6]).

In this work, I will demonstrate that the absence of a set of zeros in the zeta function, whose real part converges to 1, implies the absence of zeros along the critical strip.

2 Teoremas Fundamentais

In this section, some theorems used throughout the article will be listed.

Theorem 2.1. *If $\varphi(s)$ is analytic in the strip $a < \Re(s) < b$, and if it tends to zero uniformly as $Im(s) \to \pm\infty$ for any real value c between a and b, with its integral along such a line converging absolutely, then if*

$$f(x) = \mathcal{M}^{-1}\varphi = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s)\,ds,$$

we have that

$$\varphi(s) = \mathcal{M}f = \int_0^\infty x^{s-1}f(x)\,dx.$$

Conversely, suppose $f(x)$ is piecewise continuous on the positive real numbers, taking a value halfway between the limit values at any jump discontinuities, and suppose the integral

$$\varphi(s) = \int_0^\infty x^{s-1}f(x)\,dx$$

is absolutely convergent when $a < \Re(s) < b$. Then f is recoverable via the inverse Mellin transform from its Mellin transform φ.

Proof. [7] □

Theorem 2.2. *If $\Re(s) > 1$, we have:*

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$$

If the zeta function has no zeros in the region $\Re(s) > \rho$, we can extend the equality above to such a region.

Proof. [8] □

Theorem 2.3.

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0$$

If the zeta function is free of zeros in $\Re(s) > \rho$, we have:

$$\sum_{n=1}^{x}\frac{\mu(n)}{n} = O\left(\frac{1}{x^{1-\rho-\epsilon}}\right)$$

for all $\epsilon > 0$.

Proof. [8] □

Theorem 2.4. *If $0 < \Re(s) < 1$, we have:*

$$-\frac{\zeta(s)}{s} = \int_0^\infty \frac{\{x\}}{x^{s+1}}\, dx$$

Proof. [5] □

Theorem 2.5. *For any natural number $n > 1$, the sum of the values of the Möbius function $\mu(d)$ over all positive divisors of n is given by:*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Proof. [9] □

Theorem 2.6. *It is stated that*

$$\theta_n(x) = \int_0^x \phi_n(u) du,$$

where

$$\phi_n(x) = \int_0^x \{nu\} \frac{\mu(n)}{n} du.$$

Then:

$$\sum_{n=1}^\infty \theta_n(x) = \frac{1}{2\pi^2} \left(\frac{\sin(2\pi x)}{2\pi} - x \right)$$

and convergence occurs uniformly.

Proof. Using the fact that, for every x, we have

$$\{x\} = \frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^\infty \frac{\sin(2\pi n x)}{n}. \tag{2.1}$$

It concludes that

$$\phi_n(x) = \frac{x}{2} \frac{\mu(n)}{n} + \frac{\mu(n)}{2n^2\pi^2} \sum_{k=1}^\infty \frac{\cos(2\pi n k x) - 1}{k^2}. \tag{2.2}$$

Performing another integration we obtain the following expression:

$$\theta_n(x) = \frac{x^2}{4} \frac{\mu(n)}{n} + \frac{\mu(n)}{4n^3\pi^3} \sum_{k=1}^\infty \frac{\sin(2\pi n k x)}{k^3} - \frac{x}{2\pi^2} \frac{\mu(n)}{n^2} \sum_{k=1}^\infty \frac{1}{k^2}. \tag{2.3}$$

Thus

$$\sum_{n=1}^{\infty} \theta_n(x) = \frac{x^2}{4} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} - \frac{x}{2\pi^2} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{1}{4\pi^3} \sum_{n,k=1}^{\infty} \frac{\sin(2\pi nkx)\mu(n)}{n^3k^3}. \quad (2.4)$$

Rearranging the series, it follows that

$$\sum_{n,k=1}^{\infty} \frac{\sin(2\pi nkx)\mu(n)}{n^3k^3} = \sum_{l=1}^{\infty} \frac{\sin(2\pi lx)}{l^3} \sum_{n|l} \mu(n) = \sin(2\pi x). \quad (2.5)$$

(The rearrangement of the summations is justified by the uniform convergence of the series).

Finally, it can be observed that:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \quad (2.6)$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^3} \sum_{k=1}^{\infty} \frac{1}{k^3} = 1, \quad (2.7)$$

concluding the proof.

□

Theorem 2.7. *Suppose the Riemann zeta function has no zeros in the region $Re(s) > \rho$. In this case, for every $0 < \gamma < 1 - \rho$, we have*

$$\psi_\gamma(M) = \max\{\|\frac{1}{(\gamma + 2 + it)} \sum_{n=M}^{\infty} \frac{\mu(n)}{n^{1-\gamma-it}}\|, t \in \mathbb{R}\};$$

$$\lim_{M\to\infty} \psi_\gamma(M) = 0.$$

Proof. Since

$$\sum_{n=k}^{\infty} \frac{\mu(n)}{n^s} = \frac{M(k)}{k^s} - s\int_k^{\infty} \frac{M(x)}{x^{s+1}} dx \quad (2.8)$$

where

$$M(x) = \sum_{n=1}^{x} \mu(n). \quad (2.9)$$

The fact that the zeta function has no zeros in $\Re(s) > \rho$ is equivalent to $M(x) = O(x^\rho)$, therefore, we conclude that:

$$\lim_{M\to\infty} \psi_\gamma(M) = 0 \quad (2.10)$$

if $0 < \gamma < 1 - \rho$. □

3 Proof

Theorem 3.1. *If the zeta function does not exhibit a sequence of zeros whose real part converges to 1, then the zeta function has no zeros along the critical strip.*

Proof. By Theorem (2.4), we have:

$$-\frac{\zeta(s)}{s} = \int_0^\infty \frac{\{x\}}{x^{s+1}}\,dx \tag{3.1}$$

Making the variable change $x = n \cdot y$ in (3.1), we obtain:

$$-\frac{\zeta(s)}{s} n^s = \int_0^\infty \frac{\{nx\}}{x^{s+1}}\,dx \tag{3.2}$$

We then conclude:

$$-\frac{\zeta(s)}{s} \sum_{n=1}^{M} \frac{\mu(n)}{n^{1-s}} = \int_0^\infty \sum_{n=1}^{M} \frac{\mu(n)}{n} \frac{\{nx\}}{x^{s+1}}\,dx \tag{3.3}$$

Performing two integrations by parts in (3.3), we obtain:

$$-\frac{\zeta(s)}{s(s+1)(s+2)} \sum_{n=1}^{M} \frac{\mu(n)}{n^{1-s}} = \int_0^\infty \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{s+3}}dx, \tag{3.4}$$

where

$$\theta_n(x) = \int_0^x \phi_n(u)du, \tag{3.5}$$

and

$$\phi_n(x) = \int_0^x \{n \cdot u\} \frac{\mu(n)}{n} du. \tag{3.6}$$

Using the inverse Mellin transform in (3.4), we have:

$$\sum_{n=1}^{M} \theta_n(x) = -\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s+2} \frac{\zeta(s)}{s(s+1)(s+2)} \sum_{n=1}^{M} \frac{\mu(n)}{n^{1-s}} ds. \tag{3.7}$$

Similarly, we can conclude:

$$\sum_{n=M}^{M+P} \theta_n(x) = -\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s+2} \frac{\zeta(s)}{s(s+1)(s+2)} \sum_{n=M}^{M+P} \frac{\mu(n)}{n^{1-s}} ds. \tag{3.8}$$

Note that, by the Cauchy theorem, the integral is independent of the value of σ, provided that $\sigma \in (0, 1)$.

We will now make the following hypothesis:

Hipótese 1. *Suppose there exists a* ρ, *such that* $1 - \epsilon > \Re(\rho) > \epsilon > 0$ *for all* ϵ, *and* $\zeta(\rho) = 0$.

In this case, by equation (3.4), we have:

$$\int_0^\infty \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{\rho+3}} dx = 0. \tag{3.9}$$

Dividing this integral into two parts, we obtain:

$$\int_0^1 \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{\rho+3}} dx + \int_1^\infty \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{\rho+3}} dx = 0. \tag{3.10}$$

By Theorem (2.6), we have:

$$\int_1^\infty \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{\rho+3}} dx \to \int_1^\infty \frac{1}{x^{\rho+3}} \frac{1}{2\pi^2} \left(\frac{\sin(2\pi x)}{2\pi} - x \right) dx, \tag{3.11}$$

if $M \to \infty$.(See lemma (3.1))

For the first integral, it is initially observed that:

$$\int_0^1 \sum_{n=M}^{M+P} \frac{\theta_n(x)}{x^{\rho+3}} dx = - \int_0^1 \left(\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s-\rho-1} \frac{\zeta(s)}{s(s+1)(s+2)} \sum_{n=M}^{M+P} \frac{\mu(n)}{n^{1-s}} ds \right) dx. \tag{3.12}$$

Choosing $\sigma > \text{Re}(\rho)$, we get:

$$\int_0^1 \sum_{n=M}^{M+P} \frac{\theta_n(x)}{x^{\rho+3}} dx = - \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(s)}{s(s-\rho)(s+1)(s+2)} \sum_{n=M}^{M+P} \frac{\mu(n)}{n^{1-s}} ds. \tag{3.13}$$

(The permutation of integrals is allowed by the uniform convergence of the line integral).

It is noted that, because $\zeta(\rho) = 0$, the integrand in (3.13) has no poles in $0 < \sigma < 1$. Thus, by the Cauchy theorem, we can take σ as close to zero as desired.

We will now make another hypothesis:

Hipótese 2. *Suppose there exists an $\epsilon > 0$, such that the Riemann zeta function has no zeros for $\Re(s) > 1 - \epsilon$.*

In this case, by taking σ sufficiently close to 0 and $\sigma > 0$ in (3.13), we can conclude by Theorem (2.7) :

$$\int_0^1 \sum_{n=M}^{\infty} \frac{\theta_n(x)}{x^{\rho+3}} dx \to 0, \tag{3.14}$$

if $M \to \infty$.(See lemma (3.2))

In summary, we have proven:

$$\int_0^\infty \frac{1}{x^{\rho+3}} \frac{1}{2\pi^2} \left(\frac{\sin(2\pi x)}{2\pi} - x \right) dx = 0, \tag{3.15}$$

or equivalently:

$$\int_0^\infty \frac{1}{x^{\rho+1}} \frac{\sin(2\pi x)}{\pi} dx = 0. \tag{3.16}$$

But this cannot happen, because:

$$\int_0^\infty \frac{1}{x^{\rho+1}} \frac{\sin(2\pi x)}{\pi} dx = \frac{\zeta(\rho)}{\zeta(1-\rho)\rho}. \tag{3.17}$$

Therefore, we conclude that if hypothesis 2 is true, hypothesis 1 cannot occur, which means that the Riemann hypothesis is false as the zeta function has an infinite set of zeros along the critical line.

□

Corollary 3.1. *The Riemann Hypothesis is false.*

Lemma 1. *If* $0 < \Re(\rho) < 1$*, then:*

$$\int_1^\infty \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{\rho+3}} dx \to \int_1^\infty \frac{1}{x^{\rho+3}} \frac{1}{2\pi^2} \left(\frac{\sin(2\pi x)}{2\pi} - x \right) dx, \tag{3.18}$$

if $M \to \infty$*.*

Proof. By Theorem (2.6), we have:

$$\sum_{n=1}^{\infty} \theta_n(x) = \frac{1}{2\pi^2} \left(\frac{\sin(2\pi x)}{2\pi} - x \right) \tag{3.19}$$

and

$$\sum_{n=M}^{\infty} \theta_n(x) = \frac{x^2}{4} \sum_{n=M}^{\infty} \frac{\mu(n)}{n} - \frac{x}{2\pi^2} \sum_{n=M}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{1}{4\pi^3} \sum_{n=M,k=1}^{\infty} \frac{\sin(2\pi nkx)\mu(n)}{n^3 k^3}. \tag{3.20}$$

Note that the last two terms of (3.20) go to zero uniformly as $M \to \infty$. Moreover, they are bounded by $O(x)$, so it follows that the integral of this term divided by $x^{\rho+3}$ over the interval $[1, \infty)$ tends to zero uniformly as $M \to \infty$.

For the first term, it suffices to observe that its integral is (in case $\Re(\rho) > 0$):

$$\int_1^\infty \frac{x^{-1-\rho}}{4} \sum_{n=M}^{\infty} \frac{\mu(n)}{n} = \frac{1}{4\rho} \sum_{n=M}^{\infty} \frac{\mu(n)}{n} = O\left(\frac{1}{\ln M} \right) \tag{3.21}$$

□

Lemma 2. *If* $\zeta(\rho) = 0$ *and there exists an* $\epsilon > 0$ *such that for* $\Re(s) > 1 - \epsilon$*, the zeta function has no zeros, then:*

$$\int_0^1 \sum_{n=M}^{\infty} \frac{\theta_n(x)}{x^{\rho+3}} dx \to 0, \tag{3.22}$$

if $M \to \infty$.

This implies that:

$$\int_0^1 \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{\rho+3}} dx \to \int_0^1 \frac{1}{x^{\rho+3}} \frac{1}{2\pi^2} \left(\frac{\sin(2\pi x)}{2\pi} - x \right) dx, \tag{3.23}$$

if $M \to \infty$.

Proof. By equation (3.13):

$$\int_0^1 \sum_{n=M}^{M+P} \frac{\theta_n(x)}{x^{\rho+3}} dx = -\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(s)}{s(s-\rho)(s+1)(s+2)} \sum_{n=M}^{M+P} \frac{\mu(n)}{n^{1-s}} ds. \tag{3.24}$$

The contour integral (3.24) does not depend on σ if $0 < \sigma < 1$, because the integrand is analytic, given that by hypothesis $\zeta(\rho) = 0$.

Choosing a σ sufficiently close to zero, we can make

$$\sum_{n=M}^{M+P} \frac{\mu(n)}{n^{1-s}} \tag{3.25}$$

converge, because there exists an ϵ such that for $\Re(s) > 1 - \epsilon \to \zeta(s) \neq 0$, by hypothesis.

Thus, we have:

$$\int_0^1 \sum_{n=M}^{\infty} \frac{\theta_n(x)}{x^{\rho+3}} dx = -\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(s)}{s(s-\rho)(s+1)(s+2)} \sum_{n=M}^{\infty} \frac{\mu(n)}{n^{1-s}} ds. \tag{3.26}$$

And by the notation of Theorem (2.7):

$$\| \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(s)}{s(s-\rho)(s+1)(s+2)} \sum_{n=M}^{\infty} \frac{\mu(n)}{n^{1-s}} ds \| < \psi_\sigma(M) \int_{\sigma-i\infty}^{\sigma+i\infty} \| \frac{\zeta(s)}{s(s-\rho)(s+1)} \| dt \tag{3.27}$$

Hence, it follows that

$$\int_0^1 \sum_{n=M}^{\infty} \frac{\theta_n(x)}{x^{\rho+3}} dx \to 0, \tag{3.28}$$

if $M \to \infty$.

For the second statement of the lemma, just note that:

$$\int_0^1 \sum_{n=1}^{M} \frac{\theta_n(x)}{x^{\rho+3}} dx = \int_0^1 \frac{1}{x^{\rho+3}} \frac{1}{2\pi^2} \left(\frac{\sin(2\pi x)}{2\pi} - x \right) dx - \int_0^1 \sum_{n=M}^{\infty} \frac{\theta_n(x)}{x^{\rho+3}} dx \tag{3.29}$$

and use (3.28). $\square$

References

[1] Conrey, J.B.: Riemann's Hypothesis. American Mathematical Society, Providence, RI, USA (2003)

[2] Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On robin's criterion for the riemann hypothesis. Journal de théorie des nombres de Bordeaux **19**(2), 357–372 (2007) https://doi.org/10.5802/jtnb.591

[3] Borwein, P., Choi, S., Rooney, B., Weirathmueller, A.: The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike. CMS Books in Mathematics / Springer, New York, NY, USA (2008)

[4] Schumayer, D., Hutchinson, D.A.W.: Colloquium: Physics of the riemann hypothesis. Rev. Mod. Phys. **83**, 307–330 (2011) https://doi.org/10.1103/RevModPhys.83.307

[5] Karatsuba, A.A., Voronin, S.M., Koblitz, N.: The Riemann Zeta-Function. De Gruyter, Berlin, Germany (2011)

[6] Yang, A.: Explicit bounds on $\zeta(s)$ in the critical strip and a zero-free region. arXiv preprint arXiv:2301.03165 (2023)

[7] Lang, S.: Complex Analysis. Springer, New York, NY (1985)

[8] Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function. Oxford University Press, Oxford, UK (1986)

[9] Apostol, T.M.: Introduction to Analytic Number Theory. Springer, New York, NY, USA (1976)