The magic of mirror composite numbers. Their factorization and their relationship with Goldbag conjecture.

Emilio Sánchez. Óscar E. Chamizo.
Redonda Kingdom University, Faculty of Sciences, Department of Mathematics.

Abstract:

In this paper, continuation and completion of some previous papers, [1] we fully develop the new concept of mirror composite numbers. Mirror composite numbers are composite numbers of the form 2n-p for some n natural number and p prime. We shall show that the factorization of these numbers have interesting properties in order to face the Goldbach conjecture [2][3] by the *divide et impera* method.

Definitions:

From now on, *m* and *n* are positive integer numbers, *p* and *q* are prime numbers.

All prime numbers p ≥ 5 are of the form 6m+1 or 6m-1. A prime of the form 6m+1 is a **right prime**; a prime of the form 6m-1 is a **left prime**.

A **mirror composite number** is a composite number of the form 2n-p for some n and some prime p ≥ 5.

Given a mirror composite 2n-p, if p=6m+1, i.e., if p is a right prime, 2n-p is a **right mirror composite (r.m.c.)**.

Given a mirror composite 2n-p, If p=6m-1, i.e., if p is a left prime, 2n-p is a **left mirror composite (l.m.c.)**.

Lemma 1.

Fixed n, if 3 is a factor of some l.m.c (respectively r.m.c.), 3 is a factor of every l.m.c. (r.m.c.) and 3 is not a factor of any r.m.c. (l.m.c)

Proof:

The difference between two l.m.c. (r.m.c.) is 6n. If 3|m, 3|m±6n. On the other hand, if 3|2n-(6m-1), then 3 ∤ 2n-(6m+1) and *viceversa*.

Lemma 2.

Fixed n, if q≠3 is a prime factor of two different l.m.c. (respectively r.m.c.), the difference between them is a multiple of 6q so the minimum gap between two consecutive occurrences of factor q is 6q for all l.m.c. (r.m.c.).

Proof:

If q | 2n-(6x-1) and q | 2n-(6y-1) exists z such that zq=6(x-y), so z is multiple of 6, given that q is a prime and q ≠ 2,3.

If q | 2n-(6x+1) and q | 2n-(6y+1) exists z such that zq=6(x-y), so z is multiple of 6, given that q is a prime and q ≠ 2,3.

Goldbach conjecture states that for all n and all prime p such that 3≤p≤n, some 2n-p is a prime, i.e., not every 2n-p is composite.

Let´s assume for the sake of contradiction that exists n such that every 2n-p is composite. Then, 3 consecutive odd numbers, 2n-3, 2n-5 and 2n-7 are composite, so one and only one of them must be multiple of 3.

Case A: 3|2n-7:

3|2n-7⇒ 3|2n-(6m+1) for all m (**Lemma 1**). Every right mirror composite is a multiple of 3 and no left mirror composite is a multiple of 3. So all elements of the sequence:

2n-3, 2n-5, 2n-11, 2n-17, 2n-23, 2n-29, 2n-41, …., 2n-q

where q ≥ 5 is a left prime, must be factorized. There are k consecutive primes p_i (i=1,2,3, …, k) from p_1=5 to p_k, where p_k is the largest prime $p_k \leq \sqrt{2n-3}$, available for that factorization.

Now, given the correlative sequence of odd numbers 2n-3, 2n-5, 2n-7, 2n-9, 2n-11, 2n-13, 2n-15, 2n-a…, let be $2n-a_i$ the number containing the first occurrence of prime factor p_i in that sequence.
Notice that:

For each p_i, a_i is unique.

$3 \leq a_i \leq 2p_i+1$.

For some i, a_i = 3; for some i, a_i=5; for some i, a_i=11 MOD p_i; for some i, a_i=17 MOD p_i; for some i, a_i=23 MOD p_i and so on.

2n-q, i.e., 2n-(6m-1), is composite if and only if exists i such that $6m-1 \equiv a_1 \bmod p_i$ (**Lemma 2**).

Now, let´s state conditions in order to find some 2n-q with q=6m-1 and q inside the interval $\sqrt{2n-5}$ < q < n that can not be factorized:

1) q is a left prime, i.e., q is not multiple of any p_i, so $6m-1 \not\equiv 0 \bmod p_i$ for all i.
2) There is no p_i factor available for 2n-q, so $6m-1 \not\equiv a_1 \bmod p_i$ for all i.

Prime condition for 6m-1	No factor available condition for 2n-(6m-1)
$6m \not\equiv 1 \bmod 5$	$6m \not\equiv (a_1+1) \bmod 5$
$6m \not\equiv 1 \bmod 7$	$6m \not\equiv (a_2+1) \bmod 7$

$6m \not\equiv 1 \bmod 11$	$6m \not\equiv (a_3+1) \bmod 11$
$6m \not\equiv 1 \bmod 13$	$6m \not\equiv (a_4+1) \bmod 13$
…………..	……………….
$6m \not\equiv 1 \bmod p_k$	$6m \not\equiv (a_k+1) \bmod p_k$

Hence for each p_i there are *at least* p_i-2 remainders moduli p_i that fullfill the conditions. That amounts up to a minimum of $(p_1-2)(p_2-2)(p_3-2)\dots(p_k-2)$, id est, $3.5.9.11.\dots(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli $5.7.11.13\dots p_k$.

It´s necessary then to prove that exists at least a multiple of 6 that fullfills the preceding conditions inside the interval:

$$\sqrt{2n-5} < 6m < n$$

So let´s prove that at least one in $3.5.9.11\dots(p_k-2)$ solutions from $5.7.11.13\dots p_k$ possible systems lies inside the aformentioned interval.

Let be M the highest number of consecutive occurrences of 6m that do not fullfill the conditions.[1] Is not easy to figure out the value of M, given the unpredictable nature of prime number distribution. But we can prove that exists an upper bound S for M such that for sufficient large n:

$$S < \left[\frac{n-\sqrt{2n-5}}{6}\right] \qquad (1)$$

Given p_k, an upper bound for the total number of occurrences of each one of the two remainders moduli p are $2\left[\frac{p_k}{p}\right]$. So

$$S = 2\left(\left[\frac{p_k}{5}\right] + \left[\frac{p_k}{7}\right] + \left[\frac{p_k}{11}\right] + \left[\frac{p_k}{13}\right] + \dots + \left[\frac{p_k}{p_{k-1}}\right] + 1\right)$$

is an upper bound for M:

k	p_k	M	S
1	5	2	2
2	7	4	6
3	11	8	11
4	13	13	16

[1] For all those who, like myself, enjoy practical questions that sometimes shed light on some more abstract matter of discussion, the problem to determine an accurate value for **M** is the same as the following: Suppose you may not work on 2 predetermined days in five, 2 predetermined days in seven, 2 days in 11, 2 in 13 and so on until 2 days in p_k days. What is the maximum number, as a function of p_k, of consecutive days off?

k	p_k	M	S
5	17	19	24
6	19	22	28

In turn:

$$\left\lceil\frac{p_k}{5}\right\rceil + \left\lceil\frac{p_k}{7}\right\rceil + \left\lceil\frac{p_k}{11}\right\rceil + \left\lceil\frac{p_k}{13}\right\rceil + \ \dots + \left\lceil\frac{p_k}{p_{k-1}}\right\rceil + \ 1 <$$

$$\frac{p_k}{2} + \frac{p_k}{3} + \frac{p_k}{5} + \frac{p_k}{7} + \frac{p_k}{11} + \ \dots + \frac{p_k}{p_{k-1}} + 1 =$$

$$p_k \left\{\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} \dots + \frac{1}{p_{k-1}} + \frac{1}{p_k}\right\}$$

The series between brackets is the well known partial summation of the reciprocal of the primes whose divergence was proved by Euler in 1737 together with the relationship:

$$\sum_{p \le x} \frac{1}{p} \approx \text{loglog(x)} \qquad (2)$$

Taking $x=p_k$ and given that an upper bound for all $x>e^4$ in (2) is loglogx+6 [4] allows us to state:

$$S < 2p_k(\text{loglog}p_k+6)$$

Now it`s inmediate to conclude, since $p_k \le \sqrt{2n-3}$, that (1) holds for, let´s say, every $2n \ge 10^6$.

For every $2n<10^6$ the verification of the conjecture have alredy been settled.

That completes the demonstration.

Hence, for all 2n such that 3|2n-7, i.e., for all $2n \equiv 1 \bmod 3$, exists some 2n-q that can not be factorized, so 2n-q is prime and the conjecture holds for all $2n \equiv 1 \bmod 3$.

Case B: 3|2n-5:

3|2n-5⇒ 3|2n-(6m-1) for all m (**Lemma 1**). So every left mirror composite is a multiple of 3 and no right mirror composite is a multiple of 3…

Following the same thought process than before, with q a right prime of the form 6m+1, it´s straightforward to conclude that the conjecture holds for all 2n such that 3|2n-5, i.e., for all 2n ≡ 2 mod 3.

Case C: 3|2n-3:

3|2n-3⇒ 3 ∤ 2n-(6m±1) for all m (**Lemma 1**). No mirror composite is a multiple of 3. So all elements of the sequence:

2n-5, 2n-7, 2n-11, 2n-13, 2n-17, 2n-19, 2n-23, 2n-29, 2n-31, 2n-37, ... 2n-q

where q ≥ 5 is a prime, must be factorized. There are k consecutive primes p_i (i=1,2,3, …, k) from p_1=5 to p_k, where p_k is the largest prime $p_k \leq \sqrt{2n-5}$, available for that factorization.

Now, given the correlative sequence of odd numbers 2n-3, 2n-5, 2n-7, 2n-9, 2n-11, 2n-13, 2n-15, 2n-a…, let be 2n-a_i the number containing the first occurrence of prime factor p_i in that sequence.
Notice that:

For each p_i, a_i is unique.

$3 \leq a_i \leq 2p_i+1$.

For some i, a_i = 3; for some i, a_i=5; for some i, a_i=11 MOD p_i; for some i, a_i=17 MOD p_i; for some i, a_i=23 MOD p_i and so on.

2n-q, i.e., 2n-(6m±1), is composite if and only if exists i such that $6m\pm1 \equiv a_1 \bmod p_i$ (**Lemma 2**).

Conditions in order to find some 2n-q with q=6m±1 and q inside the interval $\sqrt{2n-5}$ < q < n that can not be factorized:

Prime condition for 6m±1	No factor available condition for 2n-(6m±1)
$6m \not\equiv \pm1 \bmod 5$	$6m \not\equiv (a_1\pm1) \bmod 5$
$6m \not\equiv \pm1 \bmod 7$	$6m \not\equiv (a_2\pm1) \bmod 7$
$6m \not\equiv \pm1 \bmod 11$	$6m \not\equiv (a_3\pm1) \bmod 11$
$6m \not\equiv \pm1 \bmod 13$	$6m \not\equiv (a_4\pm1) \bmod 13$
…………..	……………….
$6m \not\equiv \pm1 \bmod p_k$	$6m \not\equiv (a_k\pm1) \bmod p_k$

Hence for each p_i there are *at least* $2(p_i-2)$ remainders moduli p_i that fullfill the conditions. That amounts up to a minimum of $2(p_1-2)(p_2-2)(p_3-2)\ldots(p_k-2)$, id est, $2.3.5.9.11\ldots.(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli $5.7.11.13\ldots p_k$.

Interesting to note here that this result is fully consistent with the fact that there are now twice as many composite numbers to factorize with the same number of factors than before (Cases **A** and **B**)

It´s necessary then to prove that exists at least a multiple of 6 that fullfills the preceding conditions inside the interval:

$$\sqrt{2n-5} < 6m < n$$

The same considerations apply as in relation to the previous point, as to conclude that:

$$S < p_k(\log\log p_k + 6)$$

is an upper bound for the highest number of consecutive occurrences of 6m that do not fullfill the previous conditions. Hence, as before, the conjecture also holds for every 2n ≡ 0 mod 3.

December, 4, 2023.

Emilio Sánchez & Óscar Chamizo.
roundkuniversity@gmail.com
PA³.

References:

[1] *Mirror composite numbers*. https://vixra.org/abs/2311.0047

[2] Christian Goldbach, *Letter to L. Euler, June 7 (1742)*.

[3] Vaughan, Robert. Charles. *Goldbach's Conjectures: A Historical Perspective*. *Open problems in mathematics*. Springer, Cham, 2016. 479-520.

[4] Pollack, Paul. *Euler and the partial sums of the prime harmonic series*. University of Georgia. Athens. Georgia.