Proof for specific type of continued fraction

isaac mor
isaac.mor@hotmail.com

Abstract

I am going to prove the following result:

$$A_n = 2n+1+\cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

I am going to demonstrate and use two telescoping series.
Then I will combine both of them to create a new formula
and completing the proof by using an inductive proof.
(I am using Lambert's continued fraction for the base case)

Formula no. 1

$$\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}-1\right]\right]=?$$

we wanna find out what is the value of the expression above

$$\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\frac{2k}{(n+1-k)(n+k)}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=$$

we will rewrite it in this way:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=$$

This is a telescoping series!

$$k=0::\left(2^{n-(0-1)}\frac{n!}{(2n)!}\frac{(n+(0-1))!0}{(n-(0-1))!0!}-2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!1}{(n-0)!(1)!}\right)$$

$$k=1::\left(2^{n-(1-1)}\frac{n!}{(2n)!}\frac{(n+(1-1))!1}{(n-(1-1))!1!}-2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!2!}\right)$$

$$k=2::\left(2^{n-(2-1)}\frac{n!}{(2n)!}\frac{(n+(2-1))!2}{(n-(2-1))!2!}-2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!3!}\right)$$

…

$$k=n-1::\left(2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!(n-1)}{(2)!(n-1)!}-2^{1}\frac{n!}{(2n)!}\frac{(n+n-1)!n}{(n-n+1)!(n)!}\right)$$

we will write it a bit cleaner:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=\left(0-2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}\right)+$$

$$\left(2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}-2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}\right)+\left(2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}-2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}\right)+$$

$$\left(2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}-2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}\right)+\ldots+\left(2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}-2^{1}\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}\right)$$

now we will highlight the terms of the telescoping series, and we will get this:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=0+$$

$$\left(-2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}+2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}\right)+\left(-2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}+2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}\right)+$$

$$\left(-2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}+2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}\right)+\left(-2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}+2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}\right)+$$

$$\ldots+\left(-2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}+2^{2}\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}\right)-2^{1}\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}$$

only the first and last terms are left (the first term in this telescoping series is 0)

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=0-2^{1}\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}=-1$$

now we will go back to our original form that we wanted to find out that value for:

$$\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}-1\right]\right]=-1$$

$$\frac{2^{n}n!}{(2n)!}\sum_{k=0}^{n-1}\left[\left(\frac{1}{2^{k}}\right)\frac{(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+1=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{1}{2^{k}}\right)\frac{(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{1}{2^{n}}\right)\frac{(2n)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^{n}}{2^{k}}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{(-1)^{n}}{2^{n}}\right)\frac{(2n+1)!}{n!}=0$$

$$\frac{(2n+1)(n+k)!}{(n-k)!k!}\cdot\frac{2k}{(n+1-k)(n+k)}=\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}$$

(The above result is a mathematical identity obtained by simple arithmetic)

$$\boxed{\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^{n-1}}{2^{k}}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{(-1)^{n}}{2^{k}}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{(-1)^{n}}{2^{n}}\right)\frac{(2n+1)!}{n!}=0}$$

Formula no. 2

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}+1\right]\right]=?$$

we wanna find out what is the value of the expression above

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\frac{2k}{(n+1-k)(n+k)}+\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\right]=$$

we will rewrite it in this way:

$$\sum_{k=0}^{n-1}\left[-(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=$$

This is a telescoping series!

$$k=0::\left(-(-2)^{n-(0-1)}\frac{n!}{(2n)!}\frac{(n+(0-1))!0}{(n-(0-1))!(0)!}+(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!(0+1)}{(n-0)!(0+1)!}\right)$$

$$k=1::\left(-(-2)^{n-(1-1)}\frac{n!}{(2n)!}\frac{(n+(1-1))!1}{(n-(1-1))!(1)!}+(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!(1+1)}{(n-1)!(1+1)!}\right)$$

$$k=2::\left(-(-2)^{n-(2-1)}\frac{n!}{(2n)!}\frac{(n+(2-1))!2}{(n-(2-1))!(2)!}+(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!(2+1)}{(n-2)!(2+1)!}\right)$$

…

$$k=n-1::\left(-(-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-1-1))!(n-1)}{(n-(n-1-1))!(n-1)!}+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+n-1)!(n-1+1)}{(n-n+1)!(n-1+1)!}\right)$$

we will write it a bit cleaner:

$$\sum_{k=0}^{n-1}\left[-(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=\left(0+(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!(1)}{(n-0)!(1)!}\right)+$$

$$\left(-(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!1}{(n-0)!(1)!}+(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}\right)+\left(-(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}+(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!3!}\right)+$$

$$\left(-(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!(3)!}+(-2)^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!4}{(n-3)!4!}\right)+\ldots+\left(-(-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-2))!(n-1)}{(n-(n-2))!(n-1)!}+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+(n-1))!n}{(n-(n-1))!(n)!}\right)$$

now we will highlight the terms of the telescoping series, and we will get this:

$$\sum_{k=0}^{n-1}\left[(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=0+$$

$$\left((-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!1}{(n-0)!(1)!}-(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!1}{(n-0)!(1)!}\right)+\left((-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}+-(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}\right)+$$

$$\left((-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!3!}-(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!3!}\right)+\left((-2)^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!4}{(n-3)!4!}-(-2)^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!4}{(n-3)!4!}\right)+\ldots+$$

$$\left((-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-2))!(n-1)}{(n-(n-2))!(n-1)!}-(-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-2))!(n-1)}{(n-(n-2))!(n-1)!}\right)+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+(n-1))!n}{(n-(n-1))!(n)!}$$

only the first and last terms are left (the first term in this telescoping series is 0)

$$\sum_{k=0}^{n-1}\left[(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=0+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+(n-1))!n}{(n-(n-1))!(n)!}=-1$$

now we will go back to our original form that we wanted to find out that value for:

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}+1\right]\right]=-1$$

$$\frac{(-2)^n n!}{(2n)!}\sum_{k=0}^{n-1}\left[\left(\frac{1}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}+1\right]\right]+1=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{1}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{1}{(-2)^n}\right)\frac{(2n)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{e^2}{(-2)^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\frac{(2n+1)(n+k)!}{(n-k)!k!}\cdot\frac{2k}{(n+1-k)(n+k)}=\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}$$

(The above result is a mathematical identity obtained by simple arithmetic)

$$\boxed{\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}\right)\frac{(2n+1)!}{n!}=0}$$

Main Idea

Now we can start:

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}\right)\frac{(2n+1)!}{n!}=0 \quad \text{(Formula no.1)}$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0 \quad \text{(Formula no.2)}$$

Combining formula no. 1 and formula no. 2 will give us this:

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

Now lets move the 3[rd] term in the summation brackets from the LHS to the RHS

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}\right]+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}\right]$$

now let's rewrite $\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}$ a bit differently

$$\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=$$

$$\left(\frac{e^2}{(-2)^n}-\frac{(-1)^n}{2^n}+\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}-\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=$$

$$\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(2n+1)!}{n!}+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}+2\cdot\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(2n+1)!}{n!}$$

(The above result is a mathematical identity obtained by simple arithmetic)

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}\right]+(2n+1)\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(2n)!}{n!}+$$
$$\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}+\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(2n+2)!}{(n+1)!}=\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}\right]$$

Let's restructure it a bit (by splitting the summation).

$$\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+(2n+1)\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(n+n)!}{(n-n)!n!}+\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}+$$
$$\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(n+1+n)!}{(n+1-n)!n!}+\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(n+1+n+1)!}{(n+1-n-1)!(n+1)!}=\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}\right]$$

Let's restructure it a bit more:

$$(2n+1)\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}+\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}=\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}$$

we will divide by $\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}$ and we will get this:

$$2n+1+\frac{\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}=\frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

Let's make one more adjustment.

$$2n+1+\frac{1}{\dfrac{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}{\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}}=\frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}} \qquad \leftarrow \text{Main result!}$$

We will define a new series that we will call A_n as follows:

$$A_n=\frac{\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}} \quad\Rightarrow\quad A_{n+1}=\frac{\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}{\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n}}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}$$

please notice that we got $2n+1+\frac{1}{A_{n+1}}=A_n$ from our main result!

this is what we wanted to get!

(remember this because we will use this result later in the Induction step)

now lets find out what is the value of A_1

$$A_1=\frac{\sum_{k=0}^{0}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{1}}{2^k}\right)\frac{(1-1+k)!}{(1-1-k)!k!}}{\sum_{k=0}^{1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{1-1}}{2^k}\right)\frac{(1+k)!}{(1-k)!k!}}=\frac{\left(\frac{e^2}{(-2)^0}-\frac{1}{2^0}\right)\frac{(0)!}{(0)!0!}}{\left(\frac{e^2}{(-2)^0}+\frac{1}{2^0}\right)\frac{(1)!}{(1)!0!}+\left(\frac{e^2}{(-2)^1}+\frac{1}{2^1}\right)\frac{(1+1)!}{(1-1)!1!}}=\frac{e^2-1}{\left(e^2+1\right)+\left(-e^2+1\right)}=\frac{e^2-1}{2}$$

Lambert's continued fraction

$$\tanh(x) = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{x^2}{5+\cfrac{x^2}{7+\cfrac{x^2}{9+\cfrac{x^2}{11+\ddots}}}}}} = \frac{e^{2x}-1}{e^{2x}+1}$$

$$\tanh(1) = \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}} = \frac{e^2-1}{e^2+1}$$

$$\frac{e^2+1}{e^2-1} = 1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}}$$

$$\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}}} = \frac{e^2+1}{e^2-1}-1 = \frac{2}{e^2-1}$$

$$\boxed{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}} = \frac{e^2-1}{2} = A_1}$$

(just like the value we got above for A_1)

this is what we wanted!

(This is known result but I am showing how to get to it anyway)

A proof by induction

Proposition:

$$A_n = 2n+1+\cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

Base Case: $n=1$

$$A_1 = 3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{0}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^1}{2^k}\right)\frac{(1-1+k)!}{(1-1-k)!k!}}{\displaystyle\sum_{k=0}^{1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{1-1}}{2^k}\right)\frac{(1+k)!}{(1-k)!k!}} = \frac{e^2-1}{2} \quad \text{(I already showed that above!)}$$

Induction step:

$$A_{n+1} = 2(n+1)+1+\cfrac{1}{2(n+1)+3+\cfrac{1}{2(n+1)+5+\cfrac{1}{2(n+1)+7+\cfrac{1}{2(n+1)+9+\ddots}}}}$$

$$A_{n+1} = 2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\cfrac{1}{2n+11+\ddots}}}}$$

$$A_n = 2n+1+\cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}} = 2n+1+\frac{1}{A_{n+1}}$$

As previously demonstrated in the “Main result” section on page 7.

Q.E.D.