Complete Collatz directed graph

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Abstract

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches a trivial cycle 1, 2, 1, 2……The Collatz sequences can be represented as a directed graph. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity. In this paper, we construct a Collatz directed graph by connecting infinite number of basic directed graphs. Each basic directed graph relates to each natural number. We prove that the Collatz directed graph covers all positive integers and there is only a trivial cycle and no sequence goes to infinity.

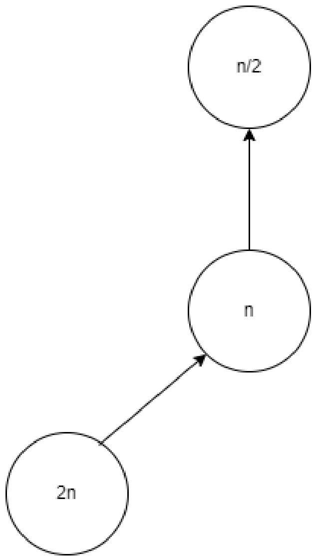
1. Introduction

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2…… The Collatz sequences can be represented as a directed graph. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity [1-2].

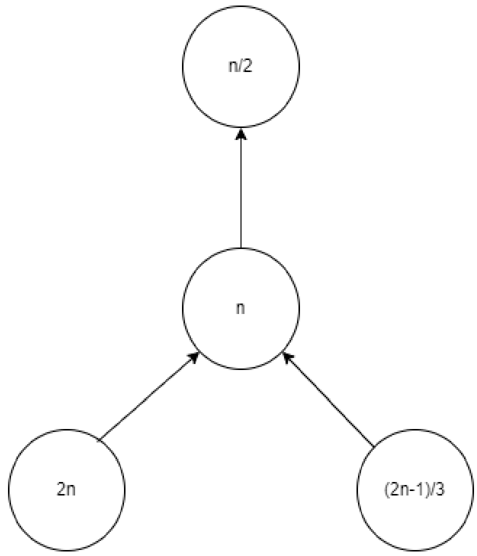
2. A basic directrd graph

The basic directed graph is constructed for each natural number as follows:

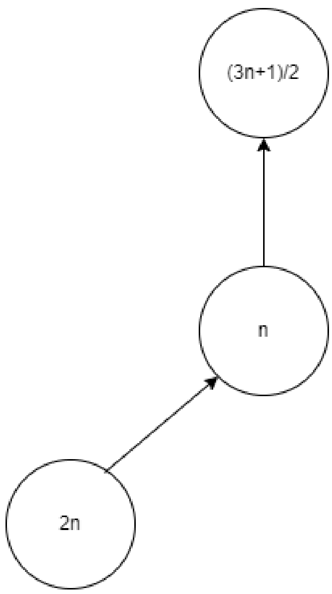
Let n be a positive integer node. Its parent node is $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. Its left child is 2n. Its right child is $\frac{2n-1}{3}$, if n $\equiv$ 2 mod 3, or no right child , if n $\neq$ 2 mod 3. Thus there are four types of basis directed graph as shown in Figure 1.



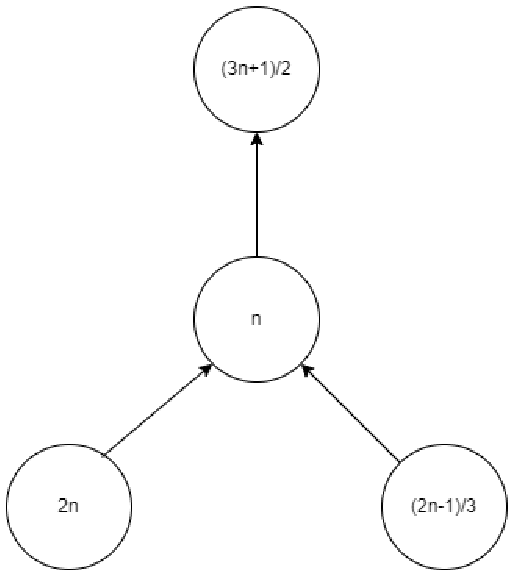
(a) n is even and not equal to 2 mod 3



(b) n is even and equals to 2 mod 3



(c) n is odd and not equal to 2 mod 3



(d) n is odd and equals to 2 mod 3

Figure 1, Four types of basic directed graphs

Examples of basic directed graphs shown in Figure 2.

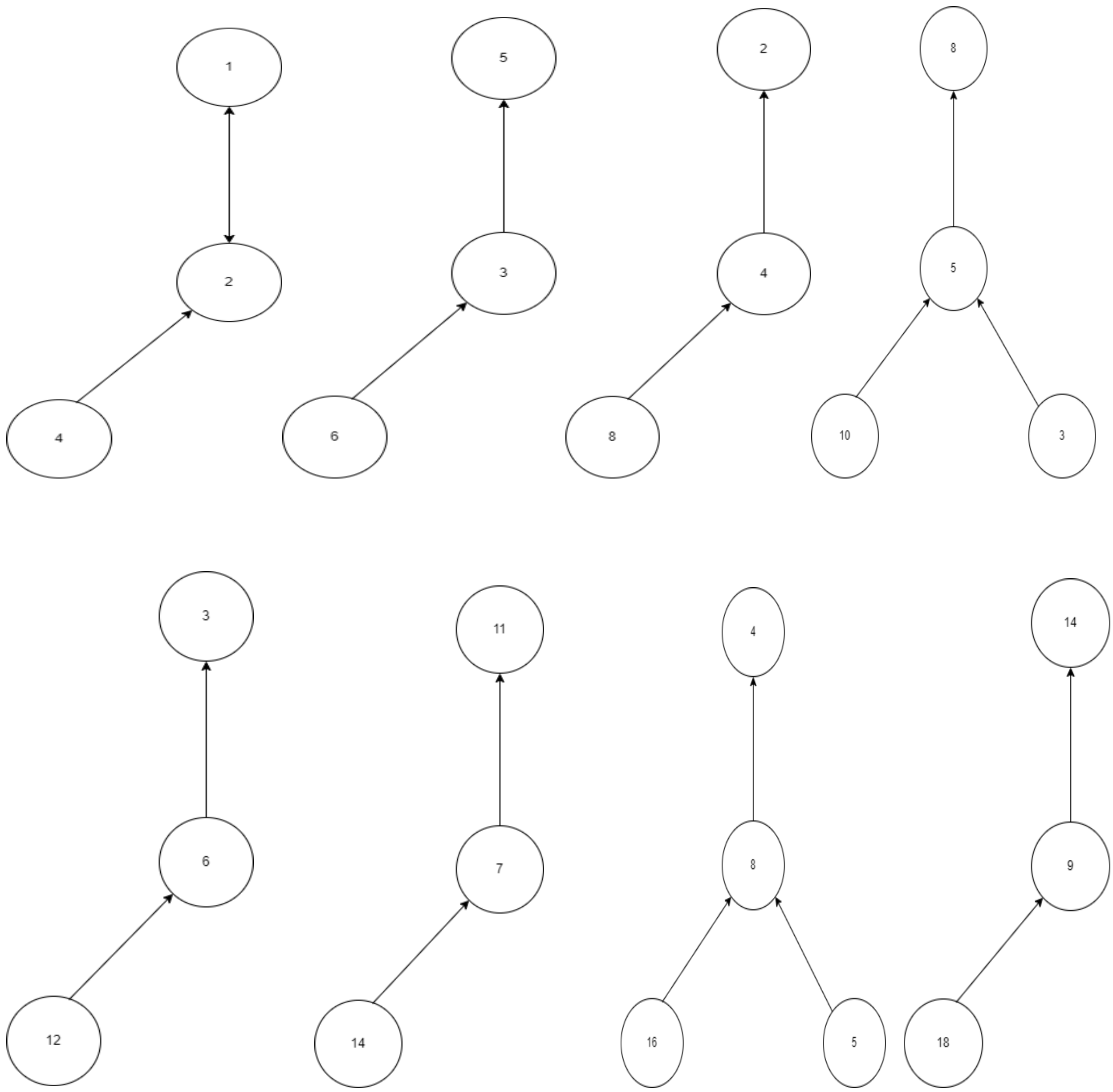


Figure 2. Basic directed graphs of positive integers 2, 3, 4, 5, 6, 7, 8, 9

2. **How to connect two basic directed graphs**

A simple rule to connect two basic directed graph is that these two basic directed graphs must have a common edge as shown in Figure 3.

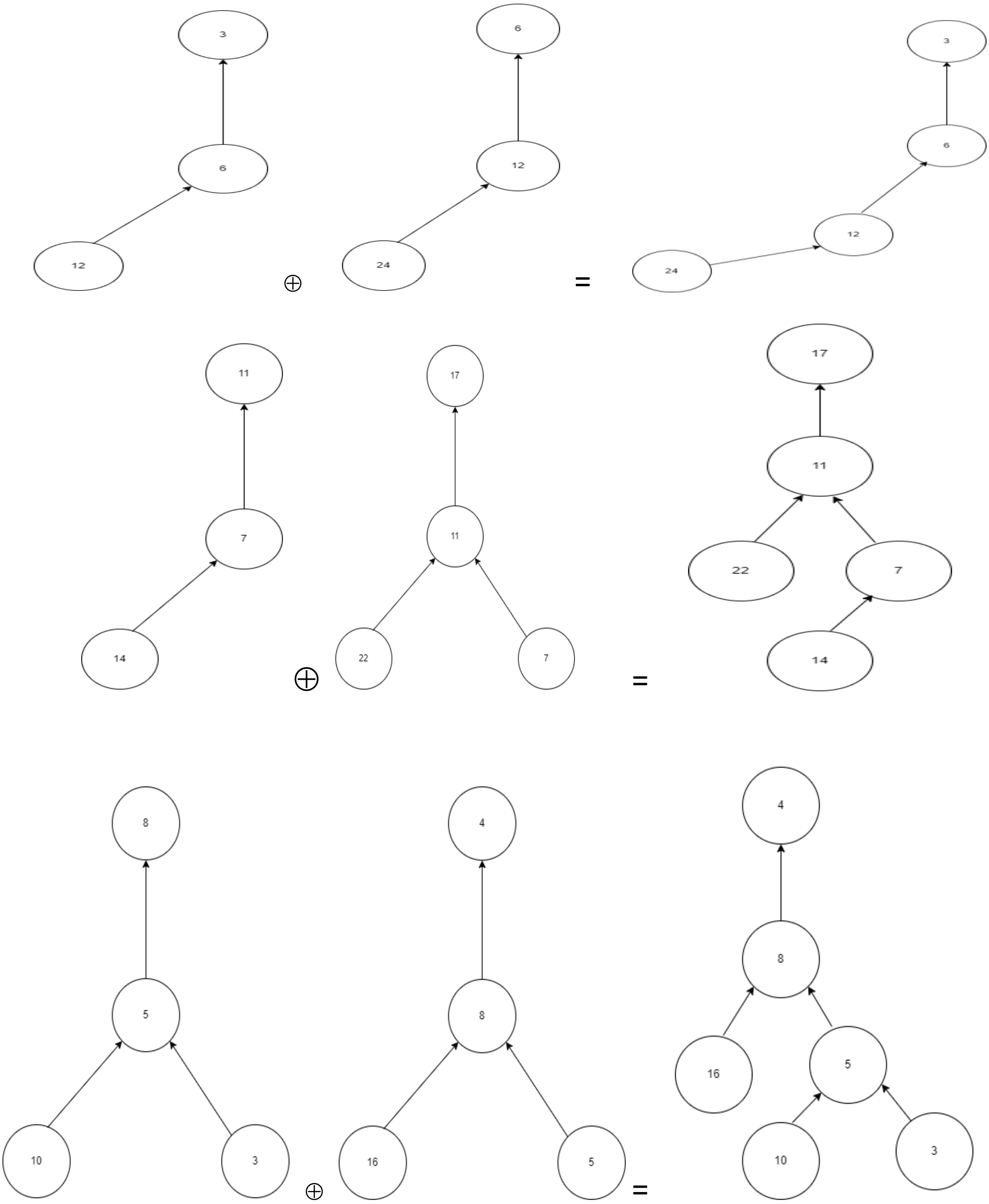


Figure 3. Various cases of basic graphs connection

3. **A Collatz directed graph**

Let start with a basic graph of node 2, connecting basic graphs according to connection rules, the complete directed graph is shown in Figure 4. There is no nontrivial cycle or divergence sequence in this graph. But in order to prove the Collatz conjecture, this graph must cover all positive integers.

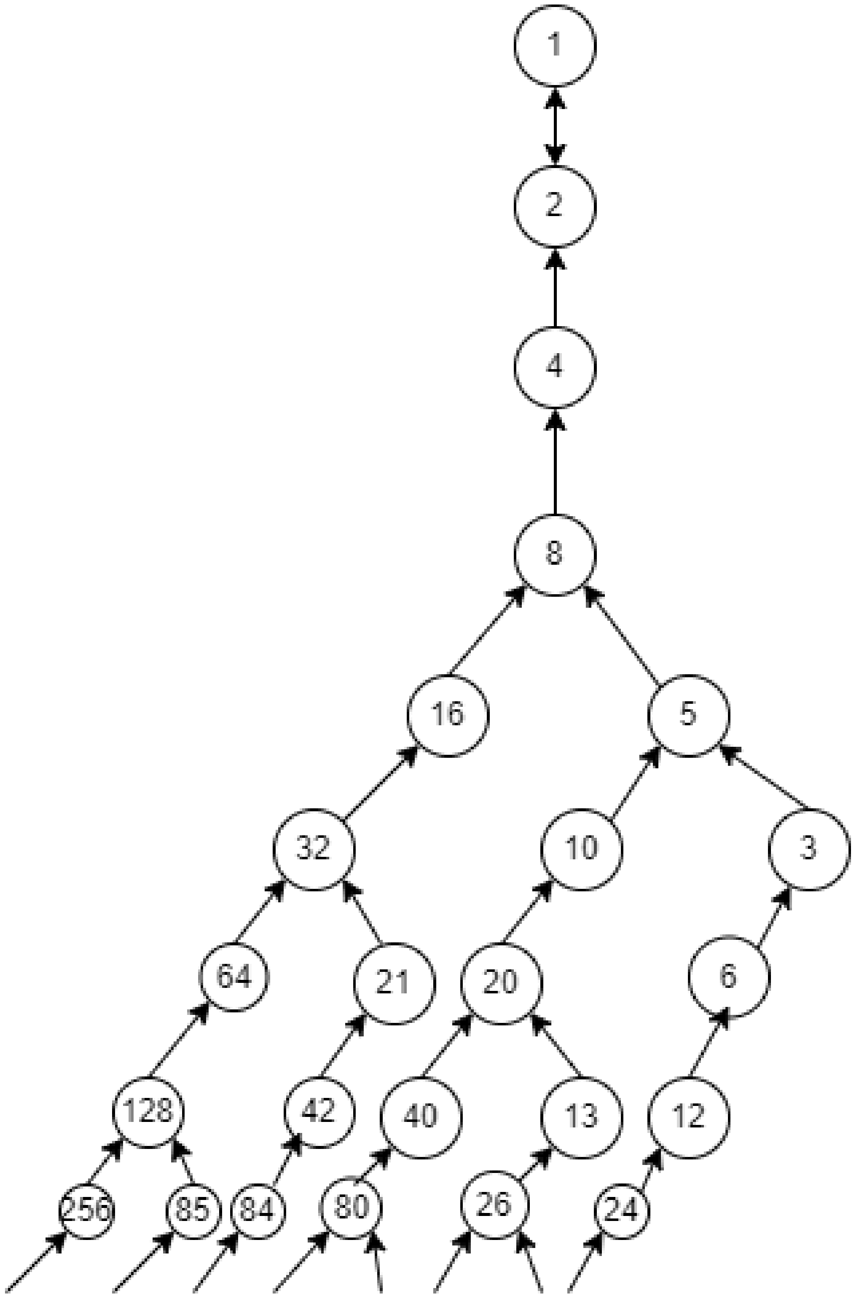


Figure 4. A complete directed graph

We can show that there is only a trivial cycle 1, 2, 1, 2….in the Colltaz directed graph. Let assume there is a nontrivial cycle which is not a part of the Collatz graph as shown in Figure 5.

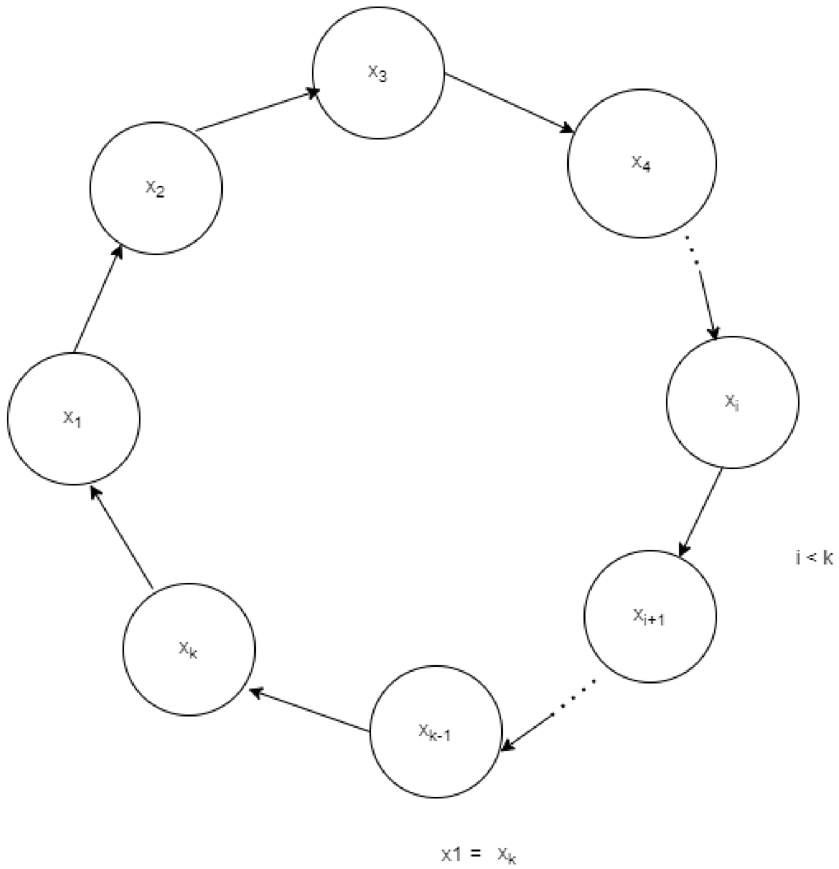


Figure 5. A nontrivial cycle, not part of the Collatz graph

According to definition of the Collatz iteration, if a number we chose at the beginning is even by continuing to divide all even number by 2, one of the odd number will remain. For this reason, it is only sufficient to consider only odd number. Cosider x_1 which is odd number in the form 6n+1 or 6m+3 , this node has only one input and one output. By apply a Collatz iteration rule, x_2 can be found as follows:

$$X_2 = \frac{3(6n+1)+1}{2} = 9n+2 \text{ , if } x_1 = 6n+1$$

Or $$X_2 = \frac{3(6n+3)+1}{2} = 9n+5 \text{ , if } x_1 = 6n+3$$

Since $x_2 \equiv 2 \bmod 3$ then node x_2 must have two inputs and one of its input must be in the Collatz directed graph. Thus, a nontrivial cycle is impossible.

We can show that there is no divergence sequences of positive integers. Let have one divergence sequence not part of the Collatz graph as shown in Figure 6.

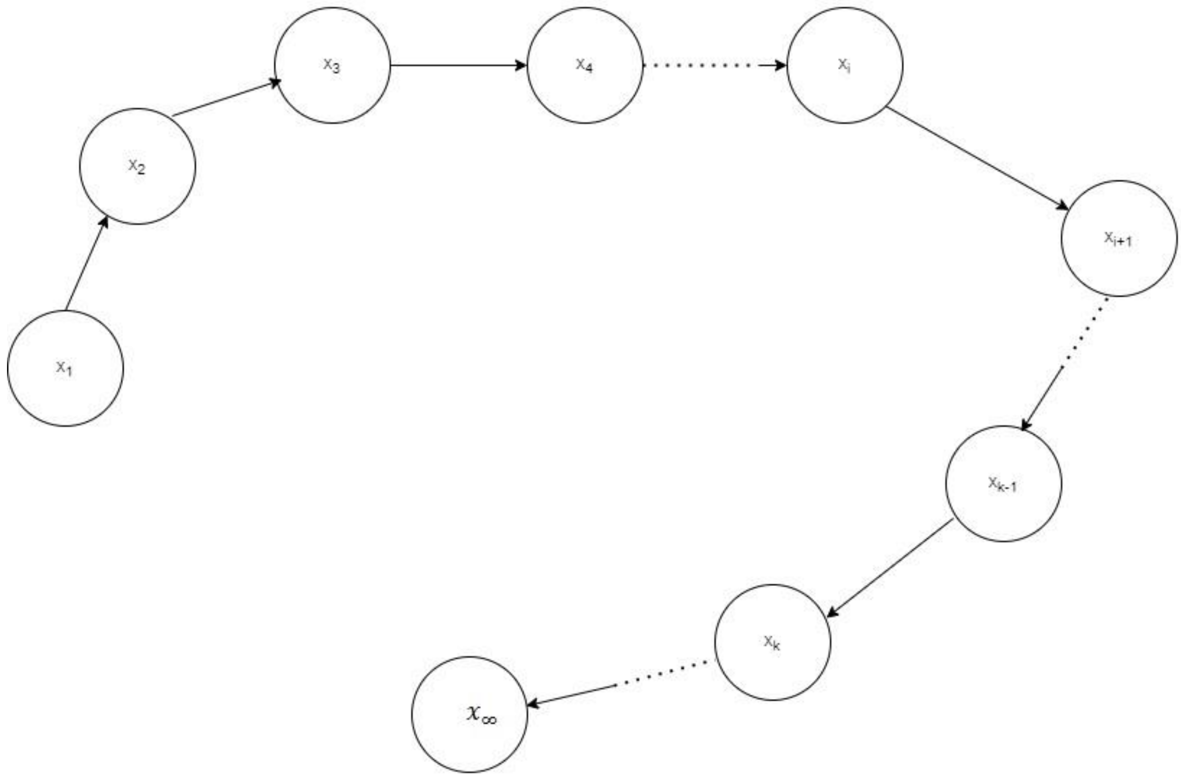


Figure 6. A divergence sequence

Let x_1 be a positive integer, there will be at least one positive integer node x_0 that can connect to x_1 and x_0 must be in the Collatz graph. Thus, a divergence sequence not part of the Collatz graph is impossible.

4. **Conclusion**

A Collatz directed graph covers all positive integers. By starting at any node in this Collatz directed graph, there is a unique path from that node to a node 1.

References

[1] R . Terras, (1976). “ A stopping time problem on the positive integers”. Acta Arithmetica, 30(3), 241-252.

[2] J. C. Lagarias. The 3x+1 Problem: An Overview. https://arxiv.org/abs/2111.02635, 2021.