Two alternative Arnowitt-Dresner-Misner(ADM) formalisms using the conventions adopted by Misner-Thorne-Wheeler(MTW) and Alcubierre applied to the Natario warp drive spacetime.

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Abstract

General Relativity describes the gravitational field using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor). However there are situations in which we need to recover the difference between space and time.The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a given spacetime the 3 dimensions of space(hypersurfaces) and the time dimension. Using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expression:

$$g_{\mu\nu}\,dx^\mu\,dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1}$$

In the equation above α is the lapse function β^i and β^j are the shift vectors and γ_{ij} is related to the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ and signature $(+,+,+)$ that measures the proper distance between two points inside each hypersurface. Expanding all the terms of this equation we will get both contravariant and covariant components. We propose two new ADM equations:
The first proposed equation is the 3+1 parallel contravariant ADM formalism with signature $(-,+,+,+)$ is given by:

$$g_{\mu\nu}\,dx^\mu\,dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{2}$$

Expanding all the terms of this equation we will get only contravariant components.
The second proposed equation is the 3+1 parallel covariant ADM formalism with signature $(-,+,+,+)$ is given by:

$$g_{\mu\nu}\,dx^\mu\,dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{3}$$

Expanding all the terms of this equation we will get only covariant components.
We apply all these equations to the Natario warp drive spacetime for both constant and variable velocities.

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1 Introduction:

The Warp Drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8],[9],[10] and [18]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7],[8] and [18]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7] and [8]).

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8],[9],[10] and [18]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

[1] do not violates Relativity

Alcubierre([12]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([11]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pgs [67(b)],[82(a)] in [12], see also eq 1 pg 3 in [1])[2][3] and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.

The Natario warp drive equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see (21.40) pg $[507(b)]$ $[534(a)]$ in [11])

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{4}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ making $\alpha = 1$ and inserting the components of the Natario vector we have:

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{5}$$

Several years ago some works appeared in the scientific literature ([16] and [17]) advocating two new parallel $3+1$ ADM formalisms.While the original ADM formalism uses mixed contravariant and covariant scripts one of the new parallel formalisms uses only contravariant scripts while the other uses only covariant scripts.

The Natario warp drive equation with signature $(-,+,+,+)$ that obeys the parallel contravariant $3+1$ ADM formalism is given below:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{6}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ making $\alpha = 1$ and inserting the components of the Natario vector we have:

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{7}$$

The Natario warp drive equation with signature $(-,+,+,+)$ that obeys the parallel covariant $3+1$ ADM formalism is given below:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{8}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ making $\alpha = 1$ and inserting the components of the Natario vector we have:

$$ds^2 = [1 - (X_{rs})^2 - (X_{\theta})^2]dt^2 + 2[X_{rs}drs + X_{\theta}rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{9}$$

[2] see Appendix E

[3] see the Remarks section on our system to quote pages in bibliographic references

However some important things must be outlined in both the Alcubierre and Natario warp drive spacetimes or the works in ([16] and [17]) :

- 1)-The warp drives as proposed by Alcubierre,Natario or the works in ([16] and [17]) always have a constant speed vs.They do not accelerate or de-accelerate and travel always with a constant speed.But a real warp drive must "know" how to accelerate for example from 0 to a speed of 200 times faster than light in the beginning of an interstellar journey and in the end of the journey it must de-accelerate again to 0 in the arrival at the destination point which means to say of course a distant star.

- 2)-The warp drives as proposed by Alcubierre or Natario always have a constant speed vs raised to the square in their equations for the negative energy density.An accelerating warp drive probably must have the terms of variable velocities or accelerations included in the expression for the negative energy density since this energy is responsible for the generation of the warp drive spacetime.

Since the Natario vector is the generator of the Natario warp drive spacetime metric in this work we expand the original Natario vector including the coordinate time as a new Canonical Basis for the Hodge star generating an expanded Natario vector and extended versions of the Natario warp drive spacetime metric which encompasses accelerations and variable velocities.

Our proposed extended Natario warp drive metric with variable velocity vs due to a constant acceleration a in the $3+1$ original ADM formalism is given by the following equation:

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{10}$$

Our proposed extended Natario warp drive metric with variable velocity vs due to a constant acceleration a in the $3+1$ parallel contravariant ADM formalism is given by the following equation:

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2 - (X^\theta)^2)dt^2 + 2(X^{rs}drs + X^\theta rsd\theta)dt - drs^2 - rs^2 d\theta^2 \tag{11}$$

Our proposed extended Natario warp drive metric with variable velocity vs due to a constant acceleration a in the $3+1$ parallel covariant ADM formalism is given by the following equation:

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2 - (X_\theta)^2)dt^2 + 2(X_{rs}drs + X_\theta rsd\theta)dt - drs^2 - rs^2 d\theta^2 \tag{12}$$

Note that in these equations when compared with the equations given in the previous page new sets of contravariant and covariant components X^t and X_t appears because in these cases as the velocity vs changes its value as times goes by due to a constant acceleration a this affects the whole spacetime geometry in all these equations.

Note that in the original ADM formalism mixed contravariant and covariant terms X^t and X_t appears together while in the contravariant parallel ADM formalism only the contravariant term X^t appears and in the covariant parallel ADM formalism only the covariant term X_t appears.

Two important things must be outlined by now:

- 1)-The Natario shape function used in the equations with constant speed is valid also in the equations with variable speed.

- 2)-All these equations satisfies the Natario criteria for a warp drive spacetime.

In this work we present the new extended equations for the Natario warp drive spacetime which encompasses accelerations and variable speeds using also both the original and parallel ADM formalisms and we arrive at the conclusion that the new equations are also valid solutions for the warp drive spacetime according to the Natario criteria.

For the study of the original ADM formalism we use the approaches of MTW([11]) and Alcubierre([12]) and we adopt the Alcubierre convention for notation of equations and scripts.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes and the International System of units for energetic purposes.

This work is organized as follows:

- Section 2)-Introduces the Natario warp drive continuous shape function able to low the negative energy density requirements when a ship travels with a speed of 200 times faster than light.
  The negative energy density for such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!.

- Section 3)-presents the original equation for the Natario warp drive spacetime with a constant velocity vs in the original $3+1$ ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendices A and E at the end of the work in order to fully understand the mathematical demonstrations.

- Section 4)-presents the extended equation for the Natario warp drive spacetime with a variable velocity vs and a constant acceleration a in the original $3+1$ ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendices B,C and F at the end of the work in order to fully understand the mathematical demonstrations.

- Section 5)-presents the original equation for the Natario warp drive spacetime with a constant velocity vs in the $3+1$ parallel contravariant ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendix H at the end of the work.

- Section 6)-presents the original equation for the Natario warp drive spacetime with a variable velocity vs and a constant acceleration a in the $3+1$ parallel contravariant ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendix I at the end of the work.

- Section 7)-presents the original equation for the Natario warp drive spacetime with a constant velocity vs in the $3+1$ parallel covariant ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendix J at the end of the work.

- Section 8)-presents the original equation for the Natario warp drive spacetime with a variable velocity vs and a constant acceleration a in the $3+1$ parallel covariant ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendix K at the end of the work.

- Section 9)-compares all these original and extended equations and we point out the fact that the shape function used in one equation is also valid in the other equations and since the derivatives of first or second order of the shape function low the negative energy density requirements in the one of these equations these derivatives may perhaps be able to low the same requirements in the other equations.We also point out that in the $1+1$ spacetime all these equations can be reduced to two equivalent forms one form for constant speeds and another form for variable speeds.

The $3+1$ original ADM formalism with signature $(-,+,+,+)$ is given by the equation (21.40) pg [$507(b)$] [$534(a)$] in [11]:(see Appendices E and F)

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{13}$$

The $3+1$ parallel contravariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:(see Appendices H and I)

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{14}$$

The $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:(see Appendices J and K)

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{15}$$

The $3+1$ original ADM formalism with signature $(-,+,+,+)$ and with the lapse function explicitly defined is given by the equation:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\gamma_{tt}(1+\beta^t)^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{16}$$

The 3+1 parallel contravariant ADM formalism with signature $(-,+,+,+)$ and with the lapse function explicitly defined is given by the equation:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -(\sqrt{\gamma_{tt}} + \beta^t)^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{17}$$

The $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ and with the lapse function explicitly defined is given by the equation:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -(\sqrt{\gamma_{tt}} + \beta_t)^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{18}$$

Note that we managed to define each lapse function according to each ADM formalism.

For the original formalism we have the term $\gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ and the lapse function $\gamma_{tt}(1+\beta^t)^2 dt^2$ using the same mathematical structure in which the terms γ_{ij},β^i or β^j from the spatial components appears with the terms γ_{tt} and β^t of the corresponding time components.

For the parallel contravariant formalism we have the term $(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt)$ and the lapse function $(\sqrt{\gamma_{tt}} + \beta^t)^2 dt^2$ using the same mathematical structure in which the terms $\sqrt{\gamma_{ii}}$,β^i,$\sqrt{\gamma_{jj}}$ and β^j from the spatial components appears with the terms $\sqrt{\gamma_{tt}}$ and β^t of the corresponding time components.

For the parallel covariant formalism we have the term $(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt)$ and the lapse function $(\sqrt{\gamma_{tt}} + \beta_t)^2 dt^2$ using the same mathematical structure in which the terms $\sqrt{\gamma_{ii}}$,β_i,$\sqrt{\gamma_{jj}}$ and β_j from the spatial components appears with the terms $\sqrt{\gamma_{tt}}$ and β_t of the corresponding time components..

The term γ_{ij} is related to the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.

$dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric with signature (+,+,+) then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating(see preface of second edition and pg $[337(a)], [337)b)]$ in [13]) and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pgs $[106(a)], [98(b)]$ in [23] and pgs $[394(a)][377(b)]$ in [24] pgs $[119(a)][226(b)]$ in [28]).

It is very important to note that if a galaxy in the other side of the Universe at a billion light-years of distance outside the range of our Particle Horizon is moving away from us at a faster than light speed then superluminal velocities may well exists in Nature.So the warp drive may not be impossible at all.Natario also points out exactly this.(see pgs [10] and [11] in [29]).

What Alcubierre and Natario did was an attempt to reproduce the expansion of the Universe in a local way creating a local spacetime distortion that expands the spacetime behind a ship and contracts spacetime in front reproducing the superluminal expansion of the Universe moving away the departure point in an expansion and bringing together the destination point in a contraction.The expansion-contraction can be seen in the abs of the original Alcubierre paper in [1].Although Natario says in the abs of his paper in [2] that the expansion-contraction does not occurs in its spacetime in pg 5 of the Natario paper we can see the expansion-contraction occurring however the expansion of the normal volume elements or the trace of the extrinsic curvature is zero because the contraction in the radial direction is exactly balanced by the expansion in the perpendicular directions.

An excellent explanation on how a spacetime distortion or a perturbation pushes away a spaceship from the departure point and brings the ship to the destination point at faster than light speed can be seen at pg 34 in [21],pgs $[260(a)260(b)][261(a)261(b)]$ in [22].Note that in these works it can be seen that the perturbation do not obeys the time dilatation of the Lorentz transformations hence the speed limit of Special Relativity cannot be applied here.

An accelerated warp drive can only exists if the astronaut in the center of the warp bubble can somehow communicate with the warp bubble walls sending instructions to change its speed.But for signals at light speed the Horizon exists so light speed cannot be used to send signals to the front of the bubble.(see pg 16 in [7] and pg 21 in [8]).Besides in the Natario warp drive the negative energy density covers the entire bubble.(see pg 52 in [7] and pg 51 in [8]).Since the negative energy density have repulsive gravitational behavior the photon of light if possible to reach the bubble walls would then be deflected by the repulsive behavior never reaching the bubble walls(see pg $[116(a)][116(b)]$ in [26])

The solution that allows contact with the bubble walls was presented in pg 28 in [7] and pg 31 in [8]. Although the light cone of the external part of the warp bubble is causally disconnected from the astronaut who lies inside the large bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro bubble so these bubbles can be "engineered" to be sent to the large bubble. This idea seems to be endorsed by pg 34 in [21],pgs $[268(a)268(b)]$ in [22] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

Although this work was written to be independent self-contained and self-consistent it must be regarded as a companion work to our works in [16],[17] and in [20].

2 The Natario warp drive continuous shape function

Introducing here $f(rs)$ as the Alcubierre shape function that defines the Alcubierre warp drive spacetime we can construct the Natario shape function $n(rs)$ that defines the Natario warp drive spacetime using its Alcubierre counterpart.Below is presented the equation of the Alcubierre shape function.[4].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{19}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{20}$$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

The Natario warp drive continuous shape function can be defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{21}$$

$$n(rs) = \frac{1}{2}[1 - [\frac{1}{2}[1 - tanh[@(rs - R)]]]] \tag{22}$$

This shape function gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

[4] $tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

Another Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \quad (23)$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \quad (24)$$

The shape function above also gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that like in the previous case the Alcubierre shape function is being used to define its Natario shape function counterpart. The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.

For the second Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

- Numerical plot for the second shape function with @ = 50000 $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99970000000E+001$	1	0	$2,650396620740E-251$	0
$9,99980000000E+001$	1	0	$1,915169647489E-164$	0
$9,99990000000E+001$	1	0	$1,383896564748E-077$	0
$1,00000000000E+002$	0,5	0,5	$6,250000000000E+008$	$3,872591914849E-103$
$1,00001000000E+002$	0	0,5	$1,383896486082E-077$	0
$1,00002000000E+002$	0	0,5	$1,915169538624E-164$	0
$1,00003000000E+002$	0	0,5	$2,650396470082E-251$	0

- Numerical plot for the second shape function with @ = 75000 $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99980000000E+001$	1	0	$5,963392481410E-251$	0
$9,99990000000E+001$	1	0	$1,158345097767E-120$	0
$1,00000000000E+002$	0,5	0,5	$1,406250000000E+009$	$8,713331808411E-103$
$1,00001000000E+002$	0	0,5	$1,158344999000E-120$	0
$1,00002000000E+002$	0	0,5	$5,963391972940E-251$	0

- Numerical plot for the second shape function with @ = 100000 $R = 100$ meters and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99990000000E+001$	1	0	$7,660678807684E-164$	0
$1,00000000000E+002$	0,5	0,5	$2,500000000000E+009$	$1,549036765940E-102$
$1,00001000000E+002$	0	0,5	$7,660677936765E-164$	0

The plots in the previous page demonstrate the important role of the thickness parameter @ in the warp bubble geometry wether in both Alcubierre or Natario warp drive spacetimes.For a bubble of 100 meters radius $R = 100$ the regions where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$(Natario warped region) becomes thicker or thinner as @ becomes higher.

Then the geometric position where both Alcubierre and Natario warped regions begins with respect to R the bubble radius is $rs = R - \epsilon < R$ and the geometric position where both Alcubierre and Natario warped regions ends with respect to R the bubble radius is $rs = R + \epsilon > R$

As large as @ becomes as smaller ϵ becomes too.

Note from the plots of the previous page that we really have two warped regions:

- 1)-The geometrized warped region where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$ (Natario warped region).
- 2)-The energized warped region where the derivative squares of both Alcubierre and Natario shape functions are not zero.

The parameter @ affects both energized warped regions wether in Alcubierre or Natario cases but is more visible for the Alcubierre shape function because the warp factor WF in the Natario shape functions squeezes the energized warped region into a very small thickness.

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{25}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression(see Appendix G):

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{26}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for(see Appendix D):

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{27}$$

In the equatorial plane(1 + 1 dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$):

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{28}$$

Note that in the above expressions the warp drive speed vs appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[5] of about $6 \times 10^{24} kg$

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths!!!

Note that if the negative energy density is proportional to 10^{48} this would render the warp drive impossible but fortunately the square derivative of the Natario shape function possesses values of 10^{-102} ameliorating the factor 10^{48} making the warp drive negative energy density more "affordable". For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

[5]see Wikipedia:The free Encyclopedia

3 The equation of the Natario warp drive spacetime metric with a constant speed vs in the original $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix E for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{29}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{30}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2])(see also Appendix A for details)

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{31}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{32}$$

The covariant shift vector components X_{rs} and X_{θ} are given by:

$$X_{rs} = 2v_s n(rs)\cos\theta \tag{33}$$

$$X_{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{34}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{35}$$

The equation above was written using both contravariant and covariant shift vector components of the Natario vector at the same time.

Since $X_{rs} = X^{rs}$ the equation in the $1+1$ spacetime can be written as given below:

- 1)-contravariant form;all the shift vector components of the Natario vector are contravariant

$$ds^2 = (1 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{36}$$

- 2)-covariant form:all the shift vector components of the Natario vector are covariant

$$ds^2 = (1 - (X_{rs})^2)dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{37}$$

The equal contravariant and covariant shift vector component X_{rs} and X^{rs} are then:

$$X^{rs} = X_{rs} = 2v_s n(rs) \tag{38}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.

4 The equation of the Natario warp drive spacetime metric with a variable speed vs due to a constant acceleration a in the original $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix F for details)

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta) dt^2 + 2(X_{rs} drs + X_\theta d\theta) dt - drs^2 - rs^2 d\theta^2 \tag{39}$$

The equation of the Natario vector nX is given by:

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \tag{40}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by(see Appendices B and C):

$$X^t = 2n(rs) rs cos\theta a \tag{41}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)] at cos\theta \tag{42}$$

$$X^\theta = -2n(rs) at [2n(rs) + rsn'(rs)] \sin\theta \tag{43}$$

The covariant shift vector components X_t,X_{rs} and X_θ are given by:

$$X_t = 2n(rs) rs cos\theta a \tag{44}$$

$$X_{rs} = 2[2n(rs)^2 + rsn'(rs)] at cos\theta \tag{45}$$

$$X_\theta = -2n(rs) at [2n(rs) + rsn'(rs)] rs^2 \sin\theta \tag{46}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx + xdvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get¿:

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs})dt^2 + 2(X_{rs} drs)dt - drs^2 \tag{47}$$

The equation above was written using both contravariant and covariant shift vector components of the Natario vector at the same time.

Since $X_t = X^t$ and $X_{rs} = X^{rs}$ the equation in the $1+1$ spacetime can be written as given below[6]:

- 1)-contravariant form;all the shift vector components of the Natario vector are contravariant

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2)dt^2 + 2(X^{rs} drs)dt - drs^2 \tag{48}$$

- 2)-covariant form:all the shift vector components of the Natario vector are covariant

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2)dt^2 + 2(X_{rs} drs)dt - drs^2 \tag{49}$$

The equal contravariant and covariant shift vector component X_t X^t X_{rs} and X^{rs} are then:

$$X^t = X_t = 2n(rs)rsa \tag{50}$$

$$X^{rs} = X_{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{51}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{52}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero[7] and the covariant shift vector $X_{rs} = 2[2n(rs)^2]at$ with $X_{rs} = 0$ inside the bubble and $X_{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

[6] in a geometrized system of units $\gamma_{tt} = 1$

[7] except in the neighborhoods of the bubble radius.See Section 2

5 The equation of the Natario warp drive spacetime metric in the parallel contravariant $3+1$ ADM formalism for a constant speed vs

The warp drive spacetime according to Natario for the coordinates rs and θ in the parallel contravariant $3+1$ ADM formalism is defined by the following equation:(see Appendix H for details)

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{53}$$

The expressions for X^{rs} and X^{θ} are given by:(see pg 5 in [2],see also Appendix A for details)

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{54}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{55}$$

Looking both the equation of the Natario warp drive and the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{56}$$

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{57}$$

We can see that the Natario vector is completely inserted twice in the non-diagonalized components of the metric of the Nayario warp drive equation which gives:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs)\cos\theta \tag{58}$$

$$g_{02} = g_{20} = X^{\theta}rs = -v_s(2n(rs) + (rs)n'(rs))rs\sin\theta \tag{59}$$

Since we have two sets of non-diagonalized components in the Natario warp drive equation and each set possesses equal components of the Natario vector nX this is the reason why the Natario vector nX appears twice in the Natario warp drive equation.

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given in the previous page satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

The statement above can be explained in the following way:

Consider the Natario vector nX(pg 2 and 5 in [2]) defined below as:

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{60}$$

The components of the Natario vector nX are X^{rs} and X^{θ}.These are the shift vectors.Then a Natario vector is constituted by one or more shift vectors.

When the Natario shape function $n(rs) = 0$ inside the bubble then $X^{rs} = 2v_s n(rs)\cos\theta = 0$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta = 0$.Then inside the bubble both shift vectors are zero resulting in a zero Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then $X^{rs} = 2v_s n(rs)\cos\theta = v_s\cos\theta$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta = -v_s\sin\theta$.Then outside the bubble both shift vectors are not zero resulting in a not zero Natario vector.

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

The Natario warp drive equation and the Natario vector nX in the equatorial plane $1 + 1$ spacetime now becomes:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2[X^{rs} drs]dt - drs^2 \tag{61}$$

$$nX = X^{rs} drs \tag{62}$$

The equation above was written using only contravariant shift vector components of the Natario vector.

Since $X_{rs} = X^{rs}$ the equation in the $1 + 1$ spacetime can be written using only covariant shift vector components of the Natario vector as given below:

$$ds^2 = (1 - (X_{rs})^2)dt^2 + 2(X_{rs} drs)dt - drs^2 \tag{63}$$

The equal contravariant and covariant shift vector component X_{rs} and X^{rs} are then:

$$X^{rs} = X_{rs} = 2v_s n(rs) \tag{64}$$

Note that the Natario vector nX is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) \tag{65}$$

When the Natario shape function $n(rs) = 0$ inside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = 0$ .Then inside the bubble the shift vector $X^{rs} = 0$ is zero resulting in a zero Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = v_s$ .Then outside the bubble both shift and Natario vectors are not zero and the shift vector is equal to the bubble speed vs.Then if $X^{rs} = vs$ this explains the Natario affirmation of $X = 0$ inside the bubble and $X = vs$ outside the bubble.(pg 4 in [2])

6 The equation of the Natario warp drive spacetime metric with a variable speed vs due to a constant acceleration a in the parallel contravariant $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the parallel contravariant $3+1$ ADM formalism is given by:(see Appendix I for details)

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2 - (X^\theta)^2)dt^2 + 2(X^{rs}drs + X^\theta rsd\theta)dt - drs^2 - rs^2 d\theta^2 \quad (66)$$

The equation of the Natario vector nX is given by:

$$nX = X^t dt + X^{rs} drs + X^\theta rsd\theta \quad (67)$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by(see Appendices B and C):

$$X^t = 2n(rs)rscos\theta a \quad (68)$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \quad (69)$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \quad (70)$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx + xdvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1 + 1$ spacetime the equatorial plane we get:

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{71}$$

The equation above was written only contravariant shift vector components of the Natario vector.

Since $X_t = X^t$ and $X_{rs} = X^{rs}$ the equation in the $1 + 1$ spacetime can be written using only covariant shift vector components of the Natario vector as given below[8]:

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2)dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{72}$$

The equal contravariant and covariant shift vector component X_t X^t X_{rs} and X^{rs} are then:

$$X^t = X_t = 2n(rs)rsa \tag{73}$$

$$X^{rs} = X_{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{74}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{75}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero[9] and the covariant shift vector $X_{rs} = 2[2n(rs)^2]at$ with $X_{rs} = 0$ inside the bubble and $X_{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

[8]in a geometrized system of units $\gamma_{tt} = 1$

[9]except in the neighborhoods of the bubble radius.See Section 2

7 The equation of the Natario warp drive spacetime metric in the parallel covariant $3+1$ ADM formalism for a constant speed vs

The warp drive spacetime according to Natario for the coordinates rs and θ in the parallel covariant $3+1$ ADM formalism is defined by the following equation:(see Appendix J for details).

$$ds^2 = [1-(X_{rs})^2-(X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{76}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs}drs + X^\theta rsd\theta \tag{77}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [2],see also Appendix A for details):

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{78}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{79}$$

But remember that $dl^2 = \gamma_{ij}dx^i dx^j = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.Then the covariant shift vector components X_{rs} and X_θ with $r = rs$ are given by:

$$X_r = \gamma_{rr}X^r = X_{rs} = \gamma_{rsrs}X^{rs} = 2v_s n(rs)\cos\theta = X^r = X^{rs} \tag{80}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = rs^2 X^\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{81}$$

It is possible to construct a covariant form for the Natario vector nX defined as n_cX as follows:

$$n_cX = X_{rs}drs + X_\theta rsd\theta \tag{82}$$

With the covariant shift vector components X_{rs} and X_θ defined as shown above:

Looking both the equation of the Natario warp drive and the equation of the covariant Natario vector n_cX;

$$ds^2 = [1-(X_{rs})^2-(X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{83}$$

$$n_cX = X_{rs}drs + X_\theta rsd\theta \tag{84}$$

We can see that the covariant Natario vector is completely inserted twice in the non-diagonalized components of the metric of the Nayario warp drive equation which gives:

$$g_{01} = g_{10} = X_{rs} = 2v_s n(rs)\cos\theta = X^r = X^{rs} \tag{85}$$

$$g_{02} = g_{20} = X_\theta rs = rs^3 X^\theta = -rs^3 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{86}$$

Since we have two sets of non-diagonalized components in the Natario warp drive equation and each set possesses equal components of the covariant Natario vector n_cX this is the reason why the Natario vector n_cX appears twice in the Natario warp drive equation.

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given in the previous page satisfies the Natario requirements for a warp bubble defined by:

any covariant Natario vector n_cX generates a warp drive spacetime if $n_cX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $n_cX = -vs(t)dx$ or $n_cX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

The statement above can be explained in the following way:

Consider again the covariant Natario vector n_cX defined below as:

$$n_cX = X_{rs}drs + X_\theta rsd\theta \tag{87}$$

The covariant components of the Natario vector n_cX are X_{rs} and X_θ.These are the covariant shift vectors.Then a covariant Natario vector is constituted by one or more covariant shift vectors.

When the Natario shape function $n(rs) = 0$ inside the bubble then $X_{rs} = 2v_s n(rs)\cos\theta = 0$ and $X_\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta = 0$.Then inside the bubble both covariant shift vectors are zero resulting in a zero covariant Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then $X_{rs} = 2v_s n(rs)\cos\theta = v_s\cos\theta$ and $X_\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta = -rs^2 v_s\sin\theta$.Then outside the bubble both covariant shift vectors are not zero resulting in a not zero covariant Natario vector.

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

The Natario warp drive equation and the covariant Natario vector n_cX in the equatorial plane $1 + 1$ spacetime now becomes:

$$ds^2 = [1 - (X_{rs})^2]dt^2 + 2[X_{rs}drs]dt - drs^2 \tag{88}$$

The equation above was written using only covariant shift vector components of the Natario vector.

Since $X_{rs} = X^{rs}$ the equation in the $1 + 1$ spacetime can be written using only contravariant shift vector components of the Natario vector as given below:

$$ds^2 = (1 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{89}$$

The equal contravariant and covariant shift vector component X_{rs} and X^{rs} are then:

$$X^{rs} = X_{rs} = 2v_s n(rs) \tag{90}$$

$$n_c X = X_{rs} drs \tag{91}$$

Note that the covariant Natario vector $n_c X$ is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

$$g_{01} = g_{10} = X_{rs} = 2v_s n(rs) \tag{92}$$

When the Natario shape function $n(rs) = 0$ inside the bubble then the covariant shift vector $X_{rs} = 2v_s n(rs) = 0$ .Then inside the bubble the covariant shift vector $X_{rs} = 0$ is zero resulting in a zero covariant Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the covariant shift vector $X_{rs} = 2v_s n(rs) = v_s$ .Then outside the bubble both covariant shift and Natario vectors are not zero and the covariant shift vector is equal to the bubble speed vs.Then if $X_{rs} = vs$ this explains the Natario affirmation of $X = 0$ inside the bubble and $X = vs$ outside the bubble.(pg 4 in [2])

8 The equation of the Natario warp drive spacetime metric with a variable speed vs due to a constant acceleration a in the parallel covariant $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the parallel covariant $3+1$ ADM formalism is given by:(see Appendix K for details)

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2 - (X_\theta)^2)dt^2 + 2(X_{rs}drs + X_\theta rsd\theta)dt - drs^2 - rs^2d\theta^2 \qquad (93)$$

The equation of the Natario vector nX is given by:

$$nX = X^t dt + X^{rs} drs + X^\theta rsd\theta \qquad (94)$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by(see Appendices B and C):

$$X^t = 2n(rs)rscos\theta a \qquad (95)$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \qquad (96)$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)] \sin\theta \qquad (97)$$

The covariant shift vector components X_t,X_{rs} and X_θ are given by:

$$X_t = 2n(rs)rscos\theta a \qquad (98)$$

$$X_{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \qquad (99)$$

$$X_\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2 \sin\theta \qquad (100)$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We must demonstrate that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx + xdvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get:

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2)dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{101}$$

The equation above was written using only covariant shift vector components of the Natario vector.

Since $X_t = X^t$ and $X_{rs} = X^{rs}$ the equation in the $1+1$ spacetime can be written using only contravariant shift vector components of the Natario vector as given below[10]:

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{102}$$

The equal contravariant and covariant shift vector component X_t X^t X_{rs} and X^{rs} are then:

$$X^t = X_t = 2n(rs)rsa \tag{103}$$

$$X^{rs} = X_{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{104}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{105}$$

Remember that Natario(pg 4 in [2]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero[11] and the covariant shift vector $X_{rs} = 2[2n(rs)^2]at$ with $X_{rs} = 0$ inside the bubble and $X_{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

[10] in a geometrized system of units $\gamma_{tt} = 1$

[11] except in the neighborhoods of the bubble radius.See Section 2

9 Differences and resemblances between all these equations with constant or variable velocity vs in the original or parallel $3+1$ ADM formalism for the Natario warp drive spacetime

The equation in $3+1$ original ADM formalism for the Natario warp drive spacetime for a variable velocity vs is given by:(see Appendices B,C and F for details)

$$ds^2 = (1 - 2X_t + X_t X^t - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{106}$$

The equation in $3+1$ original ADM formalism for the Natario warp drive spacetime for a constant velocity vs is given by:(see Appendices A and E for details)

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{107}$$

The equation of the Natario warp drive spacetime for a variable velocity vs in the parallel contravariant $3+1$ ADM formalism is given by:(see Appendix I for details)

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2 - (X^\theta)^2)dt^2 + 2(X^{rs} drs + X^\theta rs d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{108}$$

The equation of the Natario warp drive spacetime in the parallel contravariant $3+1$ ADM formalism for a constant velocity vs is given by:(see Appendix H for details)

$$ds^2 = [1 - (X^{rs})^2 - (X^\theta)^2]dt^2 + 2[X^{rs} drs + X^\theta rs d\theta]dt - drs^2 - rs^2 d\theta^2 \tag{109}$$

The equation of the Natario warp drive spacetime in the parallel covariant $3+1$ ADM formalism for a variable velocity vs is given by:(see Appendix K for details)

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2 - (X_\theta)^2)dt^2 + 2(X_{rs} drs + X_\theta rs d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{110}$$

The equation of the Natario warp drive spacetime for a constant velocity vs in the parallel covariant $3+1$ ADM formalism is give by:(see Appendix J for details).

$$ds^2 = [1 - (X_{rs})^2 - (X_\theta)^2]dt^2 + 2[X_{rs} drs + X_\theta rs d\theta]dt - drs^2 - rs^2 d\theta^2 \tag{111}$$

Looking to the equations above we can see that the original ADM formalism uses both contravariant and covariant shift vector components for the Natario vector while the parallel contravariant or covariant ADM formalisms uses respectively only contravariant or covariant components.

Also from the previous sections we know that although in the $3+1$ spacetime these equations are different between each other,but due to the equivalence between contravariant and covariant components in the $1+1$ spacetime all these equations can be reduced to the same form using contravariant or covariant shift vector components in the case of the $1+1$ spacetime however the form for variable velocity vs uses the extra terms for the Natario vector due to the constant acceleration a which do not exists in the form for fixed velocity vs

The Natario vector for the equation with variable velocity vs is given by:

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \tag{112}$$

The Natario vector for the equation with constant velocity vs is given by:

$$nX = X^{rs} drs + X^\theta rs d\theta \tag{113}$$

The contravariant and covariant components in the Natario vector for the equation with variable velocity vs are given by:

- $X^t = 2n(rs)rscos\theta a$
- $X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta$
- $X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta$
- $X_t = 2n(rs)rscos\theta a$
- $X_{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta$
- $X_\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2\sin\theta$

The contravariant and covariant components in the Natario vector for the equation with constant velocity vs are given by:

- $X^{rs} = 2v_s n(rs)\cos\theta$
- $X^\theta = -v_s(2n(rs) + (rs)n'(rs))\sin\theta$
- $X_{rs} = 2v_s n(rs)\cos\theta$
- $X_\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta$

Note that in the case of variable velocity vs a new set of contravariant and covariant components X^t and X_t appears both in the Natario warp drive equation and in the Natario vector because in this case as the velocity vs changes its value as times goes by due to a constant acceleration a this affects the whole spacetime geometry.The equation for a variable velocity vs due to a constant acceleration a is given by $vs = 2n(rs)at$.

Note also that the remaining contravariant and covariant components of both the Natario warp drive equations and the Natario vectors X^{rs},X_{rs},X^θ and X_θ are defined in function of a constant acceleration a in the case of a variable velocity vs and are defined in function of a fixed velocity vs in the case of a constant velocity vs.

Note also that the contravariant and covariant components X^t, X_t,X^{rs} and X_{rs} are equal considering separately the cases of fixed or variable velocities.This helps to illustrate why in the $1+1$ spacetime all these equations can be reduced to the same forms being one form suited for variable speeds while the other form contemplate fixed speeds.

We must outline now some very important things all these equations have in common:

- 1)- All these equations satisfies the Natario definition and condition for a warp drive spacetime using the same Natario shape function $n(rs)$ which gives 0 inside the bubble $\frac{1}{2}$ outside the bubble and $0 < n(rs) < \frac{1}{2}$ in the Natario warped region wether in the original or parallel $3+1$ ADM formalisms and also all these equations satisfies the Natario requirements for a warp drive spacetime wether using variable or fixed velocities.

- 2)- The same Natario shape function $n(rs)$ appears in the contravariant and covariant components of both Natario vectors.

- 3)- The same Natario shape function $n(rs)$ appears in the definition of the equation of the variable velocity $vs = 2n(rs)at$

Alcubierre used the original $3+1$ ADM formalism in his warp drive(see eq 1 pg 3 in [1])(see Appendix E) and we have reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to derive the original Natario warp drive equation with constant velocity vs:

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{114}$$

The negative energy density for the Natario warp drive in the original $3+1$ ADM formalism for fixed velocities in the International System of Units SI (see Appendix G) is given by(see pg 5 in [2])

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{115}$$

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$) the negative energy density for fixed velocities is given by:(see Appendix D)

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{116}$$

Since in the $1+1$ spacetime all the equations with fixed velocities reduces to the same form wether in the original or parallel ADM formalisms the equation above remains valid in all the cases

But for the case of the warp drive equations with variable velocity vs both in the original or parallel $3+1$ ADM formalism or the case of the $3+1$ parallel ADM formalism even for fixed speeds we can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pgs $[342(b)]$ or $[369(a)]$ in [11], pgs $[276(b)]$ or $[294(a)]$ in [13],pgs $[454, 457, 560(b)]$ or $[465, 468, 567(a)]$ in [14] pg $[98(a)]$ or $[98(b)]$ in [25],pgs $[183(a)]$ or $[178(b)]$ in [27]).

Appendix C pgs $[551-555(b)]$ or $[559-563(a)]$ in [14] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3+1$ spacetime metric using $Mathematica$.

But since the Natario shape function $n(rs)$ is the same for all these equations it is reasonable to suppose that derivatives of first second(or perhaps higher)order will appear in the negative energy density expression for the Natario warp drive with variable velocity and since the derivatives of first or second order for the Natario shape function possesses extremely low values these values can obliterate large terms for velocities vs or large accelerations a.For a detailed study of the derivatives of first and second order of the Natario shape function $n(rs)$ see pgs 10 to 41 in [18]

10 Conclusion:

In this work we presented the new equations for the warp drive spacetime according to Natario with constant or variable velocity vs and constant acceleration a(in the case for the variable velocity) in both the original or parallel $3+1$ ADM formalisms:

The Natario warp drive spacetime is a very rich environment to study the superluminal features of General Relativity because now we have six spacetime metrics and not only one and the geometry of the new equations in the $3+1$ spacetime is still unknown and needs to be cartographed.

The $3+1$ original ADM formalism with signature $(-,+,+,+)$ is given by the equation (21.40) pg [507(b)] [534(a)] in [11]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{117}$$

The $3+1$ parallel contravariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{118}$$

The $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{119}$$

While the Christoffel symbols,Riemann and Ricci tensors,Ricci scalar,Einstein tensors or extrinsic curvature tensors are completely known and chartered for the Natario warp drive in the original $3+1$ ADM formalism with fixed speed,these mathematical entities are completely unknown for the Natario warp drive in the original $3+1$ ADM formalism with variable speeds or the Natario warp drive in both the contravariant or covariant parallel $3+1$ ADM formalisms wether using fixed or variable speeds and this can open new avenues of research in General Relativity.

Considering constant velocities then the lapse function $\alpha = 1$ and for the $3+1$ ADM formalisms wether in the original or parallel cases we have the following results:

The $3+1$ original ADM formalism with signature $(-,+,+,+)$ without lapse function.

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{120}$$

The $3+1$ parallel contravariant ADM formalism with signature $(-,+,+,+)$ without lapse function.

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{121}$$

The $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ without lapse function.

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{122}$$

For the warp drives using variable velocities we must define explicitly the lapse function responsible for the extra terms due to the accelerations.

The $3+1$ original ADM formalism with signature $(-,+,+,+)$ and with the lapse function explicitly defined is given by the equation:

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\gamma_{tt}(1+\beta^{t})^{2}dt^{2} + \gamma_{ij}(dx^{i}+\beta^{i}dt)(dx^{j}+\beta^{j}dt) \tag{123}$$

The 3+1 parallel contravariant ADM formalism with signature $(-,+,+,+)$ and with the lapse function explicitly defined is given by the equation:

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -(\sqrt{\gamma_{tt}}+\beta^{t})^{2}dt^{2} + (\sqrt{\gamma_{ii}}dx^{i}+\beta^{i}dt)(\sqrt{\gamma_{jj}}dx^{j}+\beta^{j}dt) \tag{124}$$

The $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ and with the lapse function explicitly defined is given by the equation:

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -(\sqrt{\gamma_{tt}}+\beta_{t})^{2}dt^{2} + (\sqrt{\gamma_{ii}}dx^{i}+\beta_{i}dt)(\sqrt{\gamma_{jj}}dx^{j}+\beta_{j}dt) \tag{125}$$

Note that we managed to define each lapse function according to each ADM formalism.

For the original formalism we have the term $\gamma_{ij}(dx^{i}+\beta^{i}dt)(dx^{j}+\beta^{j}dt)$ and the lapse function $\gamma_{tt}(1+\beta^{t})^{2}dt^{2}$ using the same mathematical structure in which the terms γ_{ij},β^{i} or β^{j} from the spatial components appears with the terms γ_{tt} and β^{t} of the corresponding time components.

For the parallel contravariant formalism we have the term $(\sqrt{\gamma_{ii}}dx^{i}+\beta^{i}dt)(\sqrt{\gamma_{jj}}dx^{j}+\beta^{j}dt)$ and the lapse function $(\sqrt{\gamma_{tt}}+\beta^{t})^{2}dt^{2}$ using the same mathematical structure in which the terms $\sqrt{\gamma_{ii}}$,β^{i},$\sqrt{\gamma_{jj}}$ and β^{j} from the spatial components appears with the terms $\sqrt{\gamma_{tt}}$ and β^{t} of the corresponding time components.

For the parallel covariant formalism we have the term $(\sqrt{\gamma_{ii}}dx^{i}+\beta_{i}dt)(\sqrt{\gamma_{jj}}dx^{j}+\beta_{j}dt)$ and the lapse function $(\sqrt{\gamma_{tt}}+\beta_{t})^{2}dt^{2}$ using the same mathematical structure in which the terms $\sqrt{\gamma_{ii}}$,β_{i},$\sqrt{\gamma_{jj}}$ and β_{j} from the spatial components appears with the terms $\sqrt{\gamma_{tt}}$ and β_{t} of the corresponding time components..

A real and fully functional warp drive must encompasses accelerations or de-accelerations in order to go from 0 to 200 times light speed in the beginning of an interstellar journey and to slow down to 0 again in the end of the interstellar journey.

Both the Alcubierre and Natario original geometries encompasses warp drives of constant velocities so we expanded the Natario vector to encompass time coordinate as a new Canonical Basis for the Hodge Star generating an extended version of the original Natario warp drive equation which of course encompasses accelerations or de-accelerations.

The Natario warp drive spacetime is a very rich environment to study the superluminal features of General Relativity because now we have six spacetime metrics and not only one and the geometry of the new equation in the $3+1$ spacetime is still unknown and needs to be cartographed.

Because collisions between the walls of the warp bubble and the hazardous particles of the Interstellar Medium(IM) would certainly occurs in a real superluminal interstellar spaceflight we borrowed the idea of Chris Van Den Broeck proposed some years ago in 1999 in order to increase the degree of protection of the spaceship and the crew members in the Natario warp drive equation for constant speed vs(see pg 46 in [18],pg 3 in [19]).

Our idea was to keep the surface area of the bubble exposed to collisions microscopically small avoiding the collisions with the dangerous IM particles while at the same time expanding the spatial volume inside the bubble to a size larger enough to contains a spaceship inside.

A submicroscopic outer radius of the bubble being the only part in contact with our Universe would mean a submicroscopic surface exposed to the collisions against the hazardous IM particles thereby reducing the probabilities of dangerous impacts against large objects (comets asteroids etc) enhancing the protection level of the spaceship and hence the survivability of the crew members.

Any future development for the Natario warp drive must encompass the more than welcome idea of Chris Van Den Broeck and this idea can also be easily implemented in the Natario warp drive with variable velocity.Since the Broeck idea is independent of the Natario geometry wether in constant or variable velocity we did not covered the Broeck idea here because it was already covered in [18] and [19] and in order to discuss the geometry of a Natario warp drive with variable velocity the Broeck idea is not needed here however the Broeck idea must appear in a real Natario warp drive with variable velocity vs concerning realistic superluminal interstellar spaceflights.

But unfortunately although we can discuss mathematically how to reduce the negative energy density requirements to sustain a warp drive we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario-Broeck warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario-Broeck warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

11 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs in a R^3 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [2],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [15],eq 3.72 pg 69$(a)(b)$ in [15]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{126}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{127}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{128}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{129}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{130}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{131}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see pg 8 in [4],eq 3.72 pg 69$(a)(b)$ in [15]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{132}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{133}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{134}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{135}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{136}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{137}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{138}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{139}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{140}$$

We know that the following expression holds true(see pg 9 in [3], eq 3.79 pg 70$(a)(b)$ in [15]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{141}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{142}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{143}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{144}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{145}$$

According to pg 10 in [3],eq 3.90 pg 74$(a)(b)$ in [15] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{146}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{147}$$

Because and according to pg 10 in [3],eqs 3.90 and 3.91 pg 74$(a)(b)$ in [15],tb 3.2 pg 68$(a)(b)$ in [15]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{148}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{149}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{150}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{151}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{152}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{153}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{154}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{155}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{156}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{157}$$

$$2f(r)r^2 sin\theta \cos\theta(d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{158}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{159}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \sim -r\sin\theta(dr \wedge d\varphi) \tag{160}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{161}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr \wedge d\varphi)] \tag{162}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{163}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{164}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{165}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{166}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{167}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{168}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{169}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{170}$$

12 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs or for the first term $vsdx$ from the Natario vector $nX = vsdx + xdvs$(a variable speed) in a R^4 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [2],eqs 3.135 and 3.137 pg 82(a)(b) in [15],eq 3.74 pg 69(a)(b) in [15]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{171}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{172}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{173}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{174}$$

$$rd\theta \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{175}$$

$$r\sin\theta d\varphi \sim r(dt \wedge dr \wedge d\theta) \tag{176}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [4],eq 3.74 pg 69(a)(b) in [15]):

$$*dr = r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{177}$$

$$*rd\theta = r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{178}$$

$$*r\sin\theta d\varphi = r(dt \wedge dr \wedge d\theta) \tag{179}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta dt\wedge d\theta\wedge d\varphi + r\sin^2\theta dt\wedge dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{180}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{181}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{182}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{183}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(dt\wedge d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(dt\wedge d\varphi\wedge dr)] \tag{184}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] - [r\sin^2\theta(dt\wedge d\varphi\wedge dr)] \tag{185}$$

We know that the following expression holds true(see pg 9 in [3],eq 3.79 pg 70$(a)(b)$ in [15])):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{186}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] + [r\sin^2\theta(dt\wedge dr\wedge d\varphi)] \tag{187}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{188}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{189}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{190}$$

According to pg 10 in [3],eq 3.90 pg 74$(a)(b)$ in [15] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \tag{191}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \quad (192)$$

Because and according to pg 10 in [3],eqs 3.90 and 3.91 pg 74$(a)(b)$ in [15],tb 3.3 pg 68$(a)(b)$ in [15]::

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (193)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \quad (194)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (195)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = dt\wedge d(\sin^2\theta)\wedge d\varphi - dt\wedge\sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) \quad (196)$$

$$*[d(r^2)d\varphi] = 2rdt\wedge dr\wedge d\varphi - dt\wedge r^2\wedge dd\varphi = 2r(dt\wedge dr\wedge d\varphi) \quad (197)$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + r\sin^2\theta(dt\wedge dr\wedge d\varphi) \quad (198)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (199)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (200)$$

$$f(r)r^2 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (201)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (202)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \tag{203}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim dt\wedge(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(dt\wedge d\theta\wedge d\varphi) \tag{204}$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim dt\wedge(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(dt\wedge d\varphi\wedge dr)\sim -r\sin\theta(dt\wedge dr\wedge d\varphi) \tag{205}$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dt\wedge dr\wedge(rd\theta)\sim r(dt\wedge dr\wedge d\theta) \tag{206}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(dt\wedge d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)] \tag{207}$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \tag{208}$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \tag{209}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{210}$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{211}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{212}$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{213}$$

With our pedagogical approaches

$$nX=2vs(t)f(r)\ cos\theta dr-vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \tag{214}$$

$$nX=-2vs(t)f(r)\ cos\theta dr+vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \tag{215}$$

13 Appendix C:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nX = *(vsx)$ for a variable speed vs and a constant acceleration a

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

In the Appendices A and B we gave the mathematical demonstration of the Natario vector nX in the R^3 and R^4 space basis when the velocity vs is constant.Hence the complete expression of the Hodge star that generates the Natario vector nX for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{216}$$

$$*dx = *d(rcos\theta) = *d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) = *d[f(r)r^2\sin^2\theta d\varphi] \tag{217}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{218}$$

$$nX = X^r dr + X^\theta r d\theta \tag{219}$$

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{220}$$

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{221}$$

With the contravariant shift vector components explicitly given by:

$$X^r = 2v_s f(r)\cos\theta \tag{222}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{223}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{224}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [15] with the terms $S = u = 1$[12],eq 3.74 pg 69$(a)(b)$ in [15],eqs 11.131 and 11.133 with the term $m = 0$[13] pg 417$(a)(b)$ in [15].):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{225}$$

$$dt \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{226}$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [15]):

$$*dt = r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{227}$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \tag{228}$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [2]) and considering also that the Netario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \tag{229}$$

[12]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[13]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{230}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \tag{231}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\varphi)] \tag{232}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \tag{233}$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \tag{234}$$

The term $*dx$ was obtained in the Appendices A and B as follows:(see pg 5 in [2])

$$*dx = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta \tag{235}$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + x(2[atf'(r)e_r + f(r)ae_t]) \tag{236}$$

But remember that $x = rcos\theta$(see pg 5 in [2]) and this leaves us with:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{237}$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{238}$$

Then we can start with a warp bubble initially at the rest using the Natario vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed constant.The terms due to the acceleration now disappears and we are left again with the Natario vector for constant speeds shown below:

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \tag{239}$$

Working some algebra with the Natario vector for variable velocities we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (240)$$

$$nX = 4f(r)^2at\ cos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta + 2atf'(r)rcos\theta e_r + 2f(r)rcos\theta ae_t \quad (241)$$

$$nX = 2f(r)rcos\theta ae_t + 4f(r)^2at\ cos\theta e_r + 2atf'(r)rcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (242)$$

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (243)$$

Then the Natario vector for variable velocities defined using contravariant shift vector components is given by the following expressions:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \quad (244)$$

$$nX = X^t dt + X^r dr + X^\theta r d\theta \quad (245)$$

Or being:

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (246)$$

$$nX = 2f(r)rcos\theta adt + 2[2f(r)^2 + rf'(r)]atcos\theta dr - 2f(r)at[2f(r) + rf'(r)]r\sin\theta d\theta \quad (247)$$

The contravariant shift vector components are respectively given by the following expressions:

$$X^t = 2f(r)rcos\theta a \quad (248)$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \quad (249)$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \quad (250)$$

14 Appendix D:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[14]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{251}$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{252}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{253}$$

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg $[116(a)][116(b)]$ in [26])

[14] $n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

15 Appendix E:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg $[64(b)]$ $[79(a)]$ in [12])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg $[65(b)]$ $[80(a)]$ in [12].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg $[65(b)]$ $[80(a)]$ in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg $[66(b)]$ $[81(a)]$ in [12])
(see also fig 21.2 pg $[506(b)]$ $[533(a)]$ in [11] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg $[507(b)]$ $[534(a)]$ in [11])[15]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[15] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [11] with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [12] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{254}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{255}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix} \tag{256}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{257}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{258}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{259}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{260}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{261}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i \beta^i dt^2 = \beta_i \beta^i dt^2 \tag{262}$$

$$(dx^i)^2 = dx^i dx^i \tag{263}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i \beta^i dt^2 \tag{264}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i \beta^i dt^2 \tag{265}$$

$$ds^2 = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{266}$$

Note that the expression above is exactly the eq (2.2.4) pgs $[67(b)]$ $[82(a)]$ in [12].It also appears as eq 1 pg 3 in [1].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \quad (267)$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \quad (268)$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \quad (269)$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \quad (270)$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \quad (271)$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \quad (272)$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \quad (273)$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \quad (274)$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \quad (275)$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \quad (276)$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \quad (277)$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \quad (278)$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \quad (279)$$

Comparing all these equations

$$ds^2 = (1 - \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{280}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i \beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{281}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i \beta^i \end{pmatrix} \tag{282}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{283}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{284}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i \beta^i = X_i X^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{285}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{286}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{287}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{288}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{289}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{290}$$

But remember that $dl^2 = \gamma_{ii} dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_{rs} and X_{θ} with $r = rs$ are given by:

$$X_i = \gamma_{ii} X^i \tag{291}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs) \cos\theta = X^r = X^{rs} \tag{292}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{293}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{294}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{295}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{296}$$

The matrix components 2×2 evaluated separately for rs and θ gives the following results:[16]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0r} \\ g_{r0} & g_{rr} \end{pmatrix} = \begin{pmatrix} 1 - X_r X^r & X_r \\ X_r & -\gamma_{rr} \end{pmatrix} \tag{297}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0r} \\ g^{r0} & g^{rr} \end{pmatrix} = \begin{pmatrix} 1 & X^r \\ X^r & -\gamma^{rr} + X^r X^r \end{pmatrix} \tag{298}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0\theta} \\ g_{\theta 0} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 - X_\theta X^\theta & X_\theta \\ X_\theta & -\gamma_{\theta\theta} \end{pmatrix} \tag{299}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0\theta} \\ g^{\theta 0} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & X^\theta \\ X^\theta & -\gamma^{\theta\theta} + X^\theta X^\theta \end{pmatrix} \tag{300}$$

Then the equation of the Natario warp drive spacetime with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{301}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{302}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{303}$$

[16] Actually we know that the real matrix is a 3×3 matrix with dimensions t rs and θ.Our 2×2 approach is a simplification

16 Appendix F:mathematical demonstration of the Natario warp drive equation for a variable speed vs and a constant acceleration a in the original $3+1$ ADM Formalism according to MTW and Alcubierre

In the Appendix C we defined a variable bubble velocity vs due to a constant acceleration a as follows:

$$vs = 2n(rs)at \tag{304}$$

And we obtained the Natario vector nX for a Natario warp drive with variable velocities defined as follows:

$$nX = vs(2n(rs)\ cos\theta e_r - [2n(rs) + rsn'(rs)] \sin\theta e_\theta) + rscos\theta(2[atn'(rs)e_r + n(rs)ae_t]) \tag{305}$$

$$nX = 2n(rs)at(2n(rs)\ cos\theta e_r - [2n(rs) + rsn'(rs)] \sin\theta e_\theta) + rscos\theta(2[atn'(rs)e_r + n(rs)ae_t]) \tag{306}$$

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{307}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rsd\theta \tag{308}$$

Remember that $x = rscos\theta$(see pg 5 in [2]). Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]) we can see that the Natario vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * d(vs)$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2]).Working with some algebra we got:

$$nX = 2n(rs)rscos\theta ae_t + 2[2n(rs)^2 + rsn'(rs)]atcos\theta e_r - 2n(rs)at[2n(rs) + rsn'(rs)] \sin\theta e_\theta \tag{309}$$

$$nX = 2n(rs)rscos\theta adt + 2[2n(rs)^2 + rsn'(rs)]atcos\theta drs - 2n(rs)at[2n(rs) + rsn'(rs)]rs \sin\theta d\theta \tag{310}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs)rscos\theta a \tag{311}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{312}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)] \sin\theta \tag{313}$$

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65(b)] [80(a)] in [12].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65(b)] [80(a)] in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66(b)] [81(a)] in [12])
(see also fig 21.2 pg [506(b)] [533(a)] in [11] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg [507(b)] [534(a)] in [11])[17]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij} dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

Combining the eqs (21.40),(21.42) and (21.44) pgs [$507, 508$(b)] [$534, 535$(a)] in [11]
with the eqs (2.2.5) and (2.2.6) pgs [67(b)] [82(a)] in [12] using the signature $(-, +, +, +)$ we get the original equations of the $3 + 1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{314}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{315}$$

[17] we adopt the Alcubierre notation here

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{316}$$

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = \gamma_{tt}(1+\beta^t)^2 = \gamma_{tt}(1+2\beta^t+\beta^t\beta^t) = (\gamma_{tt}+2\gamma_{tt}\beta^t+\gamma_{tt}\beta^t\beta^t) \tag{317}$$

$$\beta_t = \gamma_{tt}\beta^t \tag{318}$$

Remember that here we are working with geometrized units in which $c=1$ so $\gamma_{tt}=1$

$$\alpha^2 = (1+2\beta_t+\beta_t\beta^t) \tag{319}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\gamma_{tt}(1+\beta^t)^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{320}$$

Since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{321}$$

$$ds^2 = -\gamma_{tt}(1+\beta^t)^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{322}$$

From the Appendix E we can write the $3+1$ metric as:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{323}$$

Note that the expression above is exactly the eq (2.2.4) pgs [67(b)] [82(a)] in [12].It also appears as eq 1 pg 3 in [1].Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{324}$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{325}$$

$$ds^2 = (1+2\beta_t+\beta_t\beta^t - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{326}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{327}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1+2\beta_t+\beta_t\beta^t - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{328}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$ and we modified the equation to insert the terms due to the lapse function α^2.(pg 2 in [2])

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{329}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{330}$$

Comparing all these equations

$$ds^2 = (\alpha^2 - \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{331}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i \beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{332}$$

$$ds^2 = \alpha^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{333}$$

$$\alpha^2 = \gamma_{tt}(1 + \beta^t)^2 \tag{334}$$

$$\alpha^2 = (1 + 2\beta_t + \beta_t \beta^t) \tag{335}$$

$$ds^2 = \gamma_{tt}(1 + \beta^t)^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{336}$$

$$ds^2 = (1 + 2\beta_t + \beta_t \beta^t - \beta_i \beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{337}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 + 2\beta_t + \beta_t \beta^t - \beta_i \beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{338}$$

With these

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{339}$$

$$ds^2 = \gamma_{tt}(1 - X^t)^2 dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{340}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt} X^t + \gamma_{tt} X^t X^t) = (1 - 2X_t + X_t X^t) \tag{341}$$

The generic equations for the Natario warp drive that obeys the 3+1 ADM formalism are given below:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{342}$$

$$ds^2 = \gamma_{tt}(1 - X^t)^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{343}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^t X^t) = (1 - 2X_t + X_t X^t) \tag{344}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i \beta^i = X_i X^i$ with X^i being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector both for the spatial components of the Natario vector.In the same way we can see that $\beta^t = -X^t$,$\beta_t = -X_t$ and $\beta_t \beta^t = X_t X^t$ with X^t being the contravariant form of the Natario shift vector and X_t being the covariant form of the Natario shift vector for the time component of the Natario vector.Hence we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{345}$$

$$ds^2 = (1 - 2X_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{346}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{347}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{348}$$

Looking to the equation of the Natario vector nX:

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{349}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \tag{350}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs) rs cos\theta a \tag{351}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{352}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{353}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Remember also that $\gamma_{tt} = 1$.Then the covariant shift vector components X_t,X_{rs} and X_θ with $r = rs$ are given by:

$$X_t = \gamma_{tt}X^t \tag{354}$$

$$X_i = \gamma_{ii}X^i \tag{355}$$

$$X_t = \gamma_{tt}X^t = 2n(rs)rscos\theta a \tag{356}$$

$$X_r = \gamma_{rr}X^r = X_{rs} = \gamma_{rsrs}X^{rs} = X^r = X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{357}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = rs^2 X^\theta = X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]rs^2 \sin\theta \tag{358}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - 2X_t + X_tX^t - X_iX^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{359}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_tX^t - X_iX^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{360}$$

Then the equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - 2X_t + X_tX^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{361}$$

$$ds^2 = (1 - 2X_t + X_tX^t - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{362}$$

17 Appendix G:Dimensional Reduction from $\frac{c^4}{G}$ to $\frac{c^2}{G}$

The Alcubierre expressions for the Negative Energy Density in Geometrized Units $c = G = 1$ are given by(pg 4 in [2])(pg 8 in [1]):[18]:

$$\rho = -\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{363}$$

$$\rho = -\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{364}$$

In this system all physical quantities are identified with geometrical entities such as lengths,areas or dimensionless factors.Even time is interpreted as the distance travelled by a pulse of light during that time interval,so even time is given in lengths.Energy,Momentum and Mass also have the dimensions of lengths.We can multiply a mass in kilograms by the conversion factor $\frac{G}{c^2}$ to obtain the mass equivalent in meters.On the other hand we can multiply meters by $\frac{c^2}{G}$ to obtain kilograms.The Energy Density($\frac{Joules}{meters^3}$)in Geometrized Units have a dimension of $\frac{1}{length^2}$ and the conversion factor for Energy Density is $\frac{G}{c^4}$.Again on the other hand by multiplying $\frac{1}{length^2}$ by $\frac{c^4}{G}$ we retrieve again ($\frac{Joules}{meters^3}$). [19].

This is the reason why in Geometrized Units the Einstein Tensor have the same dimension of the Stress Energy Momentum Tensor(in this case the Negative Energy Density)and since the Einstein Tensor is associated to the Curvature of Spacetime both have the dimension of $\frac{1}{length^2}$.

$$G_{00} = 8\pi T_{00} \tag{365}$$

Passing to normal units and computing the Negative Energy Density we multiply the Einstein Tensor (dimension $\frac{1}{length^2}$) by the conversion factor $\frac{c^4}{G}$ in order to retrieve the normal unit for the Negative Energy Density ($\frac{Joules}{meters^3}$).

$$T_{00} = \frac{c^4}{8\pi G} G_{00} \tag{366}$$

Examine now the Alcubierre equations:

$vs = \frac{dxs}{dt}$ is dimensionless since time is also in lengths. $\frac{y^2+z^2}{rs^2}$ is dimensionless since both are given also in lengths. $f(rs)$ is dimensionless but its derivative $\frac{df(rs)}{drs}$ is not because rs is in meters. So the dimensional factor in Geometrized Units for the Alcubierre Energy Density comes from the square of the derivative and is also $\frac{1}{length^2}$.Remember that the speed of the Warp Bubble vs is dimensionless in Geometrized Units and when we multiply directly $\frac{1}{length^2}$ from the Negative Energy Density in Geometrized Units by $\frac{c^4}{G}$ to obtain the Negative Energy Density in normal units $\frac{Joules}{meters^3}$ the first attempt would be to make the following:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[f'(rs)\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{367}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi} vs^2 \left[\frac{df(rs)}{drs}\right]^2 [\frac{y^2+z^2}{rs^2}] \tag{368}$$

[18]See Geometrized Units in Wikipedia

[19]See Conversion Factors for Geometrized Units in Wikipedia

But note that in normal units vs is not dimensionless and the equations above do not lead to the correct dimensionality of the Negative Energy Density because the equations above in normal units are being affected by the dimensionality of vs.

In order to make vs dimensionless again,the Negative Energy Density is written as follows:

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{369}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{370}$$

Giving:

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{371}$$

$$\rho = -\frac{c^2}{G}\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{372}$$

As already seen.The same results are valid for the Natario Energy Density

Note that from

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{373}$$

$$\rho = -\frac{c^4}{G}\frac{1}{32\pi}(\frac{vs}{c})^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{374}$$

Making $c = G = 1$ we retrieve again

$$\rho = -\frac{1}{32\pi}vs^2\left[f'(rs)\right]^2[\frac{y^2+z^2}{rs^2}] \tag{375}$$

$$\rho = -\frac{1}{32\pi}vs^2\left[\frac{df(rs)}{drs}\right]^2[\frac{y^2+z^2}{rs^2}] \tag{376}$$

18 Appendix H:The Natario warp drive and the parallel contravariant $3+1$ ADM Formalism for a constant speed vs

A $3+1$ ADM contravariant formalism parallel to the original $3+1$ ADM formalism according with the equation (21.40) pg [507(b)] [534(a)] in [11]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{377}$$

using the signature $(-,+,+,+)$ can be given by:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{378}$$

Note that in the equation above all the essential 3 elements of the original $3+1$ ADM formalism are also present[20].These elements are:

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)$.$\beta^i$ is known as the contravariant shift vector.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{379}$$

$$(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + (\beta^i dt)^2 \tag{380}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + (\beta^i dt)^2 \tag{381}$$

$$ds^2 = -\alpha^2 dt^2 + (\beta^i dt)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}(dx^i)^2 \tag{382}$$

$$ds^2 = (-\alpha^2 + [\beta^i]^2)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{383}$$

$$ds^2 = (-\alpha^2 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{384}$$

[20] see Appendix E on the original $3+1$ ADM formalism

Then the equations of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism are given by:

$$ds^2 = (-\alpha^2 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii} dx^i dx^i \tag{385}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{386}$$

The components of the inverse metric are given by the matrix inverse :[21]

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{387}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-\alpha^2 + \beta^i\beta^i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\alpha^2 + \beta^i\beta^i \end{pmatrix} \tag{388}$$

Suppressing the lapse function $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii} dx^i dx^i \tag{389}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{390}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{391}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-1 + \beta^i\beta^i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -1 + \beta^i\beta^i \end{pmatrix} \tag{392}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ we should expect for:

$$ds^2 = (1 - \beta^i\beta^i)dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{393}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{394}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{395}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i\beta^i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta^i \times -\sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i\beta^i \end{pmatrix} \tag{396}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i\beta^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i\beta^i \end{pmatrix} \tag{397}$$

[21] see Wikipedia:the free Encyclopedia on inverse or invertible matrices

The equations of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism given by:

$$ds^2 = (1 - \beta^i \beta^i) dt^2 - 2\sqrt{\gamma_{ii}} \beta^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{398}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta^i \beta^i & -\sqrt{\gamma_{ii}} \beta^i \\ -\sqrt{\gamma_{ii}} \beta^i & -\gamma_{ii} \end{pmatrix} \tag{399}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{400}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i \beta^i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}} \beta^i \times -\sqrt{\gamma_{ii}} \beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}} \beta^i \\ \sqrt{\gamma_{ii}} \beta^i & 1 - \beta^i \beta^i \end{pmatrix} \tag{401}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i \beta^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}} \beta^i \times \sqrt{\gamma_{ii}} \beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}} \beta^i \\ \sqrt{\gamma_{ii}} \beta^i & 1 - \beta^i \beta^i \end{pmatrix} \tag{402}$$

obeys the generic equation of a warp drive in the parallel contravariant $3+1$ ADM formalism:

$$ds^2 = dt^2 - (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)^2 \tag{403}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3} (dx^i - X^i dt)^2 \tag{404}$$

The Natario equation above given in oontravariant form is valid only in cartezian coordinates.For a generic coordinates system in contravariant form we must employ the equation given by the parallel contravariant $3+1$ ADM formalism as being:

$$ds^2 = dt^2 - \sum_{i=1}^{3} (\sqrt{\gamma_{ii}} dx^i - X^i dt)^2 \tag{405}$$

Note that $\beta^i = -X^i$ and $\beta^i \beta^i = X^i X^i$ with X^i being the Natario contravariant shift vectors. Hence we have:

$$ds^2 = (1 - X^i X^i) dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{406}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X^i X^i & \sqrt{\gamma_{ii}} X^i \\ \sqrt{\gamma_{ii}} X^i & -\gamma_{ii} \end{pmatrix} \tag{407}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{408}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X^i X^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}} X^i \times \sqrt{\gamma_{ii}} X^i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}} X^i \\ -\sqrt{\gamma_{ii}} X^i & 1 - X^i X^i \end{pmatrix} \tag{409}$$

For the equations of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism:

$$ds^2 = (1 - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{410}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X^i X^i & \sqrt{\gamma_{ii}} X^i \\ \sqrt{\gamma_{ii}} X^i & -\gamma_{ii} \end{pmatrix} \tag{411}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{412}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X^i X^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}} X^i \times \sqrt{\gamma_{ii}} X^i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}} X^i \\ -\sqrt{\gamma_{ii}} X^i & 1 - X^i X^i \end{pmatrix} \tag{413}$$

And looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{414}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{415}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{416}$$

But remember that $dl^2 = \gamma_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.Then the equation of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{417}$$

$$ds^2 = (1 - X^{rs} X^{rs} - X^{\theta} X^{\theta})dt^2 + 2(X^{rs} drs dt + X^{\theta} rs d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{418}$$

$$ds^2 = (1 - X^{rs} X^{rs} - X^{\theta} X^{\theta})dt^2 + 2(X^{rs} drs + X^{\theta} rs d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{419}$$

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs} drs + X^{\theta} rs d\theta]dt - drs^2 - rs^2 d\theta^2 \tag{420}$$

Note that the equation of the Natario vector nX(pg 2 and 5 in [2]) appears twice in the equation above due to the non-diagonalized shift components:

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{421}$$

As a matter of fact expanding the term

$$\sqrt{\gamma_{ii}} X^i dx^i = X^{rs} drs + X^{\theta} rs d\theta \tag{422}$$

we recover again the Natario vector since $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = rs^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = rs$

19 Appendix I:mathematical demonstration of the Natario warp drive equation for a variable speed vs and a constant acceleration a in the parallel contravariant $3+1$ ADM Formalism

In the Appendix C we defined a variable bubble velocity vs due to a constant acceleration a as follows:

$$vs = 2n(rs)at \tag{423}$$

And we obtained the Natario vector nX for a Natario warp drive with variable velocities defined as follows:

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{424}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rsd\theta \tag{425}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs)rscos\theta a \tag{426}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{427}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{428}$$

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65(b)] [80(a)] in [12].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65(b)] [80(a)] in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66(b)] [81(a)] in [12]) illustrated by the equation :[22]

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)(\sqrt{\gamma_{jj}} dx^j + \beta^j dt) \tag{429}$$

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij} dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij} dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}} dx^i + \beta^i dt)$.$\beta^i$ is known as the contravariant shift vector.

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ $dl = \sqrt{\gamma_{ii}} dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)^2 \tag{430}$$

$$ds^2 = (-\alpha^2 + \beta^i \beta^i) dt^2 + 2\sqrt{\gamma_{ii}} \beta^i dx^i dt + \gamma_{ii} dx^i dx^i \tag{431}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i \beta^i & \sqrt{\gamma_{ii}} \beta^i \\ \sqrt{\gamma_{ii}} \beta^i & \gamma_{ii} \end{pmatrix} \tag{432}$$

[22] we adopt also the Alcubierre notation here

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = (\sqrt{\gamma_{tt}} + \beta^t)^2 = (\gamma_{tt} + 2\sqrt{\gamma_{tt}}\beta^t + \beta^t\beta^t) \tag{433}$$

Remember that here we are working with geometrized units in which $c = 1$ so $\gamma_{tt} = 1$

$$\alpha^2 = (1 + 2\beta^t + \beta^t\beta^t) \tag{434}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{435}$$

$$ds^2 = (-\alpha^2 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{436}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{437}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -(1 + 2\beta^t + \beta^t\beta^t)dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{438}$$

$$ds^2 = (-(1 + 2\beta^t + \beta^t\beta^t) + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{439}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -(1 + 2\beta^t + \beta^t\beta^t) + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{440}$$

Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \alpha^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{441}$$

$$ds^2 = (\alpha^2 - \beta^i\beta^i)dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{442}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{443}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (1 + 2\beta^t + \beta^t\beta^t)dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{444}$$

$$ds^2 = ((1 + 2\beta^t + \beta^t\beta^t) - \beta^i\beta^i)dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{445}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} (1 + 2\beta^t + \beta^t\beta^t) - \beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{446}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$ and we modified the equation to insert the terms due to the lapse function α^2.(pg 2 in [2])

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{447}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the parallel contravariant $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X^i dt)^2 \tag{448}$$

The term α^2 is now defined by the following expression:

$$\alpha^2 = (\sqrt{\gamma_{tt}} - X^t)^2 = (\gamma_{tt} - 2\sqrt{\gamma_{tt}}X^t + X^t X^t) \tag{449}$$

Suppressing the summing convention then the Natario warp drive equation in the parallel contravariant $3+1$ ADM formalism now becomes:

$$ds^2 = (\sqrt{\gamma_{tt}} - X^t)^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i - X^i dt)^2 \tag{450}$$

Note that in the equation given above a remarkable mathematical structure appears:the term $\sqrt{\gamma_{tt}}$ is associated with the time coordinate while the term $\sqrt{\gamma_{ii}}$ is associated with the remaining spatial coordinates.

Remember that here we are working with geometrized units in which $c=1$ so $\gamma_{tt}=1$ hence the term α^2 can be given by:

$$\alpha^2 = (1 - X^t)^2 = (1 - 2X^t + X^t X^t) \tag{451}$$

The Natario warp drive equation in the parallel contravariant $3+1$ ADM formalism now becomes:

$$ds^2 = (1 - X^t)^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i - X^i dt)^2 \tag{452}$$

$$ds^2 = (1 - 2X^t + X^t X^t - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}}X^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{453}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X^t + X^t X^t - X^i X^i & \sqrt{\gamma_{ii}}X^i \\ \sqrt{\gamma_{ii}}X^i & -\gamma_{ii} \end{pmatrix} \tag{454}$$

The Natario warp drive equation in the parallel contravariant $3+1$ ADM formalism is then given by:

$$ds^2 = (1 - 2X^t + X^tX^t - X^iX^i)dt^2 + 2\sqrt{\gamma_{ii}}X^idx^idt - \gamma_{ii}dx^idx^i \tag{455}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X^t + X^tX^t - X^iX^i & \sqrt{\gamma_{ii}}X^i \\ \sqrt{\gamma_{ii}}X^i & -\gamma_{ii} \end{pmatrix} \tag{456}$$

Looking to the equation of the Natario vector nX defined for a variable velocity vs and a constant acceleration a according to the Appendix C :

$$nX = X^te_t + X^{rs}e_r + X^\theta e_\theta \tag{457}$$

$$nX = X^tdt + X^{rs}drs + X^\theta rsd\theta \tag{458}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs)rscos\theta a \tag{459}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{460}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{461}$$

But remember that $dl^2 = \gamma_{ij}dx^idx^j = dr^2 + r^2d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.Then the equation of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism for a variable velocity vs and a constant acceleration a is given by:

$$ds^2 = (1 - 2X^t + X^tX^t - X^{rs}X^{rs} - X^\theta X^\theta)dt^2 + 2(X^{rs}drsdt + X^\theta rsd\theta dt) - drs^2 - rs^2d\theta^2 \tag{462}$$

$$ds^2 = (1 - 2X^t + X^tX^t - X^{rs}X^{rs} - X^\theta X^\theta)dt^2 + 2(X^{rs}drs + X^\theta rsd\theta)dt - drs^2 - rs^2d\theta^2 \tag{463}$$

$$ds^2 = (1 - 2X^t + (X^t)^2 - (X^{rs})^2 - (X^\theta)^2)dt^2 + 2(X^{rs}drs + X^\theta rsd\theta)dt - drs^2 - rs^2d\theta^2 \tag{464}$$

20 Appendix J:The Natario warp drive and the parallel covariant $3+1$ ADM Formalism for a constant speed vs

A $3+1$ ADM covariant formalism parallel to the original $3+1$ ADM formalism according with the equation (21.40) pg [507(b)] [534(a)] in [11]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{465}$$

using the signature $(-,+,+,+)$ can be given by:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{466}$$

Note that in the equation above all the essential 3 elements of the original $3+1$ ADM formalism are also present[23].These elements are:

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β_i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)$.$\beta_i$ is known as the covariant shift vector defined as : $\beta_i = \gamma_{ij}\beta^j$.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{467}$$

$$(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + (\beta_i dt)^2 \tag{468}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + (\beta_i dt)^2 \tag{469}$$

$$ds^2 = -\alpha^2 dt^2 + (\beta_i dt)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}(dx^i)^2 \tag{470}$$

$$ds^2 = (-\alpha^2 + [\beta_i]^2)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{471}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{472}$$

[23]see Appendix E on the original $3+1$ ADM formalism

Then the equations of the Natario warp drive in the parallel covariant $3+1$ ADM formalism are given by:

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{473}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{474}$$

The components of the inverse metric are given by the matrix inverse :[24]

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{475}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-\alpha^2 + \beta_i\beta_i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\alpha^2 + \beta_i\beta_i \end{pmatrix} \tag{476}$$

Suppressing the lapse function $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{477}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{478}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{479}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-1 + \beta_i\beta_i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -1 + \beta_i\beta_i \end{pmatrix} \tag{480}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ we should expect for:

$$ds^2 = (1 - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{481}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{482}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{483}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta_i \times -\sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{484}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{485}$$

[24]see Wikipedia:the free Encyclopedia on inverse or invertible matrices

The equations of the Natario warp drive in the parallel covariant $3+1$ ADM formalism given by:

$$ds^2 = (1 - \beta_i \beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{486}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i \beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{487}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{488}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i \beta_i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta_i \times -\sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i \beta_i \end{pmatrix} \tag{489}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i \beta_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i \beta_i \end{pmatrix} \tag{490}$$

obeys the generic equation of a warp drive in the parallel covariant $3+1$ ADM formalism:

$$ds^2 = dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{491}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{492}$$

The Natario equation above given in contravariant form is valid only in cartezian coordinates.For a generic coordinates system in covariant form we must employ the equation given by the parallel covariant $3+1$ ADM formalism as being:

$$ds^2 = dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{493}$$

with $X_i = \gamma_{ii} X^i$

Note that $\beta_i = -X_i$ and $\beta_i \beta_i = X_i X_i$ with X_i being the covariant Natario shift vectors. Hence we have:

$$ds^2 = (1 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{494}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X_i & \sqrt{\gamma_{ii}}X_i \\ \sqrt{\gamma_{ii}}X_i & -\gamma_{ii} \end{pmatrix} \tag{495}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{496}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X_i X_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}X_i \times \sqrt{\gamma_{ii}}X_i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}}X_i \\ -\sqrt{\gamma_{ii}}X_i & 1 - X_i X_i \end{pmatrix} \tag{497}$$

For the equations of the Natario warp drive in the parallel covariant $3+1$ ADM formalism:

$$ds^2 = (1 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}} X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{498}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X_i & \sqrt{\gamma_{ii}} X_i \\ \sqrt{\gamma_{ii}} X_i & -\gamma_{ii} \end{pmatrix} \tag{499}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{500}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X_i X_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}} X_i \times \sqrt{\gamma_{ii}} X_i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}} X_i \\ -\sqrt{\gamma_{ii}} X_i & 1 - X_i X_i \end{pmatrix} \tag{501}$$

And looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{502}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{503}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{504}$$

But remember that $dl^2 = \gamma_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.Then the covariant shift vector components X_{rs} and X_{θ} with $r = rs$ are given by:

$$X_i = \gamma_{ii} X^i \tag{505}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs) \cos\theta = X^r = X^{rs} \tag{506}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{507}$$

It is possible to construct a covariant form for the Natario vector nX defined as $n_c X$ as follows:

$$n_c X = X_{rs} drs + X_{\theta} rs d\theta \tag{508}$$

With the covariant shift vector components X_{rs} and X_{θ} defined as shown above:

The equation of the Natario warp drive in the parallel covariant $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}} X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{509}$$

$$ds^2 = (1 - X_{rs}X_{rs} - X_\theta X_\theta)dt^2 + 2(X_{rs}drsdt + X_\theta rsd\theta dt) - drs^2 - rs^2 d\theta^2 \tag{510}$$

$$ds^2 = (1 - X_{rs}X_{rs} - X_\theta X_\theta)dt^2 + 2(X_{rs}drs + X_\theta rsd\theta)dt - drs^2 - rs^2 d\theta^2 \tag{511}$$

$$ds^2 = [1 - (X_{rs})^2 - (X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{512}$$

Note that the equation of the covariant Natario vector $n_c X$ appears twice in the equation above due to the non-diagonalized shift components:

$$n_c X = X_{rs}drs + X_\theta rsd\theta \tag{513}$$

As a matter of fact expanding the term

$$\sqrt{\gamma_{ii}} X_i dx^i = X_{rs}drs + X_\theta rsd\theta \tag{514}$$

we recover again the covariant form of the Natario vector since $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = rs^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = rs$

21 Appendix K:mathematical demonstration of the Natario warp drive equation for a variable speed vs and a constant acceleration a in the parallel covariant $3+1$ ADM Formalism

In the Appendix C we defined a variable bubble velocity vs due to a constant acceleration a as follows:

$$vs = 2n(rs)at \tag{515}$$

And we obtained the Natario vector nX for a Natario warp drive with variable velocities defined as follows:

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{516}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \tag{517}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs)rscos\theta a \tag{518}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{519}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{520}$$

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg $[65(b)]$ $[80(a)]$ in [12].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg $[65(b)]$ $[80(a)]$ in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg $[66(b)]$ $[81(a)]$ in [12]) illustrated by the equation :[25]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{521}$$

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.
- 3)-the relative velocity β_i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)$.$\beta_i$ is known as the covariant shift vector defined as : $\beta_i = \gamma_{ij}\beta^j$.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{522}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{523}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{524}$$

[25]we adopt also the Alcubierre notation here

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = (\sqrt{\gamma_{tt}} + \beta_t)^2 = (\gamma_{tt} + 2\sqrt{\gamma_{tt}}\beta_t + \beta_t\beta_t) \tag{525}$$

Remember that here we are working with geometrized units in which $c = 1$ so $\gamma_{tt} = 1$

$$\alpha^2 = (1 + 2\beta_t + \beta_t\beta_t) \tag{526}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{527}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{528}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{529}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -(1 + 2\beta_t + \beta_t\beta_t)dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{530}$$

$$ds^2 = (-(1 + 2\beta_t + \beta_t\beta_t) + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{531}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -(1 + 2\beta_t + \beta_t\beta_t) + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{532}$$

Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \alpha^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{533}$$

$$ds^2 = (\alpha^2 - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{534}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{535}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (1 + 2\beta_t + \beta_t\beta_t)dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{536}$$

$$ds^2 = ((1 + 2\beta_t + \beta_t\beta_t) - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{537}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} (1 + 2\beta_t + \beta_t\beta_t) - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{538}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$ and we modified the equation to insert the terms due to the lapse function α^2.(pg 2 in [2])

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{539}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the parallel covariant $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{540}$$

The term α^2 is now defined by the following expression:

$$\alpha^2 = (\sqrt{\gamma_{tt}} - X_t)^2 = (\gamma_{tt} - 2\sqrt{\gamma_{tt}}X_t + X_t X_t) \tag{541}$$

Suppressing the summing convention then the Natario warp drive equation in the parallel covariant $3+1$ ADM formalism now becomes:

$$ds^2 = (\sqrt{\gamma_{tt}} - X_t)^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{542}$$

Note that in the equation given above a remarkable mathematical structure appears:the term $\sqrt{\gamma_{tt}}$ is associated with the time coordinate while the term $\sqrt{\gamma_{ii}}$ is associated with the remaining spatial coordinates.

Remember that here we are working with geometrized units in which $c=1$ so $\gamma_{tt}=1$ hence the term α^2 can be given by:

$$\alpha^2 = (1 - X_t)^2 = (1 - 2X_t + X_t X_t) \tag{543}$$

The Natario warp drive equation in the parallel covariant $3+1$ ADM formalism now becomes:

$$ds^2 = (1 - X_t)^2 dt^2 - (\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{544}$$

$$ds^2 = (1 - 2X_t + X_t X_t - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{545}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2Xt + X_t X_t - X_i X_i & \sqrt{\gamma_{ii}}X_i \\ \sqrt{\gamma_{ii}}X_i & -\gamma_{ii} \end{pmatrix} \tag{546}$$

The Natario warp drive equation in the parallel covariant $3+1$ ADM formalism is then given by:

$$ds^2 = (1 - 2X_t + X_t X_t - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}} X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{547}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_t X_t - X_i X_i & \sqrt{\gamma_{ii}} X_i \\ \sqrt{\gamma_{ii}} X_i & -\gamma_{ii} \end{pmatrix} \tag{548}$$

Looking to the equation of the Natario vector nX defined for a variable velocity vs and a constant acceleration a according to the Appendix C :

$$nX = X^t e_t + X^{rs} e_r + X^\theta e_\theta \tag{549}$$

$$nX = X^t dt + X^{rs} drs + X^\theta rs d\theta \tag{550}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by:

$$X^t = 2n(rs) rs cos\theta a \tag{551}$$

$$X^{rs} = 2[2n(rs)^2 + rs n'(rs)] at cos\theta \tag{552}$$

$$X^\theta = -2n(rs) at [2n(rs) + rs n'(rs)] \sin\theta \tag{553}$$

But remember that $dl^2 = \gamma_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.The covariant shift vector components X_t,X_{rs} and X_θ with $r = rs$ are given by:

$$X_t = \gamma_{tt} X^t \tag{554}$$

$$X_i = \gamma_{ii} X^i \tag{555}$$

$$X_t = \gamma_{tt} X^t = 2n(rs) rs cos\theta a \tag{556}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = X^r = X^{rs} = 2[2n(rs)^2 + rs n'(rs)] at cos\theta \tag{557}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = rs^2 X^\theta = X^\theta = -2n(rs) at [2n(rs) + rs n'(rs)] rs^2 \sin\theta \tag{558}$$

Then the equation of the Natario warp drive in the parallel covariant $3+1$ ADM formalism for a variable velocity vs and a constant acceleration a is given by:

$$ds^2 = (1 - 2X_t + X_t X_t - X_{rs} X_{rs} - X_\theta X_\theta)dt^2 + 2(X_{rs} drs dt + X_\theta rs d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{559}$$

$$ds^2 = (1 - 2X_t + X_t X_t - X_{rs} X_{rs} - X_\theta X_\theta)dt^2 + 2(X_{rs} drs + X_\theta rs d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{560}$$

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_{rs})^2 - (X_\theta)^2)dt^2 + 2(X_{rs} drs + X_\theta rs d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{561}$$

22 Remarks

References [11],[12],[13],[14],[15],[22],[23],[24],[25],[26],[27] and [28] are standard textbooks used to study General Relativity or warp drive spacetimes and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention:when we refer for example the pages $[507, 508(b)]$ or the pages $[534, 535(a)]$ in [11] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader

23 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[26]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[27][28]

[26]special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[27]"Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[28]appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

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