New bounds on Mertens function

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Abstract

In this brief paper we study and bound Mertens function, which is defined for all positive integers as $M(n) = \sum_{k=1}^{n} \mu(k)$, where $\mu(x)$ is the Möbius function. The main breakthrough is the obtention of a Möbius-invertible formulation of Mertens function, which with some transformations and the application of the generalization of Möbius inversion formula, allows us to reach that

$$M(n) = O\left(\sqrt{n} \cdot \frac{\log(\sqrt{n})}{\log\log(\sqrt{n})}\right)$$

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1 Introduction

In 1859, in his paper "*On the Number of Primes Less Than a Given Magnitude*" [3], Bernhard Riemann published the assumption that all non-trivial zero-points of the zeta function extended to the range of complex numbers $\mathbf{C}$ have a real part of $\frac{1}{2}$. Ever since David Hilbert in 1900 added this problem to his list of the 23 most important problems of 20^{th} century, mathematicians have been working on finding evidence for the Riemann hypothesis.

Other hand, for any positive integer n, we define the Möbius function $\mu(n)$ as having the following values depending on the factorization of n into prime factors:

- $\mu(n) = 1$ if n is a square-free positive integer with an even number of prime factors.
- $\mu(n) = -1$ if n is a square-free positive integer with an odd number of prime factors.
- $\mu(n) = 0$ if n has a squared prime factor.

Merten's function $M(n)$ is the summatory function of the Möbius function, so it is defined for all positive integers as

$$M(n) = \sum_{k=1}^{n} \mu(k) \tag{1}$$

The value of Mertens function is closely connected to Riemann hypothesis through the identity

$$\frac{1}{\zeta(s)} = s\int_{1}^{\infty} M(x)x^{-s-1}dx \tag{2}$$

This identity is valid for $\mathrm{Re}(s) > 1$, and $\zeta(s)$ is the Riemann zeta function. If $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$, then the integral converges for $\mathrm{Re}(s) > \frac{1}{2}$, implying that $\frac{1}{\zeta(s)}$ has no poles in this region and that the Riemann hypothesis is true. Concretely, we have that

Theorem (Littlewood):[2] *Riemmann's hypothesis is equivalent to the statement: for every $\epsilon > 0$ the function $M(x)/x^{1/2+\epsilon}$ approaches zero as $x \to \infty$.*

In this brief paper we study and bound Mertens function. Using some transformations and a generalization of Möbius inversion formula, we are able to reach that

$$M(n) = O\left(\sqrt{n} \cdot \frac{\log(\sqrt{n})}{\log\log(\sqrt{n})}\right)$$

2 A new reformulation of Mertens function

From the definition of Möbius function, we have that

$$\sum_{k=1} \frac{\mu(k)}{k} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \dots \tag{3}$$

Where k runs over the square-free integers.

It is straightforward from the definition of $\mu(n)$ to note that

$$\sum_{k \le n} \frac{\mu(k)}{k} = 1 - \sum_{p_i \le n} \left(\frac{1}{p_i}\right) + \sum_{p_i < p_j \le \frac{n}{p_i}} \left(\frac{1}{p_i p_j}\right) - \sum_{p_i < p_j < p_k \le \frac{n}{p_i p_j}} \left(\frac{1}{p_i p_j p_k}\right) + \dots \tag{4}$$

Other hand, as already stated in (1), we have that Merten's function $M(n)$ is defined for all positive integers as

$$M(n) = \sum_{k=1}^{n} \mu(k)$$

Starting from (4), it is pretty straightforward to obtain that

$$M(n) = 1 - \pi(n) + \sum_{p_i \le \frac{n}{p_i}} \left(\pi\left(\frac{n}{p_i}\right) - i\right) - \sum_{p_i < p_j \le \frac{n}{p_i p_j}} \left(\pi\left(\frac{n}{p_i p_j}\right) - j\right) +$$

$$+ \sum_{p_i < p_j < p_k \le \frac{n}{p_i p_j p_k}} \left(\pi\left(\frac{n}{p_i p_j p_k}\right) - k\right) - \dots$$

Where $\pi(x)$ is the prime counting function.

It can be noted that the above expansion can be re-expressed more compactly as

$$M(n) = - \sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k) \pi\left(\frac{n}{k}\right) + S(n) \tag{5}$$

Where $p_{\pi(\sqrt{n})}$ is the greatest prime number less than $\sqrt{n}$, and where we have that

$$S(n) = - \sum_{p_i \le \frac{n}{p_i}} i + \sum_{p_i < p_j \le \frac{n}{p_i p_j}} j - \sum_{p_i < p_j < p_k \le \frac{n}{p_i p_j p_k}} k + \dots$$

It is worthy to have a look at the expansion of $S(n)$, which looks as follows:

$$S(n) = -(1 + 2 + 3 + \dots + \pi(\sqrt{n}) + ((2 + 3 + ... + \pi\left(\sqrt{\frac{n}{2}}\right)) + (3 + 4 + ... + \pi\left(\sqrt{\frac{n}{3}}\right)) + ...$$

$$+(k + (k+1) + ... + \pi\left(\sqrt{\frac{n}{k}}\right))) - \dots$$

Working on this expansion leads us to a closed form for $S(n)$, which can be expressed as:

$$S(n) = - \sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k) \left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right) + \lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right) - \lambda + 1\right)\right)}{2}\right) \tag{6}$$

Where $\lambda = \omega(k) + \pi(gpf(k))$, $\omega(k)$ counts the number of distinct prime factors of k, and $\pi(gpf(k))$ counts the number of prime numbers equal or less than the greatest prime factor of k.

At the end, from (5) and (6), we get that

$$M(n) = - \sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k) \left(\pi\left(\frac{n}{k}\right) + \left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right) + \lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right) - \lambda + 1\right)\right)}{2}\right)\right) \tag{7}$$

3 Bounding Mertens function

3.1 Transformations and simplifications

Firstly, we need to bound the difference

$$d_{n,k} = \frac{\pi\left(\frac{n}{k}\right)}{\left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right)+\lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right)-\lambda+1\right)\right)}{2}\right)} \tag{8}$$

For the purpose of this paper, it suffices to note that the minimum difference is obtained when $\lambda = 0$, and that this happens only when $k = 1$. For $k = 1$, we have that

$$\left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right)\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right)+1\right)\right)}{2}\right) = \left(\frac{\left(\pi\left(\sqrt{n}\right)\right)\left(\left(\pi\left(\sqrt{n}\right)+1\right)\right)}{2}\right)$$

Applying one of the best currently known explicit bounds for $\pi(x)$[1], we have that for $x > 6$

$$\frac{x}{\log(x)} < \pi(x) \leq \frac{x}{\log(x)}\left(1 + \frac{1}{\log(x)} + \frac{2}{\log^2(x)} + \frac{7.59}{\log^3(x)}\right) \tag{9}$$

And therefore, after substituting and operating, we have that

$$\left(\frac{\left(\pi\left(\sqrt{n}\right)\right)\left(\left(\pi\left(\sqrt{n}\right)+1\right)\right)}{2}\right) \leq \frac{n}{2\log^2(\sqrt{n})}\left(1 + \frac{1}{\log^2(\sqrt{n})} + \frac{4}{\log^4(\sqrt{n})} + \frac{7.59^2}{\log^6(\sqrt{n})}\right) + \frac{\sqrt{n}}{2\log(\sqrt{n})}\left(1 + \frac{1}{\log(\sqrt{n})} + \frac{4}{\log^2(\sqrt{n})} + \frac{7.59^2}{\log^3(\sqrt{n})}\right) \quad (11)$$

For the purpose of this paper, it is sufficient to note that, applying the explicit bounds settled before, we have that, for $x > 6$,

$$d_{n,k} \geq \frac{\log(\sqrt{n})}{2} \tag{10}$$

As a result, we have that, for $n > 6$, for all k,

$$\frac{2 \cdot \pi\left(\frac{n}{k}\right)}{\log(\sqrt{n})} > \left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right)+\lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right)-\lambda+1\right)\right)}{2}\right) \tag{11}$$

Therefore, considering (11) and (7), as $\frac{2}{\log(\sqrt{n}}$ tends to zero as x approaches infinity, we have that

$$M(n) = O\left(\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k)\,\pi\left(\frac{n}{k}\right)\right) \tag{12}$$

3.2 Application of Möbius inversion formula

In this section we will apply a generalization of Möbius inversion formula [4] and some transformations. Concretely, if $F(x)$ and $G(x)$ are complex-valued functions such that, as x grows to infinity, $G(x) = O\left(\sum_{k=1}^{x} F\left(\frac{x}{k}\right)\right)$, we have that $\sum_{k=1}^{x} \mu(k) G\left(\frac{x}{k}\right) = O(F(x))$.

The Prime Number Theorem yields that, when n grows to infinity, $p_{\pi(\sqrt{n})} = O\left(\pi(\sqrt{n}) \cdot \log(\pi(\sqrt{n}))\right)$ and $\pi(\sqrt{n}) = O\left(\frac{\sqrt{n}}{\log(\sqrt{n})}\right)$. Thus, we have that

$$p_{\pi(\sqrt{n})} = O\left(\frac{\sqrt{n}}{\log(\sqrt{n})} \cdot \log\left(\frac{\sqrt{n}}{\log(\sqrt{n})}\right) = \sqrt{n} \cdot \left(1 - \frac{\log\log(\sqrt{n})}{\log(\sqrt{n})}\right)\right) = O\left(\sqrt{n}\right)$$

Therefore, we have that, as n grows to infinity, $p_{\pi(\sqrt{n})} = O\left(\sqrt{n}\right)$, and therefore $\frac{n}{p_{\pi(\sqrt{n})}} = O\left(\sqrt{n}\right)$.

By the Prime Number Theorem we have that $\pi(x) = O\left(\frac{x}{\log(x)}\right)$. Therefore, if we set $G(x) = \frac{x}{\log(x)}$, we have that

$$\sum_{k=1}^{\sqrt{n}} \mu(k) \pi\left(\frac{n}{k}\right) = O\left(\sum_{k=1}^{\sqrt{n}} \mu(k) G\left(\frac{n}{k}\right)\right) \tag{13}$$

To apply the generalization of Möbius inversion formula to the right hand side of the above asymptotic, we need to find $F(x)$ such that we have that

$$G(n) = O\left(\sum_{k=1}^{n} F\left(\frac{n}{k}\right)\right)$$

Or, substituting,

$$\frac{n}{\log(n)} = O\left(\sum_{k=1}^{n} F\left(\frac{n}{k}\right)\right) \tag{14}$$

Using Stirling's and Riemann sums approximations, when n grows to infinity, we have that

$$\frac{n}{\log(n)} = O\left(\sum_{k=1}^{n} \frac{\log\left(\frac{n}{k}\right)}{\log\log\left(\frac{n}{k}\right)}\right)$$

As a result, applying the generalization of Möbius inversion formula, we get that

$$\sum_{k=1}^{n} \mu(k) G\left(\frac{n}{k}\right) = O\left(\frac{\log(n)}{\log\log(n)}\right) \tag{15}$$

From this result, we can just substitute n with $\sqrt{n}$ to get that

$$\sum_{k=1}^{\sqrt{n}} \mu(k) G\left(\frac{\sqrt{n}}{k}\right) = O\left(\frac{\log\left(\sqrt{n}\right)}{\log\log\left(\sqrt{n}\right)}\right)$$

Substituting, we have that

$$G\left(\frac{\sqrt{n}}{k}\right) = \frac{n}{\sqrt{n} \cdot k \cdot \log\left(\frac{n}{k}\right) - \sqrt{n} \cdot k \cdot \log\left(\sqrt{n}\right)}$$

Operating, we have that

$$\frac{1}{G\left(\frac{\sqrt{n}}{k}\right)} = \frac{\sqrt{n}}{G\left(\frac{n}{k}\right)} - \frac{k \cdot \log(\sqrt{n})}{\sqrt{n}}$$

Operating a bit more, we get that

$$G\left(\frac{\sqrt{n}}{k}\right) = G\left(\frac{n}{k}\right) \cdot \sqrt{n} \cdot \left(\frac{1}{n - k \cdot G\left(\frac{n}{k}\right)}\right)$$

As we have that
$$k \cdot G\left(\frac{n}{k}\right) = k \cdot \frac{n}{k \log\left(\frac{n}{k}\right)} = \frac{n}{\log\left(\frac{n}{k}\right)}$$

We finally get that
$$G\left(\frac{\sqrt{n}}{k}\right) = G\left(\frac{n}{k}\right) \cdot \sqrt{n} \cdot \left(\frac{1}{n \cdot \left(1 - \frac{1}{\log\left(\frac{n}{k}\right)}\right)}\right)$$
$$G\left(\frac{\sqrt{n}}{k}\right) = G\left(\frac{n}{k}\right) \cdot \frac{1}{\sqrt{n}} \cdot \left(\frac{1}{1 - \frac{1}{\log\left(\frac{n}{k}\right)}}\right)$$
$$G\left(\frac{\sqrt{n}}{k}\right) = G\left(\frac{n}{k}\right) \cdot \frac{1}{\sqrt{n}} \cdot \frac{\log\left(\frac{n}{k}\right)}{\log\left(\frac{n}{k}\right) - 1}$$
$$G\left(\frac{n}{k}\right) = O\left(G\left(\frac{\sqrt{n}}{k}\right) \cdot \sqrt{n}\right) \tag{16}$$

Therefore, we have that
$$\sum_{k=1}^{\sqrt{n}} \mu(k) G\left(\frac{n}{k}\right) = O\left(\sqrt{n} \cdot \sum_{k=1}^{\sqrt{n}} \mu(k) G\left(\frac{\sqrt{n}}{k}\right)\right)$$
$$\sum_{k=1}^{\sqrt{n}} \mu(k) G\left(\frac{n}{k}\right) = O\left(\sqrt{n} \cdot \frac{\log\left(\sqrt{n}\right)}{\log\log\left(\sqrt{n}\right)}\right)$$

And finally, we have that
$$\sum_{k=1}^{\sqrt{n}} \mu(k) G\left(\frac{n}{k}\right) = O\left(\sqrt{n} \cdot \frac{\log\left(\sqrt{n}\right)}{\log\log\left(\sqrt{n}\right)}\right) \tag{17}$$

Considering this result together with (13) we have that, as n grows to infinity,
$$\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k)\, \pi\left(\frac{n}{k}\right) = O\left(\sqrt{n} \cdot \frac{\log\left(\sqrt{n}\right)}{\log\log\left(\sqrt{n}\right)}\right) \tag{18}$$

And finally, considering this result together with (12) yields that
$$M(n) = O\left(\sqrt{n} \cdot \frac{\log\left(\sqrt{n}\right)}{\log\log\left(\sqrt{n}\right)}\right) \tag{19}$$

4 Final Remarks

The growth rate of Mertens function obtained is sufficient to prove the Riemann Hypothesis, as from the result obtained we have that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$. And we are sure that our results can be (and will be) improved to get better explicit bounds for Mertens function for sufficiently large x.

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References

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