Mirror composite numbers. Their factorization and their relationship with Goldbag conjecture.

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Abstract:

In this paper we present the concept of mirror composite numbers. Mirror composite numbers are composite numbers of the form 2n-p for some n positive natural number and p prime. We shall show that the factorization of these numbers have interesting properties in order to face the Goldbach conjecture [1][2] by the divide et impera method.

Definitions:

From now on, *m* and *n* are positive integer numbers, *p* and *q* are prime numbers.

All prime numbers p ≥ 5 are of the form 6m+1 or 6m-1. A prime of the form 6m+1 is a **right prime**; a prime of the form 6m-1 is a **left prime**.

A **mirror composite number** is a composite number of the form 2n-p for some n and some prime p ≥ 5.

Given a mirror composite 2n-p, if p=6m+1, i.e., if p is a right prime, 2n-p is a **right mirror composite (r.m.c.)**.

Given a mirror composite 2n-p, If p=6m-1, i.e., if p is a left prime, 2n-p is a **left mirror composite (l.m.c.)**.

Lemma 1.

Fixed n, if 3 is a factor of some l.m.c (respectively r.m.c.), 3 is a factor of every l.m.c. (r.m.c.) and 3 is not a factor of any r.m.c. (l.m.c)

Proof:

The difference between two l.m.c. (r.m.c.) is 6n. If 3|m, 3|m±6n. On the other hand, if 3|2n-(6m-1), then 3 ∤ 2n-(6m+1) and *viceversa*.

Lemma 2.

Fixed n, if q≠3 is a prime factor of two different l.m.c. (respectively r.m.c.), the difference between them is a multiple of 6q so the minimum gap between two consecutive occurrences of factor q is 6q for all l.m.c. (r.m.c.).

Proof:

If q | 2n-(6x-1) and q | 2n-(6y-1) exists z such that zq=6(x-y), so z is multiple of 6, given that q is a prime and q ≠ 2,3.

If q | 2n-(6x+1) and q | 2n-(6y+1) exists z such that zq=6(x-y), so z is multiple of 6, given that q is a prime and $q \neq 2,3$.

Goldbach conjecture states that for all n and all p such that $3 \leq p \leq n$, some 2n-p is a prime, i.e., not every 2n-p is composite.

Let´s assume for the sake of contradiction that exists n such that every 2n-p is composite. Then, 3 consecutive odd numbers, 2n-3, 2n-5 and 2n-7 are composite, so one and only one of them must be multiple of 3.

Case A: 3|2n-7:

$3|2n-7 \Rightarrow 3|2n-(6m+1)$ for all m (**Lemma 1**). Every right mirror composite is a multiple of 3 and no left mirror composite is a multiple of 3. So all elements of the sequence:

$$2n-3, 2n-5, 2n-11, 2n-17, 2n-23, 2n-29, 2n-41, \ldots, 2n-q$$

where $q \geq 5$ is a left prime, must be factorized. There are k consecutive primes p_i (i=1,2,3, …, k) from $p_1=5$ to p_k, where p_k is the largest prime $p_k \leq \sqrt{2n-5}$, available for that factorization.

Now, given the correlative sequence of odd numbers 2n-3, 2n-5, 2n-7, 2n-9, 2n-11, 2n-13, 2n-15, 2n-a…, let be $2n-a_i$ the number containing the first occurrence of prime factor p_i in that sequence.

Notice that:

For each p_i, a_i is unique.

$3 \leq a_i \leq 2p_i+1$.

For some i, $a_i = 3$; for some i, $a_i=5$; for some i, $a_i=11$ MOD p_i; for some i, $a_i=17$ MOD p_i; for some i, $a_i=23$ MOD p_i and so on.

2n-q, i.e., 2n-(6m-1), is composite if and only if exists i such that $6m-1 \equiv a_1 \bmod p_i$ (**Lemma 2**).

Now, let´s state conditions in order to find some 2n-q with q=6m-1 and q inside the interval $\sqrt{2n-5} \leq q \leq n$ that can not be factorized:

1) q is a prime, i.e., q is not multiple of any p_i, so $6m-1 \not\equiv 0 \bmod p_i$ for all i.
2) There is no p_i factor available for 2n-q, so $6m-1 \not\equiv a_1 \bmod p_i$ for all i.

Prime condition for 6m-1	No factor available condition for 2n-(6m-1)
$6m \not\equiv 1 \bmod 5$	$6m \not\equiv (a_1+1) \bmod 5$
$6m \not\equiv 1 \bmod 7$	$6m \not\equiv (a_2+1) \bmod 7$

$6m \not\equiv 1 \bmod 11$ $\quad$ $6m \not\equiv (a_3+1) \bmod 11$

$6m \not\equiv 1 \bmod 13$ $\quad$ $6m \not\equiv (a_4+1) \bmod 13$

………….. $\quad$ ……………….

$6m \not\equiv 1 \bmod p_k$ $\quad$ $6m \not\equiv (a_k+1) \bmod p_k$

Hence for each p_i there are *at least* p_i-2 remainders moduli p_i that fullfill the conditions. That amounts up to a minimum of $(p_1-2)(p_2-2)(p_3-2)\ldots(p_k-2)$, id est, $3.5.9.11.\ldots(p_k-2)$ different systems of linear congruences with prime moduli. The chinese remainder theorem ensures that each one of them has a different and unique solution moduli $5.7.11.13\ldots p_k$.

It´s necessary then to prove that exists at least a multiple of 6 that fullfills the preceding conditions inside the interval:

$$\sqrt{2n-5} < 6m < n$$

So let´s prove that at least one in $3.5.9.11\ldots(p_k-2)$ solutions from $5.7.11.13\ldots p_k$ systems lies inside the aformentioned interval.

Let be M the highest number of consecutive occurrences of 6m that do not fullfill the conditions.[1] Is not easy to figure out the value of M, given the unpredictable nature of prime number distribution. But we can prove that exists an upper bound S for M such that for sufficient large n:

$$S < \left[\frac{n-\sqrt{2n-5}}{6}\right] \qquad (1)$$

Given p_k, an upper bound for the total number of occurrences of each one of the two remainders moduli p are $2\left\lceil\frac{p_k}{p}\right\rceil$. So

$$S = 2\left(\left\lceil\frac{p_k}{5}\right\rceil + \left\lceil\frac{p_k}{7}\right\rceil + \left\lceil\frac{p_k}{11}\right\rceil + \left\lceil\frac{p_k}{13}\right\rceil + \ldots + \left\lceil\frac{p_k}{p_{k-1}}\right\rceil + 1\right)$$

is an upper bound for M:

k	**p_k**	**M**	**S**
1	5	2	2
2	7	4	6
3	11	8	11
4	13	13	16

[1] For all those who, like myself, enjoy practical questions that sometimes shed light on some more abstract matter of discussion, the problem to determine an accurate value for **M** is the same as the following: Suppose you may not work on 2 predetermined days in five, 2 predetermined days in seven, 2 days in 11, 2 in 13 and so on until 2 days in p_k days. What is the maximum number, as a function of p_k, of consecutive days off?

k	p_k	M	S
5	17	19	24
6	19	22	28

In turn:

$$\left\lceil\frac{p_k}{5}\right\rceil + \left\lceil\frac{p_k}{7}\right\rceil + \left\lceil\frac{p_k}{11}\right\rceil + \left\lceil\frac{p_k}{13}\right\rceil + \ \dots + \left\lceil\frac{p_k}{p_{k-1}}\right\rceil + \ 1 <$$

$$\frac{p_k}{2} + \frac{p_k}{3} + \frac{p_k}{5} + \frac{p_k}{7} + \frac{p_k}{11} + \ \dots + \frac{p_k}{p_{k-1}} + 1 =$$

$$p_k \left\{\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} \ \dots + \frac{1}{p_{k-1}} + \frac{1}{p_k}\right\}$$

The series between brackets is the well known partial summation of the reciprocal of the primes whose divergence was proved by Euler in 1737 together with the relationship:

$$\sum_{p \le x} \frac{1}{p} \approx \text{loglog}(x) \qquad (2)$$

Taking $x=p_k$ and given that an upper bound for all $x>e^4$ in (2) is loglogx+6 [3] allows us to state:

$$S < 2p_k(\text{loglog}p_k+6)$$

Now it`s inmediate to conclude, since $p_k \le \sqrt{2n-5}$, that (1) holds for, let´s say, every $2n \ge 10^6$.

For every $2n<10^6$ the verification of the conjecture have alredy been settled.

That completes the demonstration.

Hence, for all 2n such that 3|2n-7, i.e., for all $2n \equiv 1 \bmod 3$, exists some 2n-q that can not be factorized, so 2n-q is prime and the conjecture holds for all $2n \equiv 1 \bmod 3$.

Case B: 3|2n-5:

3|2n-5⇒ 3|2n-(6m-1) for all m (**Lemma 1**). So every left mirror composite is a multiple of 3 and no right mirror composite is a multiple of 3…

Following the same thought process than before, with q a right prime of the form 6m+1, it´s straightforward to conclude that the conjecture holds for all 2n such that $3|2n-5$, i.e., for all $2n \equiv 2 \mod 3$.

Case C: $3|2n-3$:

Interesting matter of forward research.

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