Convergence and Computation of sum of a series on the Riemann Zeta function

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ABSTRACT

In this paper, we present a new method of evaluating the convergence and sum of a series with the Riemann zeta function in its general term.We consider the convergence and sum of a series by means of difference other than previous methods.

Keywords: Difference, Alternating Series, Riemann Zeta function, Abel's summation formula,

1. Introduction

Riemann zeta function is defined as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \mathrm{Re}(s)>1.(1)$$

Riemann zeta function is a special function which can find wide applications in natural science, engineering as well as mathematics, and which can provide particularly powerful means for solving problems arising in physics, chemistry, probability, computer science, control, etc.

Especially, Riemann zeta function is famous for Riemann's hypothesis.

Riemann zeta function is very useful for sum computation as well, in which a lot of attention has been paid. (See [1]-[4], [6], [7], [9]-[12]

[8] validated the following problem by using Abel's summation formula and definition of Riemann zeta function.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(n\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)-1\right)=\frac{\pi^2}{16}-\frac{\ln 2}{2}-\frac{1}{2}. \tag{2}$$

This problem has been more generalized, one of which is as follows.(see[13])

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(n^{k-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)-\frac{1}{k-1}\right). \tag{3}$$

Where if $k=2$, then (2) is formedfrom equation(3).

In [5], series(3) was considered in case $k=3$, from which the following result was obtained.

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^2\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}-\frac{1}{2n^2}\right)=\frac{1}{2}\left(\ln 2+\frac{1}{2}-\frac{\zeta(2)}{2}\right). \tag{4}$$

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In our paper, we aim at eliciting convergence of series (3) and calculating its sum.
We use the difference theory in consideration of convergence and sum unlike the preceding literature.
Finally we check the correctness of our results by comparing with the previous research results.

2. Main result

We introduce some basic conceptions and lemmas for further consideration.
First we note some things relevant to difference.
Given a function $f(x)$, $f(x+1)-f(x)$ is called the first order difference of the function $f(x)$ and is denoted as

$$\Delta f(x) \equiv \Delta f \tag{4}$$

In general, the first order difference of $p$order difference of function $f(x)$ is called $p+1$ order difference and $p$order difference of function is $f(x)$ denoted as

$$\Delta^p f(x) \equiv \Delta^p f \tag{5}$$

Namely

$$\Delta^{p+1} f = \Delta(\Delta^p f(x)),$$

We arrange as $\Delta^0 f(x) = f(x)$.

Lemma 1. For function $f(x)$,

$$f(x+n) = \sum_{i=0}^{n} C_n^i \Delta^i f(x). \tag{6}$$

is formed.

Proof. It is proved by induction on *n*.
The base case $n = 1$ is clear.
For the induction step, assume that the statement holds for n. Then since

$$\begin{aligned} f(x+n+1) = f(x+1+n) &= \sum_{i=0}^{n} C_n^i \Delta^i f(x+1) \\ &= \sum_{i=0}^{n} C_n^i \left(\Delta^i f(x) + \Delta^{i+1} f(x)\right) = \sum_{i=0}^{n} \left(C_{n+1}^{i+1} - C_n^{i+1}\right)\Delta^{i+1} f(x) + \sum_{i=0}^{n} C_n^i \Delta^i f(x) \\ &= \sum_{i=0}^{n} C_{n+1}^{i+1} \Delta^{i+1} f(x) + f(x) = \sum_{i=0}^{n+1} C_{n+1}^{i} \Delta^{i} f(x). \end{aligned}$$

even in case of $n+1$ equation (6) holds good.
Hence, lemma 1 is validated.

Next, Let's define p order arithmetic progression.
$\{\Delta f(n)\}$ is called first difference progression of $\{f(n)\}$.
In general, $p$order difference progression is defined as

$$\Delta^p f(n) \equiv \Delta\left(\Delta^{p-1} f(n)\right)$$

If $p$order difference progression of $\{f(n)\}$ is notzero progression, but $p+1$ order difference progression is zero progression, then progression $\{f(n)\}$ is called $p$order arithmetic progression.

Also, arithmetic progression over second order is called higher order arithmetic progression.

Lemma 2. The following equation holds good for $p$arithmetic progression.

$$f(n)=\sum_{i=0}^{\min\{n,p\}} C_n^i \Delta^i f(0). \tag{7}$$

Proof. Substituting $x=0$ into equation(6),the following equation yields.

$$f(n)=\sum_{i=0}^{n} C_n^i \Delta^i f(0).$$

If $n>p$, then we get that

$$\Delta^{p+1} f(x)=\cdots=\Delta^n f(0)=0$$

Moreover, when $m<n$, $C_m^n=0$

So, the proof of the lemma 2 is complete.

Lemma 3. $\{f(n)\}$ is $p$order arithmetic progression, if and only if $f(n)$ is $p$order polynomial.

Proof. The necessity is derived from lemma 2.

If $\deg(f(x))=p$, then since

$$\Delta^p f(n)=a_0 p!,\ \Delta^{p+1} f(n)=0,$$

it is sufficient. Here a_0 is highest leading knot coefficient of $f(n)$.

Lemma 4. Assuming that $\{f(n)\}$ is higher arithmetic progression,the first n knot summation of progression is as follows.

$$f(1)+f(2)+\cdots+f(n)=\sum_{i=0}^{n-1} C_n^{i+1} \Delta^i f(1). \tag{8}$$

Proof. It is proved by induction on n.

When n=1, it holds.

Assume that when it is n, it holds good.Then by using lemma 1 we can get the following, which demonstrates that even in case of n+1 equation (8) holds good.

$$f(1)+f(2)+\cdots+f(n+1)=\sum_{i=0}^{n-1} C_n^{i+1} \Delta^i f(1)+f(n+1)$$

$$=\sum_{i=0}^{n-1} C_n^{i+1} \Delta^i f(1)+\sum_{i=0}^{n} C_n^i \Delta^i f(1)=\sum_{i=0}^{n-1}\left(C_n^{i+1}+C_n^i\right)\Delta^i f(1)+\Delta^n f(1)=\sum_{i=0}^{n} C_{n+1}^{i+1} \Delta^i f(1).$$

Thus, the proof is completed.

Let's define difference polynomial. In case of $r \geq 1$, it is defined as

$$p_r(x)=\frac{1}{r!}x(x-1)\cdots(x-r+1)$$

Conspicuously it is true that $\Delta^r p_r(x)=1$.

For convenience sake it is expressed as

$$\binom{x}{r}=\frac{1}{r!}x(x-1)\cdots(x-r+1).$$

Since any r order polynomial can be written as the first order combination in terms of difference polynomial, it is expressed as

$$f(x)=c_r\binom{x}{r}+c_{r-1}\binom{x}{r-1}+\ldots+c_1\binom{x}{1}+c_0,$$

where $c_0,c_1,\cdots,c_r$ areconstants.

Taking the first order, second order, ...*r*order difference to both terms of this equation in turn, and using $\Delta\binom{x}{j}=\binom{x}{j-1}$, we can obtain the following.

$$\Delta f(x)=c_r\binom{x}{r-1}+c_{r-1}\binom{x}{r-2}+\cdots+c_2\binom{x}{1}+c_1,$$

$$\Delta^2 f(x)=c_r\binom{x}{r-2}+c_{r-1}\binom{x}{r-3}+\cdots+c_3\binom{x}{1}+c_2,$$

$$\ldots$$

$$\Delta^r f(x)=c_r.$$

Here we consider convergence and sum of equation (3).

If we use the foregoing solution method,we can't solve this problem.

So we are going to solve this problem with the aid of aforementioned difference theory.

At first let's consider convergence.

Lemma 5. For a natural number $k\geq 2$, any natural number *n*satisfies the following inequality.

$$\frac{k}{(n+1)^{k+1}}<\frac{1}{n^k}-\frac{1}{(n+1)^k}<\frac{k}{n^{k+1}}. \tag{9}$$

Proof. The left-side inequality of (9) is equivalent to the following expression

$$\frac{k+n+1}{(n+1)^{k+1}}<\frac{1}{n^k}\Leftrightarrow (n+1)^{k+1}>n^{k+1}+(k+1)n^k$$

which is readily proved by binomial formula. Thus the left-side inequality is easily proved.

Next, let's prove the right-side inequality.

The right-side inequality is equivalent to the following inequality.

$$\frac{\dfrac{1}{(n+1)^k}-\dfrac{1}{n^k}}{1}>\frac{-k}{n^{k+1}} \tag{10}$$

We consider function $g(x)=x^{-k}$, $x>0$ in order to prove equation (10).

Since this function is differentiable at interval $(0,+\infty)$, mean value theorem at interval [n, n+1] can be applied. Therefore as for natural number *n*given randomly there exists $c\in(n,n+1)$ to satisfy

$$\frac{1}{(n+1)^k}-\frac{1}{n^k}=g'(c).$$

Meanwhile, as $g''(x) = k(k+1)x^{-(k+2)}$, $g'(x)$ is increasing function at $(0,+\infty)$, whereas $\frac{-k}{n^{k+1}} = g'(n)$, it is expected that $g'(c) > g'(n)$. So equation (10) holds good.

Remark. From lemma 5, we can make sure that for any natural number n , the following inequality is obtained.

$$\frac{1}{(n+1)^k} < \frac{1}{k-1}\left(\frac{1}{n^{k-1}} - \frac{1}{(n+1)^{k-1}}\right) < \frac{1}{n^k}, k \geq 3. \tag{11}$$

Lemma 6. The following inequality is available.

$$\frac{1}{(k-1)(n+1)^{k-1}} \leq \zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{n^k} \leq \frac{1}{(k-1)n^{k-1}} \tag{12}$$

Proof. From the expression

$$\zeta(k) - 1 - \frac{1}{2^k} - \frac{1}{3^k} - \cdots - \frac{1}{n^k} = \sum_{p=1}^{\infty} \frac{1}{(n+p)^k},$$

and (11), the following inequalities holds true.

$$\sum_{p=1}^{\infty} \frac{1}{(n+p)^k} \leq \sum_{p=1}^{\infty} \frac{1}{k-1}\left(\frac{1}{(n+p-1)^{k-1}} - \frac{1}{(n+p)^{k-1}}\right) = \frac{1}{(k-1)n^{k-1}},$$

$$\sum_{p=1}^{\infty} \frac{1}{(n+p)^k} \geq \sum_{p=1}^{\infty} \frac{1}{k-1}\left(\frac{1}{(n+p)^{k-1}} - \frac{1}{(n+p+1)^{k-1}}\right) = \frac{1}{(k-1)(n+1)^{k-1}}.$$

Thus, (12) is proved.

Lemma 7. $\sum_{i=1}^{n} i^{k-1}$ is expressed as *k*order polynomial in terms of *n*.

Proof. Since assuming $f(n) = n^{k-1}$, $f(n)$ is *k*order polynomial with respect to *n*, $\{f(n)\}$ is *k*orderarithmetic progression from lemma 3. Hence we can apply lemma 4 to $f(n)$. Thus since $C_m^n = 0, m < n$.,we can obtain the following.

$$\sum_{i=1}^{n} i^{k-1} = \sum_{i=0}^{n-1} C_n^{i+1} \Delta^i f(1) = \sum_{i=0}^{k-1} C_n^{i+1} \Delta^i f(1)$$

As developing C_n^i ,*k* is constant, this is *i*order polynomial in terms of *n*.

$$\sum_{i=1}^{n} i^{k-1} = \sum_{i=0}^{k-1} C_n^{i+1} \Delta^i f(1) = \sum_{i=0}^{k-1} \frac{n(n-1)\cdots(n-i)}{(i+1)!} \Delta^i f(1)$$

$$= \sum_{i=0}^{k-1} P_i n(n-1)\cdots(n-i) = \sum_{i=0}^{k-1} P_i \sum_{j=0}^{i} n^{j+1} \sum_{1 \leq u_1 \leq u_2 \leq \cdots \leq u_{i-j} \leq i} (-1)^{i-j} u_1 u_2 \cdots u_{i-j}$$

$$= \sum_{i=0}^{k-1} P_i \sum_{j=0}^{i} n^{j+1} V(i-j, i) = \sum_{i=0}^{k-1} n^{i+1} \sum_{j=i}^{k-1} P_j V(j-i, j) = \sum_{i=0}^{k-1} Q_i n^{i+1},$$

where

$$P_i=\frac{\Delta^i f(1)}{(i+1)!},\ V(i-j,i)=\sum_{1\le u_1<u_2<\cdots<u_{i-j}\le i}(-1)^{i-j}u_1u_2\cdots u_{i-j},0\le i\le k-1,0\le j\le i,$$

$$V(0,t)=1(0\le t\le k-1),\ Q_i=\sum_{j=i}^{k-1}P_jV(j-i,j),0\le i\le k-1.$$

On the other hand

$$Q_{k-1}=P_{k-1}=\frac{\Delta^{k-1}f(1)}{k!}=\frac{1}{k}\neq 0,$$

So the lemma is proved.

Now we consider convergence of series (3).

Theorem 1. Series (3) is convergent.

Proof. We rewrite series (3) as follows;

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n-1}\left(n^{k-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)-\frac{1}{k-1}\right)\\
&=\sum_{n=2}^{\infty}(-1)^{n-1}\left(n^{k-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)-\frac{1}{k-1}\right)+\zeta(k)-1-\frac{1}{k-1}\\
&=\sum_{m=1}^{\infty}(-1)^{m}\left((m+1)^{k-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(m+1)^k}\right)-\frac{1}{k-1}\right)+\zeta(k)-1-\frac{1}{k-1}\\
&=\sum_{m=1}^{\infty}(-1)^{m}\left((m+1)^{k-1}-m^{k-1}\right)\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{m^k}\right)-S+\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m+1}+\zeta(k)-1-\frac{1}{k-1}\\
&=\zeta(k)-\ln 2-\frac{1}{k-1}+\sum_{m=1}^{\infty}(-1)^{m}\left((m+1)^{k-1}-m^{k-1}\right)\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{m^k}\right)\\
&-\sum_{n=1}^{\infty}(-1)^{n-1}\left(n^{k-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)-\frac{1}{k-1}\right).
\end{aligned}\tag{13}$$

Therefore, series (3) converges iff series

$$\sum_{m=1}^{\infty}(-1)^{m}\left((m+1)^{k-1}-m^{k-1}\right)\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{m^k}\right).$$

converges.

Since $(m+1)^{k-1}-m^{k-1}$ is $k-2$ orderpolynomial in terms of*m*, we get that

$$\lim_{m\to\infty}\left((m+1)^{k-1}-m^{k-1}\right)\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{m^k}\right)=0,$$

with help of equation (12).

Let (y_m)be the sequence defined by

$$y_m=\left((m+1)^{k-1}-m^{k-1}\right)\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{m^k}\right),$$

then $y_m>0$ and the following is obtained.

$$y_m - y_{m+1} = (2(m+1)^k - m^{k-1} - (m+2)^{k-1})\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{m^k}\right)$$
$$+\frac{((m+2)^{k-1} - (m+1)^{k-1})}{(m+1)^k} \geq \frac{2(m+1)^k - m^{k-1} - (m+2)^{k-1}}{(k-1)m^{k-1}} + \frac{((m+2)^{k-1} - (m+1)^{k-1})}{(m+1)^k}$$
$$= \frac{(2(m+1)^k - m^{k-1} - (m+2)^{k-1})(m+1)^k + ((m+2)^{k-1} - (m+1)^{k-1})(k-1)m^{k-1}}{(k-1)m^{k-1}(m+1)^k}.$$

In the final expression, numerator is polynomial with respect to m.
Let's find higher order knot coefficient of this polynomial.
$2k-1, 2k-2$ order coefficient is zero. And $2k-3$ order coefficient is constant as

$$2C_{k-1}^2 - 2^2 C_{k-1}^2 + (k-1)(2 \cdot C_{k-1}^1 - C_{k-1}^1) = k-1.$$

Therefore we get the following. $\exists m_0 \in \mathbf{N}, \forall m > m_0$

$$\left(2(m+1)^{k-1} - m^{k-1} - (m+2)^{k-1}\right)(m+1)^k + \left((m+2)^{k-1} - (m+1)^{k-1}\right)(k-1)m^{k-1} > 0,$$

Namely $\exists m_0 \in \mathbf{N}, \forall m > m_0, y_m > y_{m+1}$.

Hence from Leibniz test, series $\sum_{m=1}^{\infty}(-1)^m y_m$ converges.

From the above discussion it is clear that series (3) is converges.

Now we find the sum of series (3).

Theorem 2. Sum of series (3) is as follows.

$$\frac{\zeta(k) - \ln 2}{2} - \frac{1}{2k-2} + \frac{1}{2}\left(\left(1 - \frac{1}{2^k}\right)\sum_{i=1}^{k-2}(-1)^i\left(2(1-2^{k-i})Q_{i-1} + C_{k-1}^i\right) - \frac{1}{2^k}\right)\zeta(k)$$
$$+\frac{1}{2}\sum_{i=1}^{k-2}\left(2^{i+1-k}(1-2^{k-i})Q_{i-1} + \sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1} + C_{k-1}^j\right)\left(1 - \frac{1}{2^{k-i}}\right)\right)\zeta(k-i),$$

Proof. Let S be the sumof series (3). Then from (13), we have that

$$S = \frac{1}{2}\left(\zeta(k) - \ln 2 - \frac{1}{k-1} + \sum_{m=1}^{\infty}(-1)^m\left((m+1)^{k-1} - m^{k-1}\right)\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{m^k}\right)\right).$$

Using Abel's summation formula with

$$a_m = (-1)^m\left((m+1)^{k-1} - m^{k-1}\right), \text{ and } b_m = \zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{m^k},$$

we get that

$$\sum_{m=1}^{\infty}(-1)^m\left((m+1)^{k-1} - m^{k-1}\right)\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{m^k}\right)$$
$$= \lim_{n\to\infty} A_n\left(\zeta(k) - 1 - \frac{1}{2^k} - \cdots - \frac{1}{(n+1)^k}\right) + \sum_{n=1}^{\infty}\frac{A_n}{(n+1)^k},$$

where

$$A_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (-1)^i \left((i+1)^{k-1} - i^{k-1}\right) = (-1)^n (n+1)^{k-1} + 2\sum_{i=2}^{n} (-1)^{i-1} i^{k-1} + 1$$
$$= \sum_{i=1}^{n+1} (-1)^{i-1} i^{k-1} + \left(\sum_{i=1}^{n} (-1)^{i-1} i^{k-1} \right) - 1.$$

Using the result of lemma 7, we obtain

$$\sum_{i=1}^{2m-1} (-1)^{i-1} i^{k-1} = \sum_{i=1}^{2m-1} i^{k-1} - 2^k \sum_{i=1}^{m-1} i^{k-1} = \sum_{i=1}^{2m} i^{k-1} - 2^k \sum_{i=1}^{m} i^{k-1} - 2^{k-1} m^{k-1} + 2^k m^{k-1}$$
$$= (2m)^{k-1} + \sum_{i=0}^{k-1} Q_i (2m)^{i+1} - 2^k \sum_{i=0}^{k-1} Q_i m^{i+1} = (2m)^{k-1} + \sum_{i=0}^{k-1} 2^{i+1} (1 - 2^{k-i-1}) Q_i m^{i+1}$$
$$= (1 - Q_{k-2})(2m)^{k-1} + \sum_{i=1}^{k-2} 2^i (1 - 2^{k-i}) Q_{i-1} m^i,$$

$$\sum_{i=1}^{2m} (-1)^{i-1} i^{k-1} = \sum_{i=1}^{2m} i^{k-1} - 2^k \sum_{i=1}^{m} i^{k-1} = \sum_{i=0}^{k-1} Q_i (2m)^{i+1} - 2^k \sum_{i=0}^{k-1} Q_i m^{i+1}$$
$$= \sum_{i=0}^{k-1} 2^{i+1} (1 - 2^{k-i-1}) Q_i m^{i+1} = -Q_{k-2} (2m)^{k-1} + \sum_{i=1}^{k-2} 2^i (1 - 2^{k-i}) Q_{i-1} m^i$$

Thus it follows that

$$A_{2n} = \sum_{i=1}^{2n+1} (-1)^{i-1} i^{k-1} + \sum_{i=1}^{2n} (-1)^{i-1} i^{k-1} - 1 = (2n+1)^{k-1} + 2\sum_{i=1}^{2n} (-1)^{i-1} i^{k-1} - 1$$
$$= (2n+1)^{k-1} - 2^k Q_{k-2} n^{k-1} + \sum_{i=1}^{k-2} 2^{i+1} (1 - 2^{k-i}) Q_{i-1} n^i - 1$$
$$= (2^{k-1} - 2^k Q_{k-2}) n^{k-1} + \sum_{i=1}^{k-2} \left(2^{i+1} (1 - 2^{k-i}) Q_{i-1} + 2^i C_{k-1}^i \right) n^i,$$

$$A_{2n-1} = \sum_{i=1}^{2n} (-1)^{i-1} i^{k-1} + \sum_{i=1}^{2n-1} (-1)^{i-1} i^{k-1} - 1 = (1 - 2Q_{k-2})(2n)^{k-1}$$
$$+ \left(\sum_{i=1}^{k-2} 2^{i+1} (1 - 2^{k-i}) Q_{i-1} n^i \right) - 1.$$

Now we calculate Q_{k-2}.

$$Q_{k-2} = \sum_{j=k-2}^{k-1} V(j-k+2, j) P_j = P_{k-2} + V(1, k-1) \cdot P_{k-1}$$
$$= \frac{\Delta^{k-2} f(1)}{(k-1)!} + \frac{\Delta^{k-1} f(1)}{k!} V(1, k-1) = \frac{\Delta^{k-2} f(1)}{(k-1)!} - \frac{k(k-1)}{2} \cdot \frac{\Delta^{k-1} f(1)}{k!}$$
$$= \frac{\Delta^{k-2} f(1)}{(k-1)!} - \frac{k-1}{2},$$

where

$$V(1, k-1) = \sum_{1 \le u_1 \le k-1} (-1)^1 u_1 = -\sum_{p=1}^{k-1} p = -\frac{k(k-1)}{2}.$$

Let $f(m)$ be the function defined by

$$f(m)=c_{k-1}\binom{m}{k-1}+c_{k-2}\binom{m}{k-2}+\cdots+c_1\binom{m}{1}+c_0.$$

Then, we get

$$f(m)=c_{k-1}\binom{m}{k-1}+c_{k-2}\binom{m}{k-2}+\cdots+c_1\binom{m}{1}+c_0=c_{k-1}C_m^{k-1}+c_{k-2}C_m^{k-2}+\cdots+c_0$$

$$=c_{k-1}\frac{m(m-1)\cdots(m-k+2)}{(k-1)!}+c_{k-2}\frac{m(m-1)\cdots(m-k+3)}{(k-2)!}+\cdots+c_0$$

$$=\frac{c_{k-1}}{(k-1)!}m^{k-1}+(\frac{c_{k-2}}{(k-2)!}-\frac{(k-1)(k-2)}{2})m^{k-2}+\cdots+c_0.$$

Since $f(m)=m^{k-1}$, it follows that

$$c_{k-1}=(k-1)!,c_{k-2}=\frac{(k-1)(k-2)}{2}(k-2)!.$$

then

$$\Delta^{k-2}f(1)=c_{k-1}\binom{1}{1}+c_{k-2}=c_{k-1}+c_{k-2}=\frac{k!}{2}.$$

And it follows that $Q_{k-2}=\frac{1}{2}$.

So with the result,we have that

$$A_{2n}=\sum_{i=1}^{k-2}\left(2^{i+1}(1-2^{k-i})Q_{i-1}+2^iC_{k-1}^i\right)n^i,A_{2n-1}=\left(\sum_{i=1}^{k-2}2^{i+1}(1-2^{k-i})Q_{i-1}n^i\right)-1.$$

Now we prove the following.

$$\lim_{n\to\infty}A_n\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)=0.$$

For the sake of this following should be validated.

$$\lim_{n\to\infty}A_{2n}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(2n+1)^k}\right)=0,$$

$$\lim_{n\to\infty}A_{2n-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(2n)^k}\right)=0.$$

It follows from equation (12) that

$$\frac{1}{(k-1)(2n+2)^{k-1}}<\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(2n+1)^k}<\frac{1}{(k-1)(2n+1)^{k-1}}.$$

Accordingly the above can be written as follows.

$$\frac{A_{2n}}{(k-1)(2n+2)^{k-1}}\le A_{2n}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(2n+1)^k}\right)\le\frac{A_{2n}}{(k-1)(2n+1)^{k-1}}.$$

Since A_{2n} is k-2 order polynomial in terms of n and $(2n+2)^{k-1},(2n+1)^{k-1}$ is k-1 order polynomial with respect to n, we get

$$\lim_{n\to\infty}\frac{A_{2n}}{(k-1)(2n+2)^{k-1}}=\lim_{n\to\infty}\frac{A_{2n}}{(k-1)(2n+1)^{k-1}}=0.$$

Therefore,

$$\lim_{n\to\infty} A_{2n}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(2n+1)^k}\right)=0.$$

In the same way, it holds.

$$\lim_{n\to\infty} A_{2n-1}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(2n)^k}\right)=0.$$

Hence it is true that

$$\lim_{n\to\infty} A_{n}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{(n+1)^k}\right)=0.$$

Now we calculate $\sum_{n=1}^{\infty}\frac{A_n}{(n+1)^k}$.

Since A_n is k-2 order polynomial in terms of$n$, the this series converges.

We have

$$\sum_{n=1}^{\infty}\frac{A_n}{(n+1)^k}=\sum_{n=1}^{\infty}\frac{A_{2n}}{(2n+1)^k}+\sum_{n=1}^{\infty}\frac{A_{2n-1}}{(2n)^k}.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{A_{2n-1}}{(2n)^k}&=\sum_{n=1}^{\infty}\left(\left(\sum_{i=1}^{k-2}2^{i+1}(1-2^{k-i})Q_{i-1}\right)-1\right)\frac{1}{(2n)^k}\\
&=\sum_{i=1}^{k-2}2^{i+1-k}(1-2^{k-i})Q_{i-1}\sum_{n=1}^{\infty}\frac{1}{n^{k-i}}-\frac{1}{2^k}\zeta(k)\\
&=\left(\sum_{i=1}^{k-2}2^{i+1-k}(1-2^{k-i})Q_{i-1}\zeta(k-i)\right)-\frac{1}{2^k}\zeta(k),\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{A_{2n}}{(2n+1)^k}=\sum_{n=1}^{\infty}\left(\sum_{i=1}^{k-2}\left(2^{i+1}(1-2^{k-i})Q_{i-1}+2^iC_{k-1}^i\right)n^i\right)\frac{1}{(2n+1)^k}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{i=1}^{k-2}\left(2^{i+1}(1-2^{k-i})Q_{i-1}+2^iC_{k-1}^i\right)\frac{1}{2^i}(2n+1-1)^i\right)\frac{1}{(2n+1)^k}$$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n+1)^k}\sum_{i=1}^{k-2}\left(\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)\sum_{j=0}^{i}(-1)^{i-j}C_i^j(2n+1)^j\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n+1)^k}\left(\left(\sum_{i=1}^{k-2}\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)(2n+1)^i\right)+\left(\sum_{i=1}^{k-2}(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)\right)\right)$$

$$=\sum_{i=1}^{k-2}\left(\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)\right)\sum_{n=1}^{\infty}\frac{1}{(2n+1)^{k-i}}$$

$$+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^k}\sum_{i=1}^{k-2}(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)$$

$$=\sum_{i=1}^{k-2}\left(\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)\right)\left((1-\frac{1}{2^{k-i}})\zeta(k-i)-1\right)$$

$$+\left(\sum_{i=1}^{k-2}(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)\right)\left((1-\frac{1}{2^k})\zeta(k)-1\right)$$

$$=\sum_{i=1}^{k-2}\left(2^{i+1-k}(1-2^{k-i})Q_{i-1}+\left(\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)\right)(1-\frac{1}{2^{k-i}})\right)\zeta(k-i)$$

$$-\sum_{i=1}^{k-2}\left(\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)+(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)\right).$$

and

$$\sum_{i=1}^{k-2}\left(\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)+(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)\right)$$

$$=\sum_{i=1}^{k-2}\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\,2(1-2^{k-j})Q_{j-1}+\sum_{i=1}^{k-2}\sum_{j=i}^{k-2}(-1)^{j-i}C_j^iC_{k-1}^j+\sum_{i=1}^{k-2}(-1)^i\,2(1-2^{k-i})Q_{i-1}+\sum_{i=1}^{k-2}(-1)^iC_{k-1}^i.$$

Calculating the first and second summations respectively, as we obtain

$$\sum_{i=1}^{k-2}\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\,2(1-2^{k-j})Q_{j-1}=\sum_{i=1}^{k-2}\sum_{j=1}^{i}(-1)^{i-j}C_i^j\,2(1-2^{k-i})Q_{i-1}$$

$$=\sum_{i=1}^{k-2}(-1)^i\,2(1-2^{k-i})Q_{i-1}\sum_{j=1}^{i}(-1)^j\cdot C_i^j=\sum_{i=1}^{k-2}(-1)^i\,2(1-2^{k-i})Q_{i-1}((1-1)^i-1)$$

$$=-\sum_{i=1}^{k-2}(-1)^i\,2(1-2^{k-i})Q_{i-1},$$

$$\sum_{i=1}^{k-2}\sum_{j=i}^{k-2}(-1)^{j-i}C_j^iC_{k-1}^j=\sum_{i=1}^{k-2}\sum_{j=1}^{i}(-1)^{i-j}C_i^jC_{k-1}^i=\sum_{i=1}^{k-2}C_{k-1}^i(-1)^i\sum_{j=1}^{i}(-1)^jC_i^j=-\sum_{i=1}^{k-2}C_{k-1}^i(-1)^i.$$

So it follows that

$$\sum_{i=1}^{k-2}\left(\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)+(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)\right)=0.$$

Summarizing the above results, we can get

$$S=\frac{1}{2}\left(\zeta(k)-\ln 2-\frac{1}{k-1}+\sum_{m=1}^{\infty}(-1)^{\mathrm{m}}\left((m+1)^{k-1}-m^{k-1}\right)\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{m^k}\right)\right)$$

$$=\frac{\zeta(k)-\ln 2}{2}-\frac{1}{2k-2}+\frac{1}{2}\left((1-\frac{1}{2^k})\sum_{i=1}^{k-2}(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)-\frac{1}{2^k}\right)\zeta(k)$$

$$+\frac{1}{2}\sum_{i=1}^{k-2}\left(2^{i+1-k}(1-2^{k-i})Q_{i-1}+\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)\left(1-\frac{1}{2^{k-i}}\right)\right)\zeta(k-i).$$

In a word the result is as follows.

$$\frac{\zeta(k)-\ln 2}{2}-\frac{1}{2k-2}+\frac{1}{2}\left(\left(1-\frac{1}{2^k}\right)\sum_{i=1}^{k-2}(-1)^i\left(2(1-2^{k-i})Q_{i-1}+C_{k-1}^i\right)-\frac{1}{2^k}\right)\zeta(k)$$

$$+\frac{1}{2}\sum_{i=1}^{k-2}\left(2^{i+1-k}(1-2^{k-i})Q_{i-1}+\sum_{j=i}^{k-2}(-1)^{j-i}C_j^i\left(2(1-2^{k-j})Q_{j-1}+C_{k-1}^j\right)\left(1-\frac{1}{2^{k-i}}\right)\right)\zeta(k-i),$$

where

$$P_i=\frac{\Delta^i f(1)}{(i+1)!},\ V(i-j,i)=\sum_{1\le u_1<u_2<\cdots<u_{i-j}\le i}(-1)^{i-j}u_1u_2\cdots u_{i-j},0\le i\le k-1,0\le j\le i,$$

$$V(0,t)=1(0\le t\le k-1),\ Q_i=\sum_{j=i}^{k-1}P_j\cdot V(j-i,j),0\le i\le k-1.$$

Our result in the case of k=3, is the same as that of (2).
The result obtained by *Mathematica* in case k=4 is identical with the above result.

ACKNOWLEDGMENT. We thank the referees for many useful comments and especially the referee whosuggested several improvements.

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