A proof for a generalization of the inequality from the 42nd International Mathematical Olympiad

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November 12, 2023

Abstract

In this paper, we present a proof for a generalization of the inequality from the 42nd International Mathematical Olympiad. The proved inequality relates to a sum involving square roots of fractions. It has various applications in mathematical analysis, optimization, or statistics. In the field of mathematical analysis, it can be used in the study of convergence. In terms of optimization, it may help establish bounds or relationships between the variables involved.

In this paper, we will present a proof for a generalization of the inequality from the 42nd International Mathematical Olympiad [IMO]. The original inequality corresponds to the case when $n = 3$. Specifically, we aim to prove

$$\sqrt{\frac{x_1^{n-1}}{x_1^{n-1} + (n^2-1)x_2x_3\cdots x_n}} + \sqrt{\frac{x_2^{n-1}}{x_2^{n-1} + (n^2-1)x_1x_3\cdots x_n}} + \cdots + \sqrt{\frac{x_n^{n-1}}{x_n^{n-1} + (n^2-1)x_1x_2\cdots x_{n-1}}} \geq 1$$

for all positive real numbers $x_i, i = 1, 2, \ldots, n$ with $n \geq 2$. For notational simplicity, equivalently, given an n-dimensional vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, we need to show $h(\mathbf{x}) = \sum_{i=1}^n h_i(\mathbf{x}) \geq 1$, in which $h_i(\mathbf{x})$ is defined by

$$h_i(\mathbf{x}) = \sqrt{\frac{x_i^{n-1}}{x_i^{n-1} + (n^2-1)x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_n}}.$$

We observe that every $h_i(\mathbf{x})$ is homogeneous. Therefore, without loss of generality, we can assume that $\sum_{i=1}^n x_i^p = 1$ with $p > 0$, without changing the value of the LHS of the inequality. To see this, suppose $\sum_{i=1}^n x_i^p = m$ where $m > 0$, let $\tilde{x}_i = x_i/m^{\frac{1}{p}}$ which ensures $\sum_{i=1}^n \tilde{x}_i^p = 1$. This yields a new vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$. With this, we can show

$h_i(\tilde{\mathbf{x}}) = h_i(\mathbf{x})$ as follows.

$$\begin{aligned}
h_i(\tilde{\mathbf{x}}) =& \sqrt{\frac{\left(\frac{x_i}{m^{1/p}}\right)^{n-1}}{\left(\frac{x_i}{m^{1/p}}\right)^{n-1} + (n^2-1)\cdot\frac{x_1}{m^{1/p}}\frac{x_2}{m^{1/p}}\cdots\frac{x_{i-1}}{m^{1/p}}\frac{x_{i+1}}{m^{1/p}}\cdots\frac{x_n}{m^{1/p}}}} \\
=& \sqrt{\frac{\frac{x_i^{n-1}}{m^{(n-1)/p}}}{\frac{x_i^{n-1}}{m^{(n-1)/p}} + \frac{(n^2-1)x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_n}{m^{(n-1)/p}}}} \\
=& \sqrt{\frac{x_i^{n-1}}{x_i^{n-1} + (n^2-1)x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_n}} = h_i(\mathbf{x})
\end{aligned}$$

which demonstrates that an arbitrary combination of x_i can be transformed into $\tilde{x}_i$ such that $\sum_{i=1}^n \tilde{x}_i = 1$ without changing the value of the left-hand side.

Consider the function $f(x) = \frac{1}{\sqrt{x}}$ with $x > 0$. Taking the first and second derivatives, we find that $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} < 0$ and $f''(x) = \frac{3}{4}x^{-\frac{5}{2}} > 0$. Hence, $f(x)$ is a monotonically decreasing and convex function. Using the homogeneity of the inequality, we can suppose $\sum_{i=1}^n x_i^{\frac{n-1}{2}} = 1$ and then apply Jensen's inequality as follows.

$$\begin{aligned}
&\sqrt{\frac{x_1^{n-1}}{x_1^{n-1} + (n^2-1)x_2x_3\cdots x_n}} + \sqrt{\frac{x_2^{n-1}}{x_2^{n-1} + (n^2-1)x_1x_3\cdots x_n}} \\
&+ \cdots + \sqrt{\frac{x_n^{n-1}}{x_n^{n-1} + (n^2-1)x_1x_2\cdots x_{n-1}}} \\
=& x_1^{\frac{n-1}{2}} f\left(x_1^{n-1} + (n^2-1)x_2x_3\cdots x_n\right) + x_2^{\frac{n-1}{2}} f\left(x_2^{n-1} + (n^2-1)x_1x_3\cdots x_n\right) \\
&+ \cdots + x_n^{\frac{n-1}{2}} f\left(x_n^{n-1} + (n^2-1)x_1x_2\cdots x_{n-1}\right) \\
\geq& f\Big(x_1^{\frac{n-1}{2}}\left(x_1^{n-1} + (n^2-1)x_2x_3\cdots x_n\right) + x_2^{\frac{n-1}{2}}\left(x_2^{n-1} + (n^2-1)x_1x_3\cdots x_n\right) \\
&+ \cdots + x_n^{\frac{n-1}{2}}\left(x_n^{n-1} + (n^2-1)x_1x_2\cdots x_{n-1}\right)\Big) \\
=& f\Big(x_1^{\frac{3(n-1)}{2}} + x_2^{\frac{3(n-1)}{2}} + \cdots + x_n^{\frac{3(n-1)}{2}} + (n^2-1)(x_1^{\frac{n-1}{2}}x_2x_3\cdots x_n \\
&+ x_1x_2^{\frac{n-1}{2}}x_2\cdots x_n + x_1x_2\cdots x_{n-1}x_n^{\frac{n-1}{2}})\Big).
\end{aligned}$$

Since $f(x)$ is decreasing and $f((\sum_{i=1}^n x_i^{\frac{n-1}{2}})^3) = f(1) = 1$, proving that the last line is greater than or equal to 1 is equivalent to showing the following inequality

$$\begin{aligned}
\left(\sum_{i=1}^n x_i^{\frac{n-1}{2}}\right)^3 \geq& x_1^{\frac{3(n-1)}{2}} + x_2^{\frac{3(n-1)}{2}} + \cdots + x_n^{\frac{3(n-1)}{2}} + (n^2-1)(x_1^{\frac{n-1}{2}}x_2x_3\cdots x_n \\
&+ x_1x_2^{\frac{n-1}{2}}x_2\cdots x_n + x_1x_2\cdots x_{n-1}x_n^{\frac{n-1}{2}}).
\end{aligned}$$

After expanding the LHS and subtracting the terms containing $x_i^{\frac{3(n-1)}{2}}$ from both sides, there are $n^3 - n = n(n^2-1)$ terms remaining on the LHS, which can be divided into n

groups with each group containing n^2-1 terms, as follows.

$$\sum_{i=1}^{n}\left(x_i^{n-1}\sum_{j\neq i}^{n}x_j^{\frac{n-1}{2}}+x_i^{\frac{n-1}{2}}\sum_{j\neq i}^{n}x_j^{n-1}+\sum_{j=1}^{n-1}x_{G_{i,j}}^{n-1}x_{G_{i,j+1}}^{\frac{n-1}{2}}+x_i^{\frac{n-1}{2}}\sum_{j\neq i}^{n}x_j^{\frac{n-1}{2}}\sum_{k\neq i,j}^{n}x_k^{\frac{n-1}{2}}\right)$$
$$\geq(n^2-1)\sum_{i=1}^{n}x_1x_2\cdots x_{i-1}x_i^{\frac{n-1}{2}}x_{i+1}\cdots x_n \tag{1}$$

where $G_i=(1,2,\ldots,i-1,i+1,\ldots,n)$ denotes a cyclic ordered sequence, in which $G_{i,j}=j$ for $0<j<i$, $G_{i,j}=j+1$ for $i\leq j<n$, otherwise $G_{i,n}=G_{i,1}$. In (1), each summand of the outermost summation consists of n^2-1 terms, we apply the AM-GM inequality to these n^2-1 terms for the n summands on the LHS of (1). After simple algebraic operations, we can finally get the right-hand side (RHS) of (1). For better presentation, we show the derivation for the first group and the other $n-1$ groups follow the same logic.

$$\begin{aligned}
&x_1^{n-1}x_2^{\frac{n-1}{2}}+x_1^{\frac{n-1}{2}}x_2^{n-1}+x_2^{n-1}x_3^{\frac{n-1}{2}}+x_1^{n-1}x_3^{\frac{n-1}{2}}+x_1^{\frac{n-1}{2}}x_3^{n-1}\\
&+x_3^{n-1}x_4^{\frac{n-1}{2}}+\cdots+x_1^{n-1}x_n^{\frac{n-1}{2}}+x_1^{\frac{n-1}{2}}x_n^{n-1}+x_n^{n-1}x_2^{\frac{n-1}{2}}\\
&+x_1^{\frac{n-1}{2}}x_2^{\frac{n-1}{2}}x_3^{\frac{n-1}{2}}+x_1^{\frac{n-1}{2}}x_3^{\frac{n-1}{2}}x_2^{\frac{n-1}{2}}+x_1^{\frac{n-1}{2}}x_2^{\frac{n-1}{2}}x_4^{\frac{n-1}{2}}\\
&+x_1^{\frac{n-1}{2}}x_4^{\frac{n-1}{2}}x_2^{\frac{n-1}{2}}+\cdots+x_1^{\frac{n-1}{2}}x_{n-1}^{\frac{n-1}{2}}x_n^{\frac{n-1}{2}}+x_1^{\frac{n-1}{2}}x_n^{\frac{n-1}{2}}x_{n-1}^{\frac{n-1}{2}}\\
\geq&(n^2-1)\sqrt[n^2-1]{x_1^{\frac{3(n-1)^2}{2}+(n-1)(n-2)\cdot\frac{n-1}{2}}x_2^{3(n-1)+2(n-2)\cdot\frac{n-1}{2}}\cdots x_n^{3(n-1)+2(n-2)\cdot\frac{n-1}{2}}}\\
=&(n^2-1)\sqrt[n^2-1]{x_1^{\frac{(n-1)^2(n+1)}{2}}x_2^{n^2-1}\cdots x_n^{n^2-1}}\\
=&(n^2-1)x_1^{\frac{(n-1)^2(n+1)}{2(n^2-1)}}x_2\cdots x_n\\
\geq&(n^2-1)x_1^{\frac{n-1}{2}}x_2x_3\cdots x_n
\end{aligned}$$

which completes the proof.

References

[IMO] The official website of IMO. *Problems in the 42nd International Mathematical Olympiad*. URL: `https://www.imo-official.org/year_info.aspx?year=2001`. (accessed: 09.27.2023).