Existing a prime in interval n^2 and $n^2 + \epsilon n$

Hashem sazegar
School of Mathematics, Azad university branch of Mashhad, Mashhad, Iran
E-mail:h.sazegar@gmail.com

Abstract

Oppermance' conjecture states that there is a prime number between n^2 and n^2+n for every positive integer n,first we show that , All integer numbers between x^2 and $x^2+\epsilon x$ can be written as $x^2+i > 4p$ that $1 \leq i \leq \epsilon x$ and $p=(x-m-2)^2+j$ in which j is a number in intervals $1 \leq j \leq \epsilon(x-m-2)$, and then we prove generalization of Oppermance' conjecture i.e there is a prime number in interval n^2 and $n^2+\epsilon n$ such that $0 < \epsilon \leq 1$.

Keywords: Bertrand-chebyshev theorem,landu's problems,Goldbach's conjecture,twin prime,Legendre's conjecture,oppermanc's conjecture

1. Introduction

Bertrand's postulate state for every positive integer n,there is always at least one prime p,such that $n < p < 2n$. This was first proved by chebyshev in 1850 which is why postulate is also called the Bertrand-chebyshev theorem.

Legendre's conjecture states that there is a prime between n^2 and $(n+1)^2$ for every positive integer n,which is one of the four Landu's problems.The rest of these four basic problems are:
(i) Twin prime conjecture :there are infinitely many primes p such that $p+2$ is a prime.
(ii) Goldbach's conjecture: every even integer $n > 2$ can be written as the sum of two primes.
(iii) Are there infinitely many primes p such that $p-1$ is a perfect square?
problems(i),(ii),(iii) are open till date.
Legendre's conjecture is proved(in[8])

Theorem: there is at least a prime between n^2 and $n^2+\epsilon n$,for every positive integer n such that $0 < \epsilon \leq 1$ is constant arbitrary number.
we prove it by induction that if there is at least a prime between all $(x-1)^2$ and $(x-1)^2+\epsilon(x-1)$, then there is a prime between x^2 and $x^2+\epsilon x$.
To proceed to this proof,firstly we use the following Lemmas:

2.Lemmas :
In this section ,we present several lemmas which are used in the proof of our main theorem.
*Lemma 2.1:*for a large x,All integer numbers between x^2 and $x^2+\epsilon x$ can be written as $x^2+i > 4p$ that $1 \leq i \leq \epsilon x$ and $p=(x-m-2)^2+j$ in which j is a number in intervals $1 \leq j \leq \epsilon(x-m-2)$, we Assume that $m=x/2$ if x is even and $m=(x+1)/2$ if x is odd and p is prime.
Proof:By induction there is a prime in intervals k^2 and $k^2+\epsilon k$ that $k=a,(a+1),...,(x-1)$ (for example if $\epsilon=1$ so a=2),since $m=x/2$,if x is even or $m=(x+1)/2$ if x is odd,so always $p > (x-4)^2/4$,for a large x,hence $x^2+i > 4p$,that $1 \leq i \leq \epsilon x$
Lemma 2.2:If l to be the number of $3 \leq q < x$,(q is prime) are in equation

$x^2+i=tq$ (is odd)that $1 \leq i \leq \epsilon x$ so $l < \frac{\epsilon x}{2q}$,for some $3 \leq q < x$
Proof: if $q \geq 3$,we put $i = j + 2ql$ $(j \geq 1)$,so $j + 2ql \leq \epsilon x$,then $l < \frac{\epsilon x}{2q}$,in this case l is the number of $q \geq 3$ that $x^2+i=tq$ is odd.
Lemma 2.3:If f to be the number of $N > x$ are in $x^2+i=qN$ that these numbers are odd and $1 \leq i \leq \epsilon x$
So:
for$g=3$

$$f \leq \frac{\epsilon x}{2\times 3} \ (1)$$

for $g=5$

$$f \leq \frac{\epsilon x(1-1/3)}{2\times 5} \ (2)$$

for $g=7$

$$f \leq \frac{\epsilon x(1-1/3-1/5)}{2\times 7} (3)$$

.
.
.

we continue this method to reach $1-1/3-1/5-...1/29 = almost 0$ (4)
Proof:If $N > x$ and $x^2+i=qN$ to be odd ,since $1 \leq i \leq \epsilon x$
so $x^2/q \leq N \leq (x^2+\epsilon x)/q$, in Which $3 \leq q < x$ are primes.Since the distance of between two odd numbers should be 2,so If $q=3$,the number of such N odd number is:

$$f \leq \frac{\epsilon x}{2\times 3}$$

but since$N > x$,only one $N > x$ could be in $x^2+i=qN$,so for $q=5$,

$$f \leq \frac{\epsilon x(1-1/3)}{2\times 5}$$

For $q=7$,

$$f \leq \frac{\epsilon x(1-1/3-1/5)}{2\times 7}$$

we continue this method to reach, $1-1/3-1/5-...1/29 = almost 0$

NOTE:we have only $\epsilon x/2$ composite odd numbers ,since we say about $N > x$(this is new idea)not old idea i.e $q < x$,for $q=3$,we have $\epsilon x/2/3$ such $N > x$,since we have only one such $N > x$, exist ,if we have two such primes i.e $N_1N_2q > x^2$ and this is contradiction ,so for $q=5$ $\epsilon x/2$ numbers changed to $\epsilon x/2 - \epsilon x/2/3$,for $q=7$ these numbers changed to $(\epsilon x/2)-(\epsilon x/2)/3-(\epsilon x/2)/5$we continue this method to reach $(\epsilon x/2)-(\epsilon x/2)/3-(\epsilon x/2)/5-....-(\epsilon x/2)/29 =$ *almost* 0 ,and also we have not same N for different q,for example for $q=3$,$x^2/3 \leq N \leq (x^2+\epsilon x)/3$,for $q=5$,$x^2/5 \leq N \leq (x^2+\epsilon x)/5$,we can reach to contradiction notice that we consider numbers between n^2 and $n^2 + \epsilon n$

3. The proof of main theorem

Theorem: There is at least a prime between x^2 and $x^2 + \epsilon x$

Proof:Let we have at least a prime in intervals k^2 and $k^2 + \epsilon k$ that $k = a, (a + 1), ..., (x - 1)$ .By induction ,we prove that,we have a prime between x^2 and $x^2+\epsilon x$.Assume that this is not true,so we can write $x^2+i = lq$,i.e all numbers in intervalx^2 and $x^2 + \epsilon x$ are not primes.Since $1 \leq i \leq \epsilon x$ so according to$(G.H.Hardy, E.M.Wright, Oxford, 1964)$there is a prime factor like q that for any composite number in n^2 and $n^2 + \epsilon n$ this interval $q \leq \sqrt{x^2 + \epsilon x} \leq x + 1$

now we use the above results to reach to a contradiction,notice that we use odd statements so:

$$(x^2 + 1or2)...(x^2 + ([\epsilon x] - 1)or[\epsilon x]) > (4p)^{[\frac{\epsilon x}{2}]} \ (5)$$

According to lemmas 2.2 *and* 2.3 ,we have:

$$(x^2+1or2)...(x^2+[\epsilon x]-1)or[\epsilon x]) < 3^{\frac{\epsilon x}{2\times 3}} \times \ldots \times 29^{\frac{\epsilon x}{2\times 29}} \times \frac{x^2}{3}^{\frac{\epsilon x}{2\times 3}} \times \frac{x^2}{5}^{\frac{(\epsilon x)(1-1/3)}{2\times 5}} \times \cdots \frac{x^2}{29}^{\frac{\epsilon x(1-1/3-\ldots-1/29)}{2\times 31}} \quad (6)$$

We continue to reach $1 - 1/3 - 1/5 - ... - 1/29 = almost 0$ .Hence we have:

$$\frac{\epsilon x-2}{2} log(4p) < log(x^2 + 1or2) + ... + log(x^2 + [\epsilon x] - 1)or[\epsilon x]) <$$
$$(\epsilon x/2) \sum_{3 \leq q < w} \frac{logq}{q} + (\epsilon x/2)(1/3+(1-1/3)/5+(1-1/3-1/5)/7+...+0)logx^2 (7)$$

So by refer to [3],$\sum_{3 \leq q < w} \frac{logq}{q} < logw + c$,that$c$ is positive constant number,so:

$$\frac{\epsilon x-2}{2} log(4p) < (\epsilon x/2)logw + (\epsilon x/2)c + 0.8(\epsilon x/2)logx^2 \ (8)$$

Then for a large x ,$\frac{\epsilon x-2}{2} log(4p) < 1.7 \frac{\epsilon x}{2} logx$,but since$\frac{\epsilon x-2}{2} > 0.94 \frac{\epsilon x}{2}$ for a large x so $p < x^{1.8}/4$ and this is a contradiction ,because by lemma 2.1,$p > (x - 4)^2/4$.

References:

[1]M.EI Bachraoui. prime in the interval$[2n, 3n]$. International journal of contemporary Mathematical sciences,1 (3):617-621,2006.

[2]P.Erdos. Beweis eines satzes von tschebyschef. Acta Litt.Univ.sci,Szeged,Sect. Math.,5:194-198,1932.

[3]G. H. Hardy , E .M . Wright , An introduction to the theory of numbers .Oxford 1964.

[4] Unsolved problem in number Theory , Richard K. Guy, Amazon.com.

[5] P.Erdos and J.suranyi. Topics in the theory of numbers.undergraduate texts in mathematics. Springer Verlag,2003.

[6]S. Ramanujan.A proof of Bertrand, S. Postulate. journal of the Indian Mathematical society 11:181-182,1919.

[7]Shiva kintali, A Generalization of Erdos's Proof of Bertrand-Chebyshev Theorem , http://www.cs.princeton.edu/ kintali, 2008.

[8]H.sazegar,A Method for solving Legendr's conjcture,journal of Mathematics Research,Feb.2012