Fictitious Currents as a Source of Electromagnetic Field

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Abstract

In this paper we introduce the idea of electric fictitious currents for the electromagnetic field. Electric fictitious currents are currents that arise in electrodynamics when we change the topology of space. We show, with a specific example, how fictitious currents may be the source of magnetic moment and charge of a singularity.

Key Words: gravity, electrodynamics, gauge theory.

1 Introduction

The Standard Model of particle physics is a very successful theory describing three out of the four known forces of nature. Its final formulation relay heavily on the use of gauge fields. In gauge theories the Lagrangian of the system (i.e. its dynamics) does not change under local transformations acting in a simply connected region of space-time.

However in the standard model, particle are point objects with no dimension being, as a matter of fact, space-time singularities with fields around them having sometimes infinite values. For example, the classical version of electric field around an electron goes to infinity as $1/r^2$, and in the Standard Models we start from the classical Lagrangian before quantizing.

In this paper we study what happens to a gauge field when we introduce a singularity in space such that the space is not simply connected any more. The major results is that, depending on the topology of the singularity, fictitious currents may arise as a manifestation of the inertia of the system in changing topology. In the example we studied, dealing only with the $U(1)$ symmetry of the Standard Model, these currents may be seen as sources for charge and magnetic moment of particles.

In section 2 we derive fictitious currents generated by a topological singularity in space. The reader, that does not want to go through the math, can find a simplified version of the content of this section in [3].

In sections 3, 4 and 5 we show how magnetic moment and charge of a particle can arise as a conseguence of ficticious currents.

In the sections 6 and 7 we give additional thoughts and conclusions.

2 Fictitious Currents

We start from the Lagrangian density of the electromagnetic field in units where $\mu_0 = 1$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_s^\nu A_\nu \tag{1}$$

where J_s^ν is the source four-current vector, $F_{\mu\nu}$ equal to:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

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is the electromagnetic tensor and A_μ is the four-potential. Moreover, we know that $F_{\mu\nu}$ is a gauge field and that its Lagrangian is invariant with respect to the symmetry:

$$A_\mu \to A_\mu + \partial_\mu \tilde{\theta} \tag{3}$$

where $\tilde{\theta}(x^\mu)$ is any continuous function in a simply connected space-time region Ω.

In the general case, the Lagrangian density $\mathcal{L}(A, \partial A)$, given by Eq. (1), depends on both the vector potential and its derivatives. We will consider now the case of the free electromagnetic field (i.e. $J^\nu = 0$) in which the Lagrangian density $\mathcal{L}(\partial A)$ depends only on the derivatives and it can be written as (see Appendix A.1):

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \tag{4}$$

Let $\xi_\mu = \partial_\mu \tilde{\theta}$ be the gradient of a continuous function in Ω. In this section we will apply the symmetry $A_\mu \to A_\mu + \xi_\mu$ in order to see what happens to the Lagrangian density.

For the first term in (4) we have;

$$\partial_\mu(A_\nu + \xi_\nu)\partial^\mu(A^\nu + \xi^\nu) = \partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A_\nu \partial^\mu \xi^\nu + \partial_\mu \xi_\nu \partial^\mu A^\nu + \overbrace{\partial_\mu \xi_\nu \partial^\mu \xi^\nu}^{\text{not needed}} \tag{5}$$

where the last term is not needed and can be omitted because we are interested in the equation of motion and that term does not depend on A_μ and therefore has not effect on the variation of the action with respect to the fields.

For the second term we have:

$$\partial_\nu(A_\mu + \xi_\mu)\partial^\mu(A^\nu + \xi^\nu) = \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\mu \xi^\nu + \partial_\nu \xi_\mu \partial^\mu A^\nu + \overbrace{\partial_\nu \xi_\mu \partial^\mu \xi^\nu}^{\text{not needed}} \tag{6}$$

where the last term once again can be omitted if we are interested in the equation of motion. Putting the two equation above back together, swapping some terms and rearranging the names of dummy indices of the third term in parenthesis below, we have:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A_\nu \partial^\mu \xi^\nu + \partial^\mu A^\nu \partial_\mu \xi_\nu - \partial_\mu A_\nu \partial^\nu \xi^\mu - \partial^\mu A^\nu \partial_\nu \xi_\mu) \tag{7}$$

Applying the Leibniz rule (i.e. $f'g' = (fg')' - fg''$) to the terms in parenthesis above, we have:

$$\partial_\alpha A_\beta \partial^\gamma \xi^\delta = \overbrace{\partial_\alpha(A_\beta \partial^\gamma \xi^\delta)}^{\text{not needed}} - A_\beta \partial_\alpha \partial^\gamma \xi^\delta \tag{8}$$

where in this case the first term is not needed, if we are interested in the equation of motion, because it is a divergence and therefore it depends only on the value of the tensors on the boundary of Ω and has no effect on the variation of the action with respect to fields. Eq. (7) becomes:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(-A_\nu \partial_\mu \partial^\mu \xi^\nu - A^\nu \partial^\mu \partial_\mu \xi_\nu + A_\nu \partial_\mu \partial^\nu \xi^\mu + A^\nu \partial^\mu \partial_\nu \xi_\mu) \tag{9}$$

If we are in nice flat Minkowski space we can raise and lower indices at will also on the derivative symbols, we have:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(-A_\nu \partial_\mu \partial^\mu \xi^\nu - A_\nu \partial_\mu \partial^\mu \xi^\nu + A_\nu \partial_\mu \partial^\nu \xi^\mu + A_\nu \partial_\mu \partial^\nu \xi^\mu) \tag{10}$$

and eventually:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\nu \partial_\mu(\partial^\nu \xi^\mu - \partial^\mu \xi^\nu) \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\nu A_\nu \end{aligned} \tag{11}$$

where $J^\mu = \partial_\mu(\partial^\nu \xi^\mu - \partial^\mu \xi^\nu)$ is a new term now present in the Lagrangian density due to the fact that we have changed the gauge field $\tilde{\theta}$. However, if $\xi^\nu = \partial^\mu \tilde{\theta}$, we have:

$$J^\nu = \partial_\mu(\partial^\nu \xi^\mu - \partial^\mu \xi^\nu) = (\partial_\mu \partial^\mu \partial^\nu - \partial_\mu \partial^\mu \partial^\nu)\tilde{\theta} = 0 \tag{12}$$

Note that:

$$\mathcal{L}(\partial A) \neq \mathcal{L}(\partial(A + \xi)) \tag{13}$$

only because we have removed terms from the Lagrangian. Otherwise, the two Lagrangians would have been identical since we are applying a symmetry. However, the way we have removed the terms has left the equation of motion unchanged, and in fact, if Ω is simply connected we have $J^\mu = 0$ and this restores the correctness of the equation of motion and leaves the field unchanged.

However, if Ω is not simply connected, then the currents J^ν may be different from zero and act as sources for the fields. We will call J^ν **Fictitious Currents** (Pseudo Currents). Note that these are electric ficticious currents not to be confused with the magnetic fictitious currents sometimes used in computational electrodynamics as a trick for solving complex problems.

The reason why we call J^ν fictitious, it is due to an analogy with discrete systems (see Appendix A.2). For discrete systems, if we act on a symmetry (e.g. shift in space) while the system is evolving, this will result in fictitious forces. For continuous systems, if we act on a symmetry, this will result in fictitious currents.

The analogy is not perfect though. For mechanical systems fictitious forces appear when we have a change of symmetry during the evolution of the system. For the electromagnetic field, fictitious currents appear when we have a change a of the gauge symmetry fields in conjunction with a specific space topology.

With abuse of terminology, we may say that fictitious currents are due to the inertia of the system in changing space topology.

3 Magnetic Moment of One Half Spin Particles

Now that we have defined fictitious currents, we want to find an example where the theory may be used. Let us consider the Lagrangian density of quantum electrodynamics:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{14}$$

and suppose that we have no particles and the electromagnetic field is zero everywhere (at least from a classical field theory point of view). We know that we have the following symmetry:

$$\psi \to \psi e^{i\tilde{\theta}(x^\mu)} \; ; \;\; A \to A + \partial^\mu \tilde{\theta} \;\Rightarrow\; \Delta\mathcal{L} = 0 \tag{15}$$

where A is the four-vector potential.

Suppose now we hit the field with the creation operator and we create a particle. Before the particle is present, the gauge field $\tilde{\theta}$ is a continuous function, and since we can choose, we choose it to be zero everywhere. Suppose finally that in the process, $\tilde{\theta}$ goes from being continuous everywhere to be discontinuous on a line segment $\overline{AB}$ lying on the z axis with $A = (0, 0, L/2)$ and $B = 0, 0, -L/2$. The topology of space has changed, and now $\tilde{\theta}$ has to adapt to it and it cannot be zero everywhere. Now $\tilde{\theta}$ will be described by the following function:

$$\tilde{\theta}(x^\mu) = \theta(x, y, z)u(t) \tag{16}$$

where $u(t)$ is the Heaviside unitary step function and θ is a function depending from special coordinates but independent from time.

For θ we give the following boundary conditions expressed in cylindrical coordinates (r, ϕ, z):

$$\begin{cases} \lim\limits_{r\to\infty} e^{i\theta(r,\phi,z)} = e^{i\theta_\infty} \\ \\ \lim\limits_{r\to 0} e^{i\theta(r,\phi,z)} = e^{i(\pm\alpha\phi + \theta_0)} \quad \text{for} \quad -\frac{L}{2} < z < \frac{L}{2} \end{cases} \tag{17}$$

where α, θ_0 and θ_∞ are constants and you can have the plus or the minus sign before α. The second condition above is the one that defines the discontinuity and, in words, it means that going around the line segment discontinuity, the phase of $e^{i\theta}$ goes around the circle α times, by the time we go around the z axis once. In particular, on any line s parallel to the (x, y) plane and crossing $\overline{AB}$ we want the function θ to have a jump of $\alpha\pi$ (i.e. $e^{i\theta}$ to change phase by $\alpha\pi$).

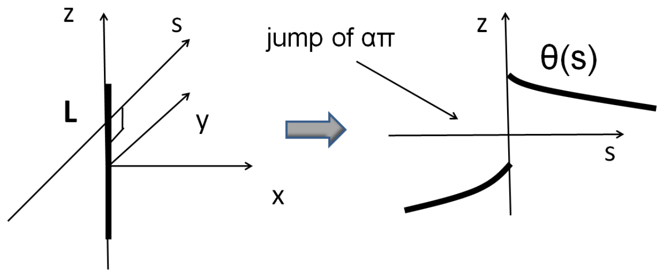

Figure 1: Boundary Conditions in the Singularity

Although we do not need to know θ in space, we want to spend a few words on the way it may look like. We assume that two close points want to have the same phase, which means that there is some energy associated to difference in phase for nearby points. As usual with these problems of finding a minimum for energy in space, we end up with a Laplace equation. We propose the following equation for θ:

$$\nabla^2\theta = 0 \tag{18}$$

that together with the boundary condition allows to evaluate θ in space.

Whatever equation we use for θ, its important to note that θ will be a multivalued function shaped like an helicoid and spiralling along the z axis.

Now let $\xi^\nu = \partial^\nu\theta$. Given the above boundary conditions, we have that $\xi^0 = 0$ and $\xi^3 = 0$ and therefore the J^μ lay on the (x, y) plane. From Eq. (12) and after the due calculations (see Appendix A.3) we have:

$$J^\nu = \begin{cases} J^0 &= 0 \\ J^1 &= \partial_2(\partial^1\xi^2 - \partial^2\xi^1) \\ J^2 &= \partial_1(\partial^2\xi^1 - \partial^1\xi^2) \\ J^3 &= 0 \end{cases} = \begin{cases} J^0 &= 0 \\ J^1 &= \partial_y(\nabla\times\hat{\xi})_z \\ J^2 &= -\partial_x(\nabla\times\hat{\xi})_z \\ J^3 &= 0 \end{cases} \tag{19}$$

where $\hat{\xi}$ is the three-vector (ξ^1, ξ^2, ξ^3), $(\cdot)_z$ means component along the z axis, and $\nabla\times\hat{\xi}$ has clearly only the z component while the x and y components vanish.

If we integrate $\hat{\xi}$ on any loop on the (x, y) plane not containing the origin, we get always a vanishing integral because $\nabla\times\nabla\theta = 0$. However, if we integrate $\hat{\xi}$ on any loop on the (x, y) plane containing the origin, given Eq. (17) we get $2\pi\alpha$. This is because, given the expression of the gradient in the cylindrical coordinate (r, ϕ, z) and picking the plus sign before α, we have;

$$\lim_{r\to 0}\hat{\xi} = \lim_{r\to 0}\nabla\theta = \frac{1}{r}\alpha\hat{i}_\phi \tag{20}$$

This means we have a discrete curl in the origin or, another way to say it, we have an impulsive curl in the origin described by a Dirac delta function of amplitude $2\pi\alpha$:

$$(\nabla\times\hat{\xi})_z = \frac{2\pi\alpha}{\mu_0}\delta(x, y) \quad [A][m] \tag{21}$$

In the above equation we have taken into account that $\hat{\xi}$ has the same units of A^μ ($[kg][m][s^{-2}][A^{-1}]$), the curl adds an unit of $[m]$ and we have put back in the equation μ_0 ($[kg][m][s^{-2}][A^{-2}]$) that was previously set to 1. Applying this reasoning to the expressions of J^1 and J^2, we have:

$$J^\nu = \begin{cases} J^1 &= \partial_y(\nabla \times \hat{\xi})_z &= \frac{\alpha}{\mu_0}\partial_y \delta(x,y) &= \frac{2\pi\alpha}{\mu_0}\delta(x)\delta'(y) \quad [A] \\ J^2 &= -\partial_x(\nabla \times \hat{\xi})_z &= -\frac{\alpha}{\mu_0}\partial_x \delta(x,y) &= -\frac{2\pi\alpha}{\mu_0}\delta(y)\delta'(x) \quad [A] \end{cases} \tag{22}$$

Since there is nothing special about the x and y axis, the above is true for any axis on to the (x, y) plane and crossing the origin. The derivative of the Dirac delta functions of amplitude $\frac{2\pi\alpha}{\mu_0}$ represents two opposite currents on opposite sides of the origin end it can be described by a current circulating around the z axis along a circle of radius $d \to 0$ and having amplitude:

$$J_m = \frac{1}{2d}\frac{2\pi\alpha}{\mu_0} \quad [A] \tag{23}$$

This current is generating a magnetic moment. Note that the above equation gives a new and nice physical meaning to μ_0. The reason why we need to divide by $2d$ is due to the definition of d delta derivative (see figure below).

To evaluate the magnetic moment M of our singularity, we can use the magnetic moment of a single coil element of a solenoid. To do that, we have to abandon our line segment model $\overline{AB}$ and use instead a cylinder of height L and radius d. The current can be distributed inside the cylinder in many ways. Among others, the current may be concentrated on the surface of the cylinder or uniformly distributed along the radius.

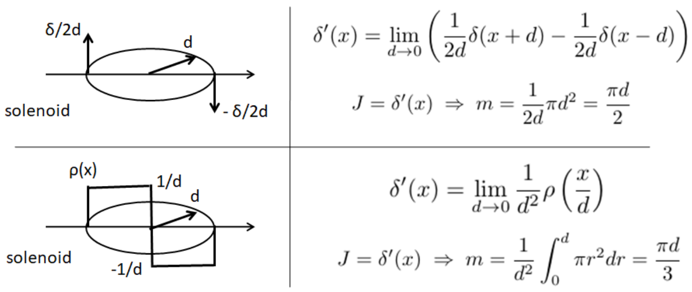

Figure 2: Magnetic Moment per Unit Length

We choose to use a current concentrated on the external surface as given by Eq. (23). The magnetic moment m per unit length of the cylinder in the case of a current J_m concentrated on the surface is $m = J_m S$ where S is the surface of the solenoid. We have:

$$m = J_m \pi d^2 = \frac{1}{2d}\frac{2\pi\alpha}{\mu_0}\pi d^2 = \frac{\pi^2 \alpha d}{\mu_0} \quad [A][m] \tag{24}$$

The total magnetic moment M of the cylinder of length L is therefore:

$$M = mL = \frac{\pi^2 \alpha d L}{\mu_0} \quad [A][m^2] \tag{25}$$

Now we need to give a value to α and we will use the value $\alpha = 3$. This choice may seem odd. However, we will justify it in a later paragraph. Assuming the radius of the particle $R = d \approx \frac{L}{2}$ (round particle) we may evaluate the radius of the particle from M. For example, taking the value of the magnetic moment[1] μ_e of the electron and the value of the permeability[2] μ_0, we have:

$$R_M = \sqrt{\frac{\mu_0 \mu_e}{2\pi^2\alpha}} = 1.4 \times 10^{-15} \quad [m] \tag{26}$$

Which is not too far from the value of the classical electron radius[3] r_e. In Appendix A.4 we have evaluated the same radius with a more refined model.

[1] $\mu_e = -9.2847647043(28) \times 10^{-24}$ $[A][m^2]$
[2] $\mu_0 = 4\pi \times 10^{-7}$ $[kg][m][s^{-2}][A^{-2}]$
[3] $r_e = 2.8179403227(19) \times 10^{-15}$ $[m]$

4 Charge of One Half Spin Particle

Now we go back to see in more detail what happens during the transition at time $t = 0$ in which we go from a zero gauge to our gauge $\theta(x, y)$:

$$\tilde{\theta}(t) = \theta(x, y)u(t) \tag{27}$$

From Eq. (12) and after the due calculations (see Appendix A.3) we have:

$$\begin{aligned} J^0 &= \partial_1(\partial^0\xi^1 - \partial^1\xi^0) + \partial_2(\partial^0\xi^2 - \partial^2\xi^0) \\ &= \partial_x(\nabla \times \tilde{\xi})_y - \partial_y(\nabla \times \tilde{\xi})_x \end{aligned} \tag{28}$$

Where $\tilde{\xi}$ is the three-vector (ξ^1, ξ^2, ξ^0) and $(\cdot)_i$ means component along the i axis.

From the above equation we see that J^0 is always zero even during the transition and therefore it cannot be used to determine the variation of charge in space. However, to evaluate the quantity of charge generate in the process we can just integrate the density of current on the surface of our cylinder. We know that at time zero the current that generates the magnetic moment goes from zero to J_m (see Eq. (23)). To get the charge we need to divide this current by c to get the correct units (as it happens for the zero component of the currents four-vector). We have:

$$J_m = \frac{1}{2d}\frac{2\pi\alpha}{c\mu_0} \quad [A][s][m^{-1}] \tag{29}$$

The total charge generated in the singularity is therefore the above current multiplied for the lateral surface of the cylinder:

$$Q = \frac{1}{2d}\frac{2\pi\alpha}{c\mu_0} \times 2\pi dL = \frac{2\pi^2\alpha L}{c\mu_0} \quad [C] \tag{30}$$

Once again, assuming the radius of the particle $R \approx \frac{L}{2}$ (round particle), given the speed of light[4] c and using the value of the elementary charge[5] e of the electron, we can evaluate the radius of our particle:

$$R_Q = \frac{c\mu_0 e}{4\pi^2\alpha} = 5.0 \times 10^{-18} \quad [m] \tag{31}$$

Which this time a bit too far from the expected value of the classical electron radius. In Appendix A.4 we have evaluated the same radius with a more refined model.

5 Choosing a Value for α

In this paper we have shown that:

$$\text{Ficticious Currents Momentum} \Leftrightarrow \text{Topological Singularity} \tag{32}$$

However, no much it can be said on the nature of the topological singularity that may match or model the behaviour of a particle. The example of the line segment singularity is just a study example that we have used to develop our reasoning. However, if we stick to this kind of singularity and to the relevant cylindrical model needed to evaluate the momentum and charge, we need to explain why we used a value of $\alpha = 3$.

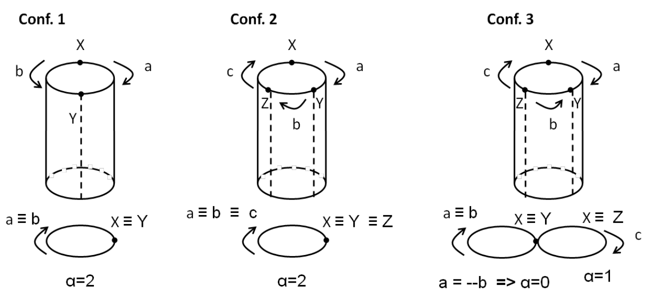

Figure 3: Space Deficiency Model

[4] $c = 2.99792458 \times 10^8 \quad [m][s]$

[5] $e = 1.602176634 \times 10^{-19} \quad [C]$

To make a topological singularity out of a cylinder, we have many possibilities. One way is to split the lateral surface in rectangles and identify them. The first possibility that come to mind is to split the surface in two rectangle (a and b) and identify the two rectangle with the orientation shown in the figure above for Configuration 1. In this case we get $\alpha = 2$ because we need to go around twice while going around the original cylinder once. If we changed the orientation of one of the lateral surfaces before identifying them, we would get $\alpha = 0$ because the would need to go around twice as before but this time walking in opposite directions and the current of the first loop would be cancelled by the current of the second loop.

If we apply the same reasoning to configuration 2 and 3, we find $\alpha = 3$ and $\alpha = 1$. In configuration 3 we go around 3 times one of which walking in the opposite direction and therefore the current of two loops cancel each other. Since the charge of the particle, keeping the same radius for the cylinder, is propositional to α, if we assign to the charge $Q_2 = e$, the elementary charge of an electron, we can summarise what we have found as follows:

$$\begin{array}{lll} \text{Configuration 1} & Q_1 \propto 2\alpha = \frac{2}{3}e & \text{Candidate for Quarks} \\ \text{Configuration 2} & Q_2 \propto 3\alpha = e & \text{Candidate for Electrons} \\ \text{Configuration 3} & Q_3 \propto \alpha = \frac{1}{3}e & \text{Candidate for Quarks} \end{array} \quad (33)$$

These are just examples. For sure, further research is needed for trying to match topological singularities with the characteristics of particles (if this is even possible) taking into account the 3 symmetries of standard model.

6 Space Deficiency Model

In [2] we have shown that if we model space as an elastic material, a deficiency in the material (i.e. in space) is equivalent to gravity. This is because, if we remove a ball of material making a hole in it and we identify the boundary of the hole to a point, the material will stretch and the strain field is equivalent to gravitational field. Moreover, two deficiencies in the material will experience an attraction force to each other proportional to $1/r^2$ where r is the distance between the two space deficiencies.

Since space should be conserved, we wonder how a deficiency may be created in space. A possibility is that when a particle is created in a point P, the configuration of space changes from flat Euclidean space to a 3-dimensional manifold attached to the point P by means of a connected sum.

We will illustrate this with a 2-dimensional example.

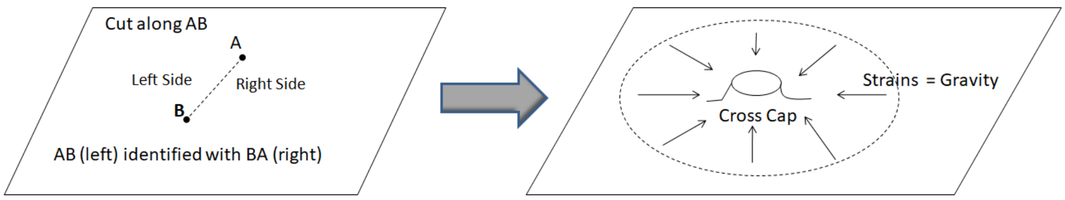

Figure 4: Space Deficiency Model

Given a sheet of elastic material representing space, if we cut along a line segment between two points A and B and we identify the two sides of the cut with opposite orientations, we get a cross cap (i.e. a real projective plane) attached to the sheet by means of a direct sum.

The sheet will pull the cross cap which will shrink till the bending forces inside the cap will balance the pulling forces of the sheet. At the equilibrium, the cross cap will protrude from the sheet and since the quantity of elastic material is conserved in the process, this will be equivalent to a space deficiency.

The sheet will be stretched around the cross cap and the strain field will be equivalent to a gravitation field (see [2]). Note that, if the sheet represents space, fields will change phase when crossing the line segment as in the example we gave in the previous sections.

7 Conclusions

In this paper, we have shown that a sufficient condition to have magnetic moment and charge in a point in space (i.e. particle) is to introduce a topological singularity designed to twist the fields in a specific way. Although the paper has been dealing with the $U(1)$ symmetry only, the very same approach may be used to explore what happens with the other two symmetries of the standard Model. We believe that it is worth to further research in order to trying to match particle characteristics to topological characteristics of space singularity (i.e. 3D compact manifolds connected by direct sum to space) with respect to the three symmetries of nature, in an effort to make a one-to-one correspondence between particles and manifolds (or topological singularities).

Although we know that this approach is very unlikely to be a theory that fully describe particles, however, there is also a chance that things may partially match just by mathematical chance, end this would allow to exploit the huge variety of 3D-compact manifold to explain some of the complex characteristics of particles.

We may for example be able to give to some constants of the Standard Model, which now are known by direct measurement, a theoretical derivation. An example that come to mind is the three different families of particles with different mass (e.g. electron, muon and tau). They may just be three different stationary state of the same manifold (i.e. same particle characteristics) like Willmore spheres that are all the same manifold in different stationary state of energy and with different sizes (i.e. mass from a particle point of view).

Another example is the fractional charge of some particles that come in 1/3 and 2/3 the charge of the electron. This may be explained if we find the correct topological singularity to describe them.

A final example may come from dark matter. Maybe dark matter particles are simply manifolds that do not twist the fields (e.g. oriented manifolds) and therefore do not interact to ordinary matter.

Appendix

A.1 Lagrangian Density of Electromagnetic Field

Given the Lagrangian density of the free electromagnetic field:

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{34}$$

we have:

$$\begin{aligned} \mathcal{L} &= \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) && (35) \\ &= \frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\mu A_\nu \partial^\mu A^\nu) && (36) \end{aligned}$$

the above terms are equal in pairs, we have:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \tag{37}$$

A.2 Fictitious Forces

Given the Lagrangian of a one variable discrete system:

$$L = L(q, \dot{q}) \tag{38}$$

having the following symmetry:

$$q \to q + \phi \ \Rightarrow \ \Delta L = 0 \tag{39}$$

then from the Noether's theorem we know that the quantity $\frac{\partial L}{\partial \dot{q}}$ is conserved.

The above quantity is conserved when we let the system evolve without applying the symmetry to it. However, if we apply the symmetry by changing some symmetry parameter (e.g. for a meccanical system this parameter may be position) while the system is evolving (e.g. for a mechanical system it may correspond to a shift of the whole system in space or to moving the relative position of parts of the system in a way the Lagrangian is not affected), then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ is not a conserved quantity any more. This physically corresponds to having fictitious forces (pseudo forces) in the system depending from the way we change the symmetry parameter as a function of time.

To address the above case, we need to change the Lagrangian in order to take into account the dependency from symmetry parameters (see [1]). To illustrate that, we will use the same example of [1], where it is shown how to get a new Lagrangian just with a change of coordinates from q_0, the coordinate of the inertial system, to $q = q_0 + \phi$, the coordinate of the non inertial one:

Given the Lagrangian of a particle in an external field:

$$L = \frac{1}{2}m(\dot{x}_0)^2 - U(x_0) \tag{40}$$

where x_0 is the coordinate in the inertial frame and x is the coordinate of the moving frame with velocity $\dot{\phi}$, we consider the change of variable $x_0 = x + \phi(t)$. We have:

$$L'(x+\phi, \dot{x}+\dot{\phi}) = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{\phi} + \frac{1}{2}m\dot{\phi}^2 - U(x) \tag{41}$$

The term $\frac{1}{2}m\dot{\phi}$ does not depends on x, gives no contribution to $\frac{\partial S}{\partial q}$ and, if we are interested to the equation of motion, it can be dropped. Regarding the term $m\dot{x}\dot{\phi}$, using the Leibniz rule we have:

$$m\dot{x}\dot{\phi} = m\frac{d}{dt}(x\dot{\phi}) - mx\ddot{\phi} \tag{42}$$

the term $m\frac{d}{dt}(x\dot{\phi})$ is the derivative of a function. Its contribution to the action depends only on its value at the two ends of the time integral. Once again, it gives no contribution to $\frac{\partial S}{\partial q}$ and it can be dropped. We are left with the following Lagrangian:

$$L' = \frac{1}{2}m\dot{x}^2 - mx\ddot{\phi} - U(x) \tag{43}$$

which, by using the Euler-Lagrange equation, gives the following equation of motion:

$$m\ddot{x} = -\frac{\partial U}{\partial x} - m\ddot{\phi} \tag{44}$$

The term $m\ddot{\phi}$ is what we call a fictitious force (pseudo force) and it is a force experienced by the particle m because it is in a non inertial reference frame.

A.3 Evaluation of J^ν

From Eq. (12) we have:

$$J^\nu = \partial_\mu T^{\mu\nu} \tag{45}$$

where:

$$T^{\mu\nu} = \begin{pmatrix} 0 & \partial^0\xi^1 - \partial^1\xi^0 & \partial^0\xi^2 - \partial^2\xi^0 & \partial^0\xi^3 - \partial^3\xi^0 \\ \partial^1\xi^0 - \partial^0\xi^1 & 0 & \partial^1\xi^2 - \partial^2\xi^1 & \partial^1\xi^3 - \partial^3\xi^1 \\ \partial^2\xi^0 - \partial^0\xi^2 & \partial^2\xi^1 - \partial^1\xi^2 & 0 & \partial^2\xi^3 - \partial^3\xi^2 \\ \partial^3\xi^0 - \partial^0\xi^3 & \partial^3\xi^1 - \partial^1\xi^3 & \partial^3\xi^2 - \partial^2\xi^3 & 0 \end{pmatrix} \tag{46}$$

Suppose $\partial^0\tilde{\theta} = \partial^3\tilde{\theta} = 0$, if we ignore all the terms that get hit by ∂^0 and ∂^3, we have:

$$\partial_\mu T^{\mu\nu} = \partial_\mu \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial^1\xi^2 - \partial^2\xi^1 & 0 \\ 0 & \partial^2\xi^1 - \partial^1\xi^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{47}$$

Suppose $\partial^3\theta = 0$, if we ignore all the terms that get hit by ∂^3 only, we have:

$$\partial_\mu T^{\mu\nu} = \partial_\mu \begin{pmatrix} 0 & \partial^0\xi^1 - \partial^1\xi^0 & \partial^0\xi^2 - \partial^2\xi^0 & 0 \\ \partial^1\xi^0 - \partial^0\xi^1 & 0 & \partial^1\xi^2 - \partial^2\xi^1 & 0 \\ \partial^2\xi^0 - \partial^0\xi^2 & \partial^2\xi^1 - \partial^1\xi^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{48}$$

A.4 A More Refined Model

In this paper we have proposed a mechanism for creation of magnetic moment in a particle and we have evaluated the magnetic moment using a simplified model made of a cylinder with currents circulating on the lateral surface.

In this appendix, we want to use a more refined model where currents are circulating on the surface of a sphere and the source are two discrete vortices. Discrete vortices will be represented as Dirac delta functions of the curl of the currents placed at the two poles of the sphere.

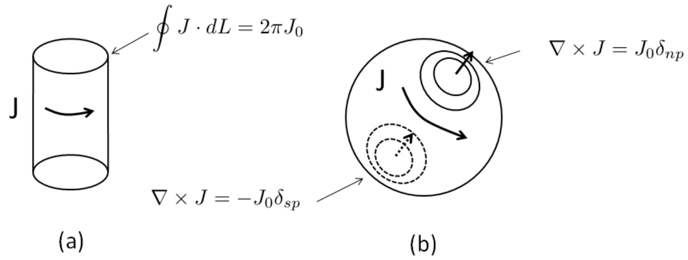

Figure 5: Models for Currents Generating Magnetic Model of a Particle

We want to evaluate the currents field J on a sphere $\mathbf{S}^2$ of radius R with current sources give by two discrete vortices of amplitude $J_0 = \frac{2\pi\alpha}{\mu_0}$ given by Eq. (22), placed at the two poles and spinning with the same orientation with respect to the axis of the sphere. We want also vanishing curl everywhere else on the sphere.

We will use a system of spherical coordinates (θ, ϕ) on $\mathbf{S}^2$ with $\theta \in [0, \pi]$ (polar angle) and $\phi \in [0, 2\pi]$ (azimuthal angle). The reader should not confuse this θ, which is one of the coordinates variables, with the θ used in the main body of the paper which was a gauge field. We have:

$$\nabla \times J = \frac{2\pi}{\mu_0}(\delta_{np} - \delta_{sp}) = J_0(\delta_{np} - \delta_{sp}) \tag{49}$$

where δ_{np} and δ_{sp} are Dirac delta functions placed on the sphere at the north and south pole.

We can use Stoke's theorem applied to a circle $\gamma(\theta)$ of constant θ. The curves γ splits the sphere in 2 surfaces Γ_{np} and Γ_{sp} booth having as boundary γ but one containing the north pole and one the south pole. We have:

$$\int_{\Gamma_{np}} \nabla \times J \cdot dS = J_0 = \oint_{\gamma} J \cdot dL = 2\pi R \sin\theta J_\phi \tag{50}$$

where J_ϕ is the component of the current on the sphere along the ϕ axis. We have:

$$J_\phi = \frac{J_0}{2\pi R \sin\theta} \tag{51}$$

Note that we can do the same calculation using the surface Γ_{sp}, and taking into account signs in the correct way. Since the flows of the curl through Γ_{np} and Γ_{sp} add up to zero (i.e. δs have opposite signs) we get the same result.

For J_θ, we know that $\frac{\partial J_\phi}{\partial_\phi} = 0$ and $\nabla \cdot J_\theta = 0$, we have:

$$\nabla \cdot J = \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(J_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial J_\phi}{\partial\phi} = 0 \tag{52}$$

from which:

$$\frac{\partial}{\partial\theta}(J_\theta \sin\theta) = -\frac{\partial J_\phi}{\partial\phi} \tag{53}$$

which leads to:

$$\sin\theta J'_\theta + \cos\theta J_\theta = 0 \tag{54}$$

The above equation is verified for $J_\theta = 0$ and $J_\theta = \frac{A}{\sin\theta}$. We go for $J_\theta = 0$.

We are now ready to evaluate the magnetic moment M of a particle in this model. Along the axis of the sphere, for each ring of current of radius $R\sin\theta$, we have a contribution to the magnetic moment given by:

$$dM = \pi d^2 \times J_\phi R d\theta = \pi(R\sin\theta)^2 J_\phi R d\theta = \frac{\pi\alpha R^2}{\mu_0}\sin\theta d\theta \tag{55}$$

$$M = \int dm = \frac{\pi\alpha R^2}{\mu_0}\int_0^\pi \sin\theta d\theta = \frac{2\pi\alpha R^2}{\mu_0} \tag{56}$$

As we did in the main body of the paper, using the value of the magnetic moment of the electron μ_e we can evaluate the radius R of the particle:

$$R_M = \sqrt{\frac{\mu_0\mu_e}{2\pi\alpha}} = 2.5\times 10^{-15} \ \ [m] \tag{57}$$

The element charge dQ of the particle can be determined multiplying the current J_π by the length of each circle $\gamma(\theta)$. This time we have to divide J_ϕ by c, the speed of light, as we do for the time component of the current four-vector. We have:

$$dQ = 2\pi d \times \frac{J_\phi}{c} = 2\pi R\sin\theta\frac{2\pi\alpha}{c\mu_0 R\sin\theta}Rd\theta = \frac{4\pi^2\alpha R}{c\mu_0}d\theta \tag{58}$$

and therefore:

$$Q = \frac{4\pi^2\alpha R}{c\mu_0}\int_0^\pi d\theta = \frac{4\pi^3\alpha R}{c\mu_0} \tag{59}$$

Once again, as we did in the main body of the paper, by using the elementary charge of the electron e, we can evaluate R. We have:

$$R_Q = \frac{c\mu_0 e}{4\pi^3\alpha} = 1.6\times 10^{-18} \ \ [m] \tag{60}$$

References

[1] L. D. Landau, E. M. LifShitz - Mechanics, Vol. 1 of Course in Theoretical Physics, Third Edition - Butterworth Heinemann (1976) - Pages: 126 - 129

[2] V. Nardozza. *A Geometrical Model of Gravity* - https://vixra.org/abs/1806.0251 (2018).

[3] V. Nardozza. *A Classical Mechanism for Creation of Magnetic Moment in a Particle* - https://vixra.org/abs/2310.0065 (2023).