Flows of the Riemann Hypothesis

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The Riemann hypothesis is a mathematical conjecture that relates to the calculation of prime numbers through the Riemann product formula, which represents the product of Riemann zeta function and factorial. There were flows in deriving $\int \frac{x^{s-1}}{e^x-1} dx$ from Riemann product formula and, in attempting to represent the negative region by substituting x with –x. Furthermore, asserting that the Riemann zeta function, in the absence of a definition for negative factorial, obtains trivial zeros for negative even numbers through the Bernoulli exponential generating formula in the negative domain is also incorrect.

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Riemann product formula[1] is consisted of the product of Gauss Π function and Riemann zeta function, which is read as

$$\int_0^\infty e^{-nx} x^{s-1} dx = \frac{\Pi(\mathrm{s}-1)}{n^s}, \tag{1}$$

where $\Pi(\mathrm{s}-1) = \Gamma(s)$, and $\zeta(\mathrm{s}) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Riemann has partially induced the Riemann zeta function as follows.

$$\int_0^\infty e^{-nx} dx = \lim_{n\to\infty} \sum_{k=1}^{n} e^{-kx}. \tag{2}$$

Therefore,

$$\sum_{k=1}^{n} e^{-kx} = e^{-x} + e^{-2x} + e^{-3x} + \cdots + e^{-nx} = \frac{e^{-x}(1 - e^{-nx})}{1 - e^{-x}}. \tag{3}$$

And, in case $e^{-x} < 1$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} e^{-kx} = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1} \tag{4}$$

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The equation (1) can be rephrased as per Riemann as below.

$$\Pi(\mathrm{s}-1)\zeta(\mathrm{s}) = \int \frac{x^{s-1}}{e^{x}-1}dx \tag{5}$$

However, Riemann's derivation of the formula as described above is incorrect. The Riemann zeta function is not composed of a geometric progression as shown in equation (3); instead, it is composed of a sum of base n to the power of s exponentiations, i.e.,

$$\lim_{n\to\infty}\sum_{k=1}^{n} e^{-kx} \neq \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{s}} . \tag{6}$$

In equation (5), Riemann substituted x with -x in the right numerator, leading to the expression $(-1)^{s-1}$ as $\left(e^{-is\pi} - e^{is\pi}\right)$. However, as evident from equation (5), the left-hand side is a function of s only, and when we solve the partial integral on the right, the variable x disappears, rendering x merely a parameter. In other words, whether x is replaced with -x or not, the result remains unchanged.

Letting $\mathrm{x} = -\mathrm{y}$ and simplifying, we obtain the following:

$$\Pi(\mathrm{s}-1)\zeta(\mathrm{s}) = \int \frac{y^{s-1}}{e^{y}-1}dy \tag{7}$$

In the equation (3) the value only converges in case of $e^{-x} < 1$, and, when $e^{-x} > 1$ it diverges. [3] [4]

Extended Riemann zeta function can be defined as follows,

$$\zeta(\mathrm{s}) = \sum_{n=1}^{\infty}\frac{1}{n^{s}} = \frac{1}{1^{s}} + \frac{1}{2^{s}} + \frac{1}{3^{s}} + \cdots , \qquad s \geq 0 \tag{8}$$

where $\zeta(\mathrm{s})$ diverges if s = 0 or s = 1 and otherwise, if s > 1, all $\zeta(s)$ converge.

In case s < 0,

$$\begin{aligned}\zeta(-\mathrm{s}) = \sum_{n=1}^{\infty}\frac{1}{n^{-s}} &= \frac{1}{1^{-s}} + \frac{1}{2^{-s}} + \frac{1}{3^{-s}} + \cdots , \qquad s > 0 \\ &= 1^{s} + 2^{s} + 3^{s} + \cdots .\end{aligned} \tag{9}$$

all $\zeta(-s)$ diverge.

Riemann derived the equation (5) to claim that when s are negative even numbers, all provide trivial zeros, which are known as Bernoulli number that comes from the Bernoulli's exponential generating formula[2], which is given as follows,

$$\frac{t}{e^{t}-1} = \frac{t}{2}\left(\coth\frac{t}{2} - 1\right) = \sum_{m=0}^{\infty}\frac{B_{\bar{m}}t^{m}}{m!} \tag{10}$$

In order for this claim to be true, both the sum of the equation (3) in case the first term is to be

te^{-t} and the Bernoulli series (10) which is approximation of the former should fundamentally be the same. Since the equations (3) and (10) represent the sum of a geometric sequence, while the equations (8) and (9) are the sum of the base n of s exponentiation, so they cannot be said to be identical to each other. Therefore, it cannot be said that the equation (9) has trivial zeros in negative even numbers of s.

From this perspective, it can be inferred that Riemann might have attempted to perform the operation of changing s to -s. In this case, as seen in equation (9), the Riemann zeta function diverges for all values in the negative plane.

Therefore, the correct form of the Riemann product formula can be written as

$$\Pi(s-1)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-nx}x^{s-1}dx, \qquad s \geq 1. \tag{11}$$

This holds true only when the domain of s is greater than or equal to 1. For example, in case where s is 0 or a negative number, Gauss' Π function or Adrien-Marie Legendre's Γ function does not have definition for negative factorial[2] and thus cannot be used in this context.

Using Euler Y function[3] which is integrating Gauss' Π function and Legendre's Γ function, the Riemann product formula in the equation (1) can be rewritten as follows.

$$Y(s)\zeta(s+1) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-nx}x^{s}dx, \qquad s \geq 0. \tag{12}$$

According to the definition of Euler Y function, in case s is a negative number (i.e. –s), it can be summarized as follows.

$$Y(-s)\zeta(-s-1) = (-1)^s\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\frac{x}{n}}x^{s}dx, \qquad s > 0. \tag{13}$$

In conclusion, the Riemann zeta function holds for all integer values $\pm s$, and, also for complex values[3] of s in the form $s=p\pm iq$. Therefore, it holds in the negative plane as well, and for $s < 0$, all $\zeta(-s)$ diverge. Consequently, there are no trivial zeros.

While the Riemann product formula does not account for $\zeta\left(\frac{1}{2}\right)$, Riemann zeta function holds for $\zeta\left(\frac{1}{2}\right)$ in the complex plane.

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[2] Refer "On the Euler Integral for the positive and negative factorial." [3]

[3] For reference, Leonhard Euler demonstrated the values in the complex plane for $n = p \pm iq$ in his posthumously published paper[5] in 1794.

References

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