Discrete Space, Continuous Time: A Study of Relativistic and Quantum Mechanics on a Discrete Non-Hausdorff Manifold

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Abstract

We seek to present a novel model of motion which underlies, and provides interpretations for, both the theory of relativistic motion (including gravity) and quantum mechanics. We reject the continuity of space and instead treat motion and space as discrete phenomena. Crucially, we also allow space to have "holes" and "branches" in it in the sense of it being a non-Hausdorff manifold. Thus space is not an all-pervading background "stage" but rather a dynamic entity built up as particles interact with each other. Only paths a particle may physically take enter into building this manifold. We reject the idea of space existing for a particle even where it cannot physically be present, such as inside the barrier in the double-slit experiment. We present our theory of motion then show the key phenomena and equations of special and general relativity as well as quantum mechanics can be recovered from it.

1. Introduction

The unifying theme of all physics is that it consists of, in essence, the careful observation and analysis of various types of motion. The study of motion is very old, having been discussed by the ancient philosophers. We particularly wish to draw attention to the ancient "Dichotomy" paradox of Zeno [1]. The argument goes as follows: "Suppose Atalanta wishes to walk to the end of a path. Before she can get there, she must get halfway there. Before she can get halfway there, she must get a quarter of the way there. Before travelling a quarter, she must travel one-eighth; before an eighth, one-sixteenth; and so on." [2] Thus, Atalanta must complete an infinite sequence of steps to cover a finite distance, something that seems nonsensical. Also, it is unclear how big Atalanta's first step is, as any "first" step must be completed by first covering half of it, thus it is not being a "first" step after all. The standard way out of the paradox is through the observation that a sum of infinitely many decreasing fractions can, and in this case does, converge to a finite sum. This we call the *continuum* solution to Zeno's paradox, as in it we posit a finite interval of space is made up of an infinite number of infinitesimally closely spaced points.

But the solution to Zeno's paradox we wish to study in the present paper is that of *discrete* space. We posit that the points in space exist a small, but finite, distance apart. Therefore Atalanta reaches the end of the path by completing a large, but finite, number of hops from point to successive point in space.

Modern physics follows the continuum solution, hence we talk of the spacetime *continuum*. The tools of calculus, which are based on continuous functions, are all-pervasive in the formulation of the basic laws of motion in quantum mechanics, quantum field theory and relativity.

As successful as the continuum approach has been, it has not yielded a solution to all problems of motion. Quantum phenomena, such as path interference in the double-slit experiment [3] and the superposition and collapse of the wavefunction are notoriously hard to interpret. More seriously, trying to combine the quantum world of superpositions and stochastic collapses with the continuum of general relativity has not been successful due to the non-linearity of the equations involved. Whereas general relativity deals in the "world-lines" of particles, there is no such concept in quantum mechanics where a particle's wavepacket will disperse during its motion [4].

In the present paper we examine the discrete solution to Zeno's paradox in detail. We start by studying one dimensional discrete motion along a path. We then study the case where multiple paths exist between the same start and end positions. We do this by gluing parts of the paths together in a discrete analogy of how continuous coordinate patches are glued together to build up a continuous manifold. We then show how we can embed such glued-together paths in a Euclidean space and assign Euclidean coordinates to positions on the discrete paths. We then discuss more complex cases, where multiple particles moving along multiple discrete paths can interact with each other. We then discuss the statistics and probabilities of different choices of path when multiple paths are available to a particle to travel on after an interaction. We then turn to applications of our theory, by making connections with the existing theories of special relativity, general relativity and quantum mechanics.

2. One-Dimensional Discrete Motion

Let us consider one-dimensional motion at a constant velocity v. In the familiar case of continuous space and time, the position coordinate x is a real number, as is the time t and velocity v. The equation of motion is then

$$x = vt + x_0 \qquad\qquad x, x_0, v, t \in \mathbb{R} \tag{1}$$

where x_0 is a real number being the position at time zero.

In the discrete approach to motion we propose in this work, we make space discrete, but keep time continuous. Thus we keep the time t and velocity v as real numbers, but use integers only for the position coordinates x and x_0. The equation of motion becomes:

$$x = \lfloor vt \rfloor + x_0 \qquad\qquad \begin{matrix} x, x_0 \in \mathbb{Z} \\ v, t \ \in \mathbb{R} \end{matrix} \tag{2}$$

where $\lfloor vt \rfloor$ is the *floor* of vt, that is, the largest integer not greater than vt.

The difference between the continuous and discrete motions given by (1) and (2) is shown in figure 1 for the case when v = 2 and x_0 = 0.

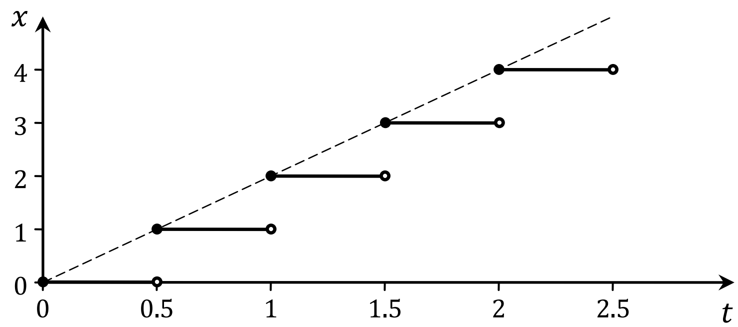

Fig. 1: Position vs. time. Continuous motion (dashed line) and discrete motion (solid line segments).

In the discrete case, the position coordinate x increases in discrete hops of one unit as time increases continuously. The position remains constant for a period of time τ in between these jumps, given by:

$$\tau = \frac{1}{v} \tag{3}$$

In the example of figure 1, v = 2 so τ = 0.5 and we see each discrete step in space has a duration of 0.5 units in time. Using (3) the equation of discrete motion (2) can also be written as:

$$x = \left\lfloor \frac{t}{\tau} \right\rfloor + x_0 \qquad \begin{matrix} x, x_0 \in \mathbb{Z} \\ t, \tau \ \in \mathbb{R} \end{matrix} \tag{4}$$

We call a one-dimensional discrete space in which a particle, associated with a continuous time coordinate, moves according to (2) or (4) a *path*. We call $v = 1/\tau$ the *path velocity*. We call the finite section of a path between a given minimum position x_{min} and maximum position x_{max}, or, equivalently, between a minimum time t_{min} and maximum time t_{max} a *path segment*.

t_{max} is defined to be the biggest real number, and t_{min} the smallest real number, such that the following two equations hold:

$$x_{max} = \left\lfloor \frac{t_{max}}{\tau} \right\rfloor + x_0 \qquad x_{min} = \left\lfloor \frac{t_{min}}{\tau} \right\rfloor + x_0 \tag{5}$$

We define the *length* of a path segment to be $\Delta x = x_{max} - x_{min}$ and the *duration* to be $\Delta t = t_{max} - t_{min}$. Note that $\Delta x = \lfloor v \, \Delta t \rfloor$ as expected. We define the distance between two positions with spatial coordinates y and z along a path to be $|y - z|$.

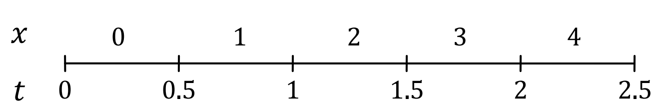

Fig. 2: A path segment of length 4, duration $2.4\overline{9}$ and path velocity 2, with x_0 = 0.

We can represent a path segment in a diagram as shown in figure 2, for a path segment of length 4, x_0 = 0, path velocity 2 and duration $2.4\overline{9}$. The path segment, a one-dimensional space, is drawn as a line segment. It is divided with tick marks into five positions, labeled above the line with their coordinates 0, 1, 2, 3 and 4. Note that in discrete space a "point" is a single spatial coordinate which would be drawn as a finite line segment between two successive tick marks, even though its length is zero. That is why we refer to *positions* rather than *points* in discrete space.

The tick marks in figure 2 are labeled below the line with the times they begin and end. So we can imagine a particle starting at tick mark t = 0, and moving to the right continuously so that its time corresponds to the labels underneath the line, which form a continuous timeline. Meanwhile, the spatial position coordinate of the particle at any given time is given by the label above the line segment between whose tick marks the particle lies at this time. So figure 2 can be thought of as a vertically collapsed version of figure 1.

3. Multiple Paths

Suppose we have two path segments A and B with equal path velocity. The equations of motion are:

$$x_A = \left\lfloor \frac{t_A}{\tau} \right\rfloor + x_{0A} \qquad\qquad x_B = \left\lfloor \frac{t_B}{\tau} \right\rfloor + x_{0B} \tag{6}$$

For example, let us consider the case where τ = 0.5 and $x_{0A} = x_{0B}$ = 0. Let us say path segment A has length 6 (x_{Amin} = 0, x_{Amax} = 6) while B has length 5 (x_{Bmin} = 0, x_{Bmax} = 5). It follows $t_{Amin} = t_{Bmin}$ = 0 and $t_{Amax} = 3.4\overline{9}$ and $t_{Bmax} = 2.\overline{9}$.

So far these paths are totally separate and exist in no relation to each other—the positions x_A and x_B are independent and there is no way to translate one into the other.

We will now proceed to *glue* the starts of segments A and B together, and similarly their ends. We do this by setting up an identification or 1:1 map of positions with some values of x_A with those of some values of x_B. For example, we make the following map:

$$\begin{array}{lcl} x_A = 0 & : & x_B = 0 \\ x_A = 1 & : & x_B = 1 \\ x_A = 5 & : & x_B = 4 \\ x_A = 6 & : & x_B = 5 \end{array} \tag{7}$$

Where the colon : indicates the positions on either side of it are to be considered equivalent. The process of gluing two paths together is shown in figure 3.

We have thus glued the starts and ends of two path segments with each other. We now have two possible paths connecting the common start and end positions, so that a particle has a choice of which of two paths to take for travelling from the start to the end. In the language of manifolds, what we have created is a non-Hausdorff [5], [6] space, that is a manifold with "branches". We may in general glue together any number of paths into a manifold, thus giving the particle a large choice of paths to take between the same starting and ending positions.

a)

b)

c)

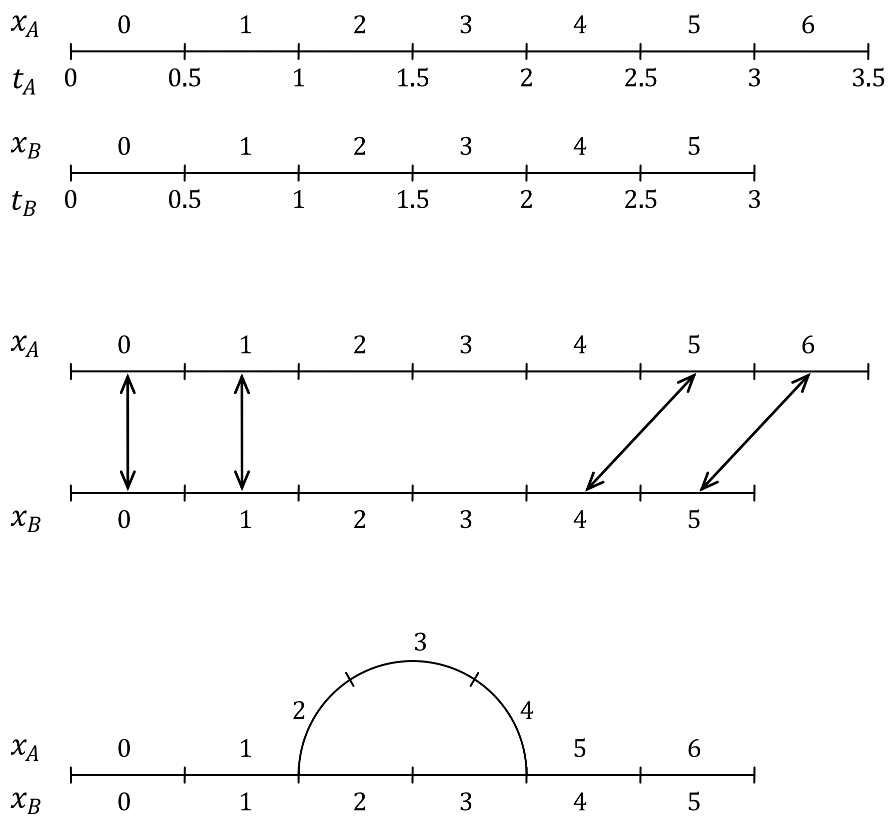

Fig. 3: a) two path segments, of length 6 and 5; b) an identification of some positions along both paths is made, t_A and t_B omitted for clarity; c) the resulting glued paths constitute a non-Hausdorff manifold.

It is crucial to note that the laws and intuitions we have developed regarding motion in ordinary Hausdorff spacetime no longer hold in non-Hausdorff spaces, especially in regards to inertia. A particle travelling along either path segment of figure 2c does not "feel" that it is moving along some, possibly curved, trajectory in a space in which these path segments are embedded. Rather, the two path segments which branch and join to form the manifold are the ENTIRE reality - there is no "outside" space in which these path segments are embedded, in which their curvature can be defined, or in which a particle can "feel" a change in direction of motion as inertia. The particle's ENTIRE reality consists of the time elapsed, the path velocity, and one bit of information about which of the two path segments is being traversed. So we could say informally that the manifold of figure 2c has dimension "1 and 1 bit". The "1" being the continuous time coordinate and the "1 bit" being information about which path segment is being traversed.

It is a general result that in any case where we glue together the starts and the ends of multiple path segments, the difference between the lengths of any two path segments must be an integer. This follows naturally from the spatial coordinates, and thus all lengths, being integers. It follows also that the difference in durations of any pair of such path segments must be an integer multiple of the period τ, since each spatial coordinate has a duration τ in time.

4. Embedding in Euclidean Space

Given a non-Hausdorff space built up by gluing together a number of discrete path segments of given lengths, we wish to discuss how these can be embedded in an ordinary Euclidean n-dimensional space. This is a purely mathematical exercise, necessary to introduce a system of real coordinates into our model to match the spacetime of relativity and quantum mechanics. However, all physics continues to take place in a discrete non-Hausdorff manifold, where particles "feel" no inertia as discussed above. In an embedding the discrete spatial coordinates of the positions along the path segments are mapped to corresponding coordinates $(x_1, x_2, \ldots\ x_n)$ where x_1 ... x_n are all real numbers. This is a well-studied constraint satisfaction problem, of the type which involves embedding a graph with given edge lengths in a Euclidean space, see for example [7]. We will illustrate this embedding process through an example.

Consider three path segments A, B and C with the same path velocity, v = 2, and length 5. We glue them together by making the following identifications:

$$\begin{array}{lcl} x_A = 0 & : & x_B = 0 \\ x_A = 1 & : & x_B = 1 \\ x_A = 4 & : & x_C = 6 \\ x_A = 5 & : & x_C = 0 \\ x_B = 4 & : & x_C = 4 \\ x_B = 5 & : & x_C = 5 \end{array} \tag{8}$$

The result is shown in figure 4.

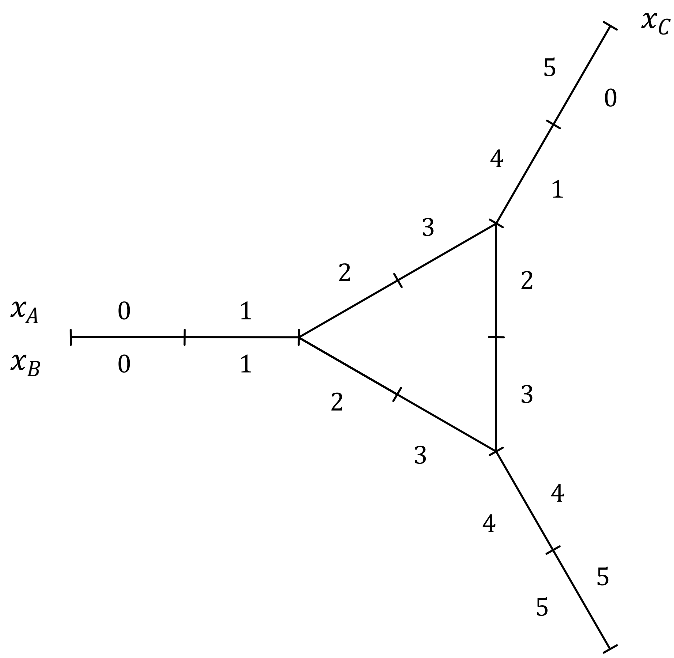

Fig. 4: Path segments A, B and C of length 5 glued together by the map (8).

Let us assign new labels a, b and c to three of the positions on the path segments shown in figure 4. These labels are just new names for some of the positions thus far named by their spatial coordinates. Specifically, a is the position $x_A = 1 \;:\; x_B = 1$, b is the position $x_A = 4 \;:\; x_C = 1$ and c is the position $x_B = 4 \;:\; x_C = 4$. Note that the distance between positions a and b is 3, as is that between b and c and between a and c.

Now we will embed our example in a Euclidean space. The space with two dimensions and coordinates (X, Y) where X, Y are real numbers will suffice. We wish to identify the three positions a, b and c of our discrete non-Hausdorff manifold with three points α, β and γ of a 2-dimensional Euclidean space. The key requirement for an embedding is that the distances between the positions a, b and c in the discrete space must match the Euclidean distances between the points α, β and γ in the Euclidean space.

The following points satisfy this criterion:

$$\alpha = (0,0) \qquad \beta = (2.598, 1.5) \qquad \gamma = (2.598, -1.5) \tag{9}$$

These points are plotted in figure 5.

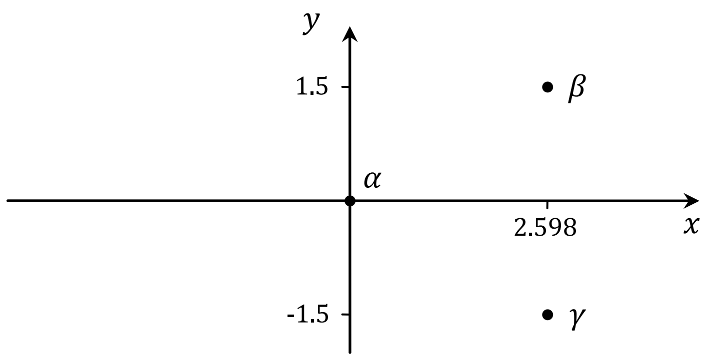

Fig. 5: The three points α, β and γ plotted in Euclidean space corresponding to the positions a, b and c of the discrete non-Hausdorff space shown in figure 4.

An embedding is not unique. We could have chosen any three points in the Euclidean space to correspond to a, b and c, subject to their mutual distances being 3 in this space. If we also wished to map the position $x_A = 0 \;:\; x_B = 0$ of figure 4 to a point in our Euclidean space, we could have chosen any point lying a distance 1 from the point α. There are many such points, again highlighting that an embedding is not unique. In practice, we will not need to map every position of the discrete space to the Euclidean space, only doing this for a few positions of interest, corresponding to the sites of particle interactions.

5. Interactions

So far, we have only considered motion of constant velocity. We also implicitly assumed that we are only dealing with the motion of a single object or particle. Thus, we considered "the position" changing in time, implying there is only one particle we are keeping track of.

Now we turn to the case where the velocity of motion can vary. We posit, analogously to Newton's first law of motion, that the only thing that can cause the change in velocity of a particle is a disturbance to it. So an undisturbed particle will always travel at a constant velocity along a path. We posit that the only way a particle can be disturbed is by an *interaction* with other particles. We also posit that during any interaction, the sum of the path velocities of the incoming (mother) particles is equal to the sum of the path velocities of the outgoing (daughter) particles. That is, the sum of path velocities is a conserved quantity for every interaction.

An example will help illustrate this. Let us consider the interaction where two incoming particles A and B interact to produce one outgoing particle C. Outside the interaction, each particle moves along a path with constant path velocity with the equation of motion (4). Let us say that the path velocity of particle A is v_A = 3, and that of particle B is v_B = 2. As before, we set x_{0A} = x_{0B} = x_{0C} = 0. The equations of motion for particles A and B are:

$$x_A = \lfloor 3t_A \rfloor \qquad\qquad x_B = \lfloor 2t_B \rfloor \tag{10}$$

We posited that the sum of incoming path velocities is equal to the sum of outgoing path velocities. We know therefore that the outgoing particle C must have path velocity v_C = 3 + 2 = 5, with the equation of motion:

$$x_C = \lfloor 5t_C \rfloor \tag{11}$$

While we take the conservation of total path velocity as a postulate, we feel it is a reasonable one. If particle A moves 3 units in space during one unit of time, and particle B moves 2 units in space during one unit of time, it seems reasonable that the union of particles A and B will move 5 units in space in one unit of time. This is equivalent to the classical postulate of the conservation of momentum, where all particles are taken to have the same mass, and the motion is in one dimension, corresponding to our one-dimensional paths.

We can draw our example as in figure 6, where the interaction is represented by a dot and the incoming and outgoing particles by lines or curves with arrows indicating direction. We can label the lines or curves with the name of the particles A, B and C and/or their velocities v_A, v_B and v_C.

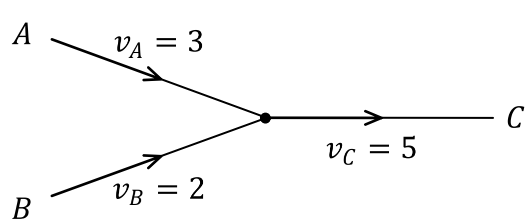

Fig. 6: The interaction of two incoming particles A and B joining into one outgoing particle C.

As another example, consider the case where one incoming particle C with v_C = 5 splits spontaneously into two daughter particles D and E with v_D = 4 and v_E = 1. Note that the total path velocity is conserved. This example is shown in figure 7.

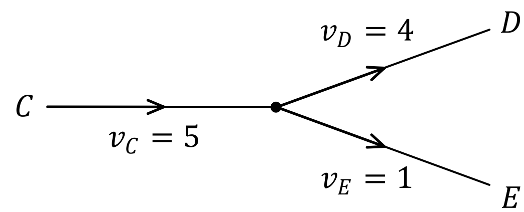

Fig. 7: One incoming particle C splitting into two outgoing particles D and E.

We can combine the two examples of figures 6 and 7 into one, where the incoming particles A and B join into one particle C, which then splits into two particles D and E. This is shown in figure 8.

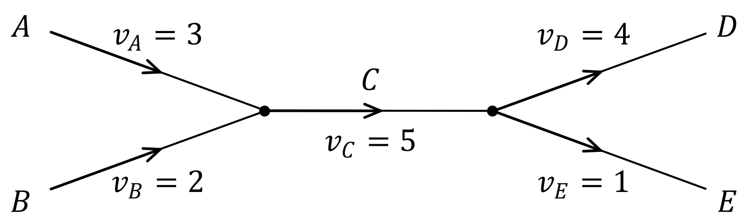

Fig. 8: A process involving two interactions: particles A and B join into C, which then splits into D and E.

In this way, we can build up more complicated processes, involving many particles and interactions. Each such process can be represented as a graph, with each interaction being a vertex and each particle's path segment an edge of the graph. Note well that these processes and graphs correspond to Feynman diagrams in the field theory of scalar particles with the path velocity equal to the rest frame energy of each particle. Also, since in nature most if not all paths are of finite length, we will use the terms "path segment" and "path" interchangeably where this leads to no misunderstanding.

Now we discussed in section 3 the case where a single undisturbed particle moving with constant path velocity may be able to travel along one of multiple possible paths. We discussed that this is done by gluing the starts and the ends of the multiple paths together. Consider the example shown in figure 3, where a particle can travel along a path of length 6 or of length 5. Let us suppose this is the case for a particle C with v_C = 5 travelling between two interactions. We have drawn this in figure 9. As before, we represent the interactions by dots. We indicate the multiple paths by multiple curves between the interactions, each labeled by its length Δx_{C1} = 6 and Δx_{C1} = 5. To indicate those curves represent alternative paths for *one* particle between two interactions, we draw an arc of a circle centered on each interaction vertex, intersecting each possible path.

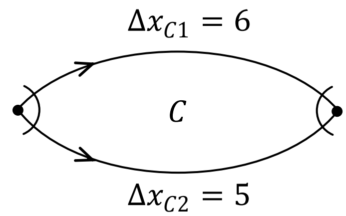

Fig. 9: A particle C can travel along one of two paths between two interactions, with path lengths 6 and 5.

Now we can make a more complicated example based on that of figure 8, where particle C can travel along one of two paths. We thus combine figures 8 and 9 to obtain the process shown in figure 10.

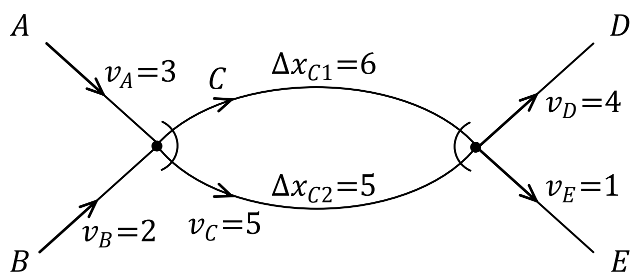

Fig. 10: Two incoming particles A and B produce daughter particle C which can travel along one of two alternate paths of different lengths to an interaction where it splits into two particles D and E.

In this way we can draw diagrams for very complicated processes, where multiple particles have multiple interactions and can travel between them along one of multiple possible paths.

As one last example, consider the process illustrated in figure 11. Two particles A and B combine to form a particle C, which can travel along one of two paths of length 6. One path results in C splitting into two particles D and E with v_D = 4 and v_E = 1. The other path segment results in C splitting into F and G with v_F = 3 and v_G = 2. Even more complicated processes are possible, where there are multiple paths on either or both sides of an interaction, that is on the incoming and/or outgoing side. In all cases, however, all paths in a set of multiple incoming or outgoing paths must be of the same path velocity, with their start and/or segments glued together in the way discussed in section 3.

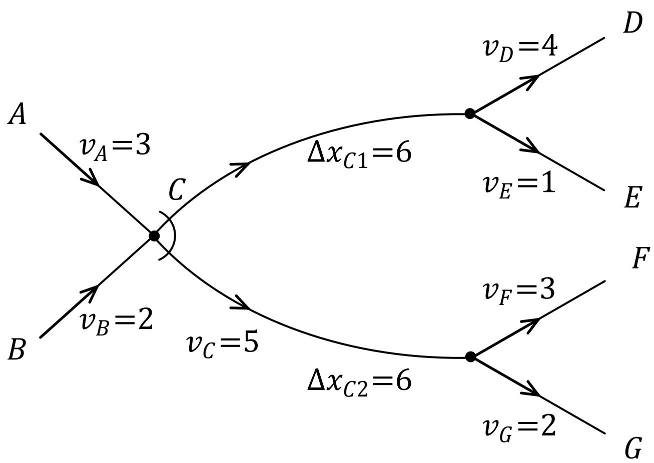

Fig. 11: Two incoming particles A and B produce C, which can split into D and E, or F and G.

In summary, in our model the ENTIRE reality of motion is composed of which particles interact with which particles, and the path velocity and duration of motion in a discrete one-dimensional space for each particle between interactions. These one-dimensional paths can be glued together into non-Hausdorff manifolds in the manner discussed in section 3. From the duration and path velocity, we can compute the distance between any pair of interactions, and through trilateration (graph embedding) assign coordinates to each interaction. We completely reject the idea of spacetime as an all-pervasive neutral "stage" upon which motion happens; rather, spacetime (the locations of particle interactions) is built up dynamically as particles interact with each other.

6. Statistics

Whenever there are multiple outgoing paths from an interaction vertex, we posit that the particle will choose one at random, with equal probability of choosing each possible path. An important detail, however, is that due to the process of gluing parts of paths together as described in section 3, some path segments may be degenerate. That is to say, even though we are ostensibly dealing with a single path segment, there may actually be several path segments which have been glued together into one, and this will affect the probability of such a composite path segment being chosen as a result of an interaction.

If we glue together N path segments together, we say the resulting composite path segment has *amplitude* $\mathcal{A}$ = N. A simple path segment with no gluing has an amplitude $\mathcal{A}$ = 1.

A given spatial position on a path of amplitude $\mathcal{A}$ has degeneracy $\mathcal{A}$. That is, it is made up of $\mathcal{A}$ equivalent, but distinct, positions glued together. As we posited that space is discrete, and motion proceeds by discrete hops from one position to the next, it follows that any such hop can occur from one of $\mathcal{A}$ distinct starting positions to one of $\mathcal{A}$ distinct ending positions one step ahead. That is, the hop, and thus the motion, along a path of amplitude $\mathcal{A}$ itself has a degeneracy of $\mathcal{A}^2$.

We will illustrate this with the example of three path segments A, B and C of length 5 with v = 2 and x_{0A} = x_{0B} = x_{0C} = 0. We make the following identifications of the paths:

$$\begin{array}{lcl} x_A = 0 & : & x_B = 0 \\ x_A = 1 & : & x_B = 1 \\ x_A = 2 & : & x_B = 2 \\ x_A = 3 & : & x_B = 3 \\ x_A = 4 & : & x_B = 4 \\ x_A = 5 & : & x_B = 5 \\ x_C = 0 & : & x_A = 0 \\ x_C = 1 & : & x_A = 1 \\ x_C = 4 & : & x_A = 4 \\ x_C = 5 & : & x_A = 5 \end{array} \tag{12}$$

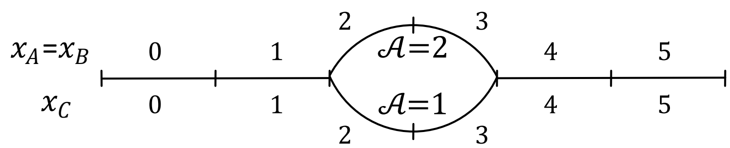

Fig. 12: Two path segments of length 5 between two interactions. The upper path has amplitude 2, with degeneracy 2^2 = 4, while the lower one has an amplitude 1 and degeneracy 1^2 = 1.

We have drawn the result in figure 12. Path segments A and B have been glued together along their entire length by (12), yielding a path segment of amplitude 2 and degeneracy 2^2 = 4. Path C has only been glued to A and B along its starting and ending portions. The remainder has amplitude 1 and degeneracy 1^2 = 1.

Since we posited a particle will choose any outgoing path with equal probability and the degeneracy of one path is 4 while that of the other is 1, it follows it will choose the first path with probability 4/5 and the latter with probability 1/5. The result is general: a particle will choose an outgoing path of amplitude $\mathcal{A}$ with probability proportional to $\mathcal{A}^2$. The proportionality factor, or *normalization factor*, is chosen so that the sum of all probabilities of all outgoing paths is one; that is, the particle is certain to choose *some* path.

This concludes our presentation of our theory with discrete space and continuous time on a non-Hausdorff manifold. Next we turn to the application of the theory, specifically making connections with the existing formalisms of special relativity, general relativity and quantum mechanics.

7. Special Relativity

We discussed in section 5 that in general a graph of particle paths and interactions may be quite complex, involving many interactions among many particles. We also discussed in section 4 how we may embed the discrete non-Hausdorff path segments that particles travel along between interactions in a Euclidean space. In this section, we consider the case of a simple yet non-trivial graph in the form of a triangle of path segments between three interactions and how such a graph may be embedded in a Euclidean space. In this way, we recover the formalism of the special theory of relativity.

The form of the graph we will study is shown in figure 13. Incoming particles L and M interact at vertex a to produce two particles A and B. Particle B travels to vertex c where it interacts with another particle N to produce particles O and C. Particle C then interacts with particle A at vertex b to produce particles P and Q.

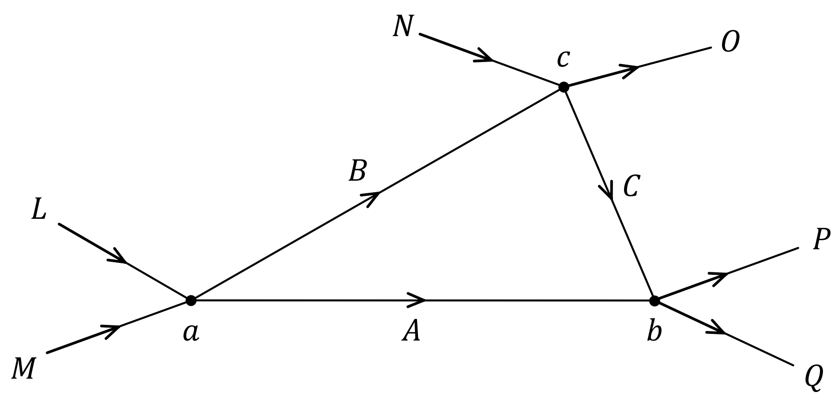

Fig. 13: A triangle of path segments among three interactions a, b and c.

We will now consider how the three vertices a, b and c may be embedded in a Euclidean space. We will make this space three dimensional, as we seem to inhabit a world of three spatial dimensions, and also the space part of the spacetime of special relativity is three dimensional. So, we will assign three points in Euclidean 3-space to the vertices a, b and c respectively:

$$\vec{x}_a = \begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \end{bmatrix} \qquad \vec{x}_b = \begin{bmatrix} x_{b1} \\ x_{b2} \\ x_{b3} \end{bmatrix} \qquad \vec{x}_c = \begin{bmatrix} x_{c1} \\ x_{c2} \\ x_{c3} \end{bmatrix} \tag{13}$$

Note we use a superposed arrow to indicate a 3-vector, to distinguish it from the 4-vectors we will use later.

Let us say the particles A, B and C travel with path velocities v_A, v_B and v_C along path segments of lengths x_{ab}, x_{ac} and x_{cb} and durations t_{ab}, t_{ac} and t_{cb} respectively:

$$x_{ab} = v_A t_{ab} \qquad x_{ac} = v_B t_{ac} \qquad x_{cb} = v_C t_{cb} \tag{14}$$

So, the path segment from interaction a to b has length x_{ab}, and that from a to c, x_{ac}, and from c to b, x_{cb}. When we embed the vertices a, b and c, the following equations must hold between the lengths measured along the discrete path segments and the corresponding Euclidean distances in 3-space:

$$x_{ab} = |\vec{x}_b - \vec{x}_a| \qquad x_{ac} = |\vec{x}_c - \vec{x}_a| \qquad x_{cb} = |\vec{x}_b - \vec{x}_c| \tag{15}$$

Next, we define the *perigee point* p of $\overline{ab}$ with respect to c with coordinates $\vec{x}_p$ as a point in our Euclidean 3-space along the line segment $\overline{ab}$ from which a line segment can be drawn to point c, this line segment being perpendicular to $\overline{ab}$. Thus, p is the point lying on $\overline{ab}$ that is closest to c. The length of the line segment from p to c is denoted by x_{pc} and is given by:

$$x_{pc} = \left|\vec{x}_c - \vec{x}_p\right| \tag{16}$$

We define:

$$t_{pc} \equiv \frac{1}{v_B} x_{pc} \tag{17}$$

Note that we have used the path velocity of particle B in the definition of the time t_{pc}. That is, t_{pc} is the time a particle travelling at the same path velocity as B would take to get from p to c if a path segment existed between these points.

Similarly, the distance from a to p is denoted by x_{ap}:

$$x_{ap} = \left|\vec{x}_p - \vec{x}_a\right| \tag{18}$$

and we define:

$$t_{ap} \equiv \frac{1}{v_B} x_{ap} \tag{19}$$

The definition of the perigee point and the above distances are shown in figure 14.

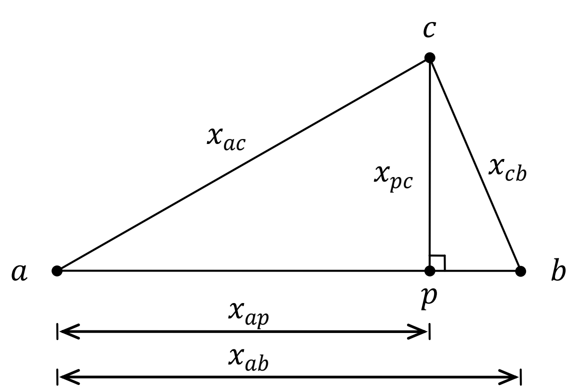

Fig. 14: The triangle abc of figure 13 with the perigee point of $\overline{ab}$ with respect to c labeled p.

Now let us define the following six 3-vectors:

$$\begin{aligned} &\vec{x}_{ac} \equiv \vec{x}_c - \vec{x}_a && \vec{x}_{ap} \equiv \vec{x}_p - \vec{x}_a && \vec{x}_{pc} \equiv \vec{x}_c - \vec{x}_p \\ &\vec{t}_{ac} \equiv \frac{1}{v_B}\vec{x}_{ac} && \vec{t}_{ap} \equiv \frac{1}{v_B}\vec{x}_{ap} && \vec{t}_{pc} \equiv \frac{1}{v_B}\vec{x}_{pc} \end{aligned} \tag{20}$$

Consider the right-angle triangle apc in figure 14. Using the Pythagorean theorem, we have:

$$\vec{x}_{ac}^{\ 2} = \vec{x}_{pc}^{\ 2} + \vec{x}_{ap}^{\ 2} \tag{21}$$

or,

$$\frac{\vec{x}_{ap}^{\ 2}}{\vec{x}_{ac}^{\ 2}} = 1 - \frac{\vec{x}_{pc}^{\ 2}}{\vec{x}_{ac}^{\ 2}} \tag{22}$$

Using (20), we have:

$$\frac{\vec{t}_{ap}^{\ 2}}{\vec{t}_{ac}^{\ 2}} = 1 - \frac{\vec{t}_{pc}^{\ 2}}{\vec{t}_{ac}^{\ 2}} \tag{23}$$

Let us define:

$$\vec{v}_R \equiv \frac{\vec{t}_{pc}}{|\vec{t}_{ac}|} \tag{24}$$

so we can write (23) as:

$$\frac{\vec{t}_{ap}^{\ 2}}{\vec{t}_{ac}^{\ 2}} = 1 - \vec{v}_R^{\ 2} \tag{25}$$

Now let us discuss the significance of the 3-vector $\vec{v}_R$ and its magnitude $v_R = |\vec{v}_R|$. Refer to figure 14. Particle B travels from interaction vertex a to c in time t_{ac}, while particle A travels from a to b. When particle B is at vertex c, it lies a distance x_{pc} from the nearest point on the path followed by particle A. So, t_{pc} is the time it would take for particle B, having reached vertex c, to then travel to the nearest point visited by particle A. The quantity v_R is just the ratio of t_{pc} to t_{ac}. It can range from 0 to 1, and we call it the *magnitude of the relative velocity*. If it is 0, the line segments $\overline{ac}$ and $\overline{ab}$ are parallel, and the perigee point p is identical with c. If v_R is 1, it means $\overline{ac}$ and $\overline{ab}$ are perpendicular. Thus v_R is a measure of how fast particle B moves away from the path followed by particle A. The *relative velocity* $\vec{v}_R$ is a 3-vector parallel to $\vec{x}_{pc}$ with magnitude v_R.

Note that the relative velocity is an abstract quantity. We shouldn't think that particle A is "really" at point p when particle B is at c. Also, there is no direct physical way to measure, or even define, the relative velocity while B is at c; instead, a daughter particle of B must interact with A before this is possible. The relative velocity is derived from the embedding of a triangle of path segments abc, and is a function of the angle between $\overline{ac}$ and $\overline{ab}$. The relative velocity we have defined is identical to the relative velocity of two observers in special relativity, an observer being equated with a particle. As in special relativity, there is a maximum possible magnitude of the relative velocity, v_R = 1, corresponding to the speed of light c. (Our choice of units throughout this paper is such that the speed of light c = 1).

To make further connection with the special theory of relativity consider the ratio of the magnitude of $\vec{t}_{ac}$ to $\vec{t}_{ap}$, from (25):

$$\frac{|\vec{t}_{ac}|}{|\vec{t}_{ap}|} = \frac{1}{\sqrt{1 - \vec{v}_R^{\,2}}} \tag{26}$$

This equation (26) is the equation for time dilation in special relativity. The time $|\vec{t}_{ac}|$ is the time it takes for particle or observer B to travel from a to c, while $|\vec{t}_{ap}|$ is the time particle B would take to travel from a to p, that is if it were travelling with zero relative velocity with respect to particle or observer A, or in other words, if it were "at rest".

Now consider the distance $x_{ac'} = v_B t_{ap} = x_{ap}$ as illustrated in figure 15.

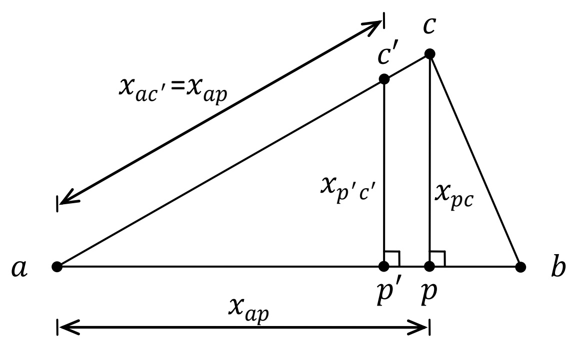

Fig. 15: Two points c' and p' are added to the diagram of figure 14. Note that the time it takes for B to get from a to c' is equal to that it would take it to get from a to p, that is if it were "at rest" with respect to A.

The distance $x_{p'c'}$ is the perigee distance of point c' with respect to $\overline{ab}$. Point c' is reached by particle B in time t_{ap}, the same time it would take to reach point p if it were "at rest" with respect to particle A. Referring to figure 15, we see that the triangles apc and ap'c' are similar, so we have:

$$\frac{|\vec{x}_{pc}|}{|\vec{x}_{ac}|} = \frac{|\vec{x}_{p'c'}|}{|\vec{x}_{ac'}|} = v_R \tag{27}$$

Thus the ratio of the magnitude of $x_{p'c'}$ to x_{pc} is given by:

$$\frac{|\vec{x}_{p'c'}|}{|\vec{x}_{pc}|} = \frac{|\vec{x}_{ac'}|}{|\vec{x}_{ac}|} = \frac{v_B|\vec{t}_{ap}|}{v_B|\vec{t}_{ac}|} = \sqrt{1 - \vec{v}_R^{\,2}} \tag{28}$$

where we used (26).

We recover in (28) the equation for Lorentz–FitzGerald contraction of special relativity. Thus in equations (26) and (28) we recover the basis for the Lorentz transform of space and time coordinates of special relativity, which we can summarize as follows. Consider two observers A, the "lab frame", and B, with observer B travelling at relative velocity v_R with respect to A. If A and B agree on the perigee distance between them, as shown in figure 14, they will disagree on the times at which this perigee distance holds, with these times being related by (26). In contrast, if the observers agree on the time they have travelled, they will disagree on the perigee distance between them, with these distances being related by (28). Here, we assume $v_A \approx v_B$ so we can ignore the stretching of space due to gravity, discussed in the next section.

To make further connections to the special theory of relativity, we will now define 4-vectors. Consider the situation illustrated in figure 14. The distance particle B travels from point a to c is given by $x_{ac} = v_B t_{ac}$. Relative to the nearest point visited by particle A, the displacement to particle B while it is at c is given by the vector $\vec{x}_{pc} = v_B \vec{t}_{pc}$. So we can describe the motion of particle B with respect to particle A by the 4-vector

$$\boldsymbol{x}_{BA} \equiv \begin{bmatrix} t_{ac} \\ \vec{t}_{pc} \end{bmatrix} \tag{29}$$

We use a bold letter to denote a 4-vector, to distinguish it from a 3-vector.

Now recall equation (21) which we can combine with (20) to obtain:

$$\vec{t}_{ap}^{\ 2} = t_{ac}^{\ 2} - \vec{t}_{pc}^{\ 2} \tag{30}$$

Thus, if we are given two fixed points a and p along the path travelled by particle A, the time particle B would have to travel for from a to p, were particle B at rest with respect to A, is always given by (30), irrespectively of the actual relative velocity of the two particles. In special relativity, the quantity $\left|\vec{t}_{ap}\right|$ given by (30) is called the *proper time*. It is an *invariant* quantity, in that it does not depend on how fast the particles are moving with respect to each other. Also, somewhat confusingly, in special relativity the proper time between two points is used as the definition of the *distance* between those points in Minkowskian spacetime. This is not to be confused with the distance in Euclidean 3-space in which our graph is embedded, this distance being given by the path velocity multiplied by the path duration.

The form of equation (30) inspires us to use the Lorentz metric to define the norm of a 4-vector, as is done in special relativity. That is, for a 4-vector (29) we define its norm to be:

$$\boldsymbol{x}_{BA}^{\ 2} \equiv t_{ac}^{\ 2} - \vec{t}_{pc}^{\ 2} \tag{31}$$

and using (30) we have:

$$\boldsymbol{x}_{BA}^{\ 2} = \vec{t}_{ap}^{\ 2} \tag{32}$$

Note that in (19) we defined $t_{ap} = (1/v_B)x_{ap}$. That is, t_{ap} is the time particle B would take to travel from a to p, were its relative velocity zero. Now we define the quantity T_{ap} which is the time particle A actually takes to travel from a to p. Note that particles A and B travel with path velocities v_A and v_B which in general may be different.

$$T_{ap} \equiv \frac{1}{v_A} x_{ap} \tag{33}$$

We define the 4-vector $\boldsymbol{p}_{BA}$ as follows:

$$\boldsymbol{p}_{BA} \equiv \frac{\boldsymbol{x}_{BA}}{T_{ap}} \tag{34}$$

which is the change in the position of particle B with respect to particle A ($\boldsymbol{x}_{BA}$) divided by the time particle A takes to travel from a to p, that is the time in the lab frame or the "lab time". Now we can write:

$$\boldsymbol{p}_{BA} = \frac{t_{ap}}{T_{ap}} \frac{t_{ac}}{t_{ap}} \begin{bmatrix} 1 \\ \vec{v}_R \end{bmatrix} = \frac{v_A}{v_B} \frac{1}{\sqrt{1 - \vec{v}_R{}^2}} \begin{bmatrix} 1 \\ \vec{v}_R \end{bmatrix} \tag{35}$$

where we used (26). Using our definition of the norm of a 4-vector (31):

$$\boldsymbol{p}_{BA}{}^2 = \left(\frac{v_A}{v_B}\right)^2 \tag{36}$$

We recognize $\boldsymbol{p}_{BA}$ as the energy-momentum of special relativity, and the quantity on the right hand side of (26) to be the square of the rest mass μ:

$$\mu \equiv \frac{v_A}{v_B} \tag{37}$$

So in our theory we interpret the rest mass as the ratio of the path velocity of particle A to that of particle B. It measures how "shrunk" or "stretched out" the interval $\overline{ap}$ looks to particle A compared with particle B, that is how much less time particle A takes to cover this interval than particle B would take were it at rest with respect to A. This is intimately related to the role of mass as the curvature of spacetime in general relativity, a topic to which we will now turn.

8. General Relativity

In our approach to motion there is no fixed outside space and/or time "in" which motion occurs. Normally, general relativity presupposes an external spacetime, and equates its curvature with mass. This however leads to a chicken-and-egg type problem when trying to combine general relativity with the non-determinism of quantum mechanics. After all, if a massive particle can be in a superposition of being in two locations, then the curvature of the spacetime must also exist in superposition, so the two superposed locations must themselves exist in a superposition of different curvatures. The non-linearity of the equations involved means that this process does not converge, and the combination of quantum mechanics and general relativity is an open problem.

In our approach, space is built up as particles interact by gluing together paths into a discrete non-Hausdorff manifold and then embedding it in a Euclidean space. There is no presupposed external space whose curvature we are to compute. Instead, particles can randomly choose any one of the outgoing paths from an interaction vertex. Thus the path chosen, the path velocity, and thus by (37) the mass, is decided locally at each vertex. So the curvature of the resulting space emerges as interactions occur, avoiding the chicken-and-egg problem of an infinite regress of superpositions.

We now wish to show that, given a certain approximation, our discrete approach to motion can recover the key equations of general relativity. We do this by first defining a metric on a graph of paths and interactions. We then show that this metric is correctly related to the mass of a particle in that particle's rest frame.

Given a vertex i in the graph, we look at the set of outgoing path segments from it, which we denote P_{ij}, where j is the ending vertex of the path segment. The graph is embedded in a Euclidean 3-space, such that vertex i has coordinates $\vec{x}_i$ and vertex j has coordinates $\vec{x}_j$. The vector from i to j is denoted $\vec{x}_{ij}$ and is given by:

$$\vec{x}_{ij} \equiv \vec{x}_j - \vec{x}_i \tag{38}$$

We define:

$$\vec{t}_{ij} \equiv \frac{1}{v_{ij}} \vec{x}_{ij} \tag{39}$$

where v_{ij} is the path velocity of the path segment from i to j.

Let us denote the distance from i to j by d_{ij} in accordance with the discussion immediately following (30):

$$d_{ij} \equiv \left|\vec{t}_{ij}\right| = \frac{1}{v_{ij}} \left|\vec{x}_{ij}\right| \tag{40}$$

Now the distance d_{ij} can be written as a function $d_i(\vec{x}_{ij})$ of the direction in which the path goes from i to j. So we have:

$$d_i(\vec{x}_{ij}) \equiv \frac{1}{v_{ij}} \left|\vec{x}_{ij}\right| \tag{41}$$

Or, squaring both sides:

$$\left(d_i(\vec{x}_{ij})\right)^2 = \frac{1}{{v_{ij}}^2} {\vec{x}_{ij}}^2 \tag{42}$$

Now we can make the *continuum approximation* to (42) to recover the metric if the spacetime continuum in general relativity. We assume that $d_i(\vec{x}_{ij})$ varies smoothly and slowly as a function of the vector $\vec{x}_{ij}$ and that $\left|\vec{x}_{ij}\right|$ is very small. This implies that the space we are dealing with is continuous and a Hausdorff space, and thus does not exhibit the phenomenon of quantum interference among different paths between the same places. Given this information, we can write (42) in terms of its power expansion with constant and linear terms equal to zero due to the translational invariance of space, and terminated after the quadratic terms. We obtain:

$$\left(d_i(\vec{x}_{ij})\right)^2 \approx {\vec{x}_{ij}}^T \boldsymbol{G}_i \, \vec{x}_{ij} \tag{43}$$

where $\boldsymbol{G}_i$ is a 3×3 matrix of real numbers associated with the vertex i.

This suffices for a metric in a given particle's rest frame. If we use another frame of reference where the particle is not at relative rest, we can describe the particle's motion by the 4-vector $\boldsymbol{x}_{ij}$ as we did in the previous section. Then we can write (43) as:

$$\left(d_i(\boldsymbol{x}_{ij})\right)^2 \approx \boldsymbol{x}_{ij}{}^T \boldsymbol{G'}_i \, \boldsymbol{x}_{ij} \tag{44}$$

where $\boldsymbol{G'}_i$ is a 4×4 matrix. We recognize $\boldsymbol{G'}_i$ is what is called the metric in general relativity, that is (44) corresponds to the equation for distance in general relativity:

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \tag{45}$$

Now we wish to show that the metric is related to the mass in the rest frame of a particle in the same way in our theory as in general relativity. Feynman discussed the meaning of general relativity in a lecture [8]: "Consider a small three-dimensional sphere, of a given surface area. Its actual radius exceeds the radius calculated by Euclidean geometry $\left([r' =]\sqrt{area/4\pi}\right)$ by an amount proportional to the amount of matter inside the sphere $\left(r - \sqrt{area/4\pi} = G/3c^2 m_{inside}\right)$." A similar interpretation is discussed in [9].

We can recover this same relation from the discrete non-Hausdorff model. Let us imagine a path segment P_{ab} from vertex a located in the embedding of a graph in the center of a small 3-sphere to a vertex b on the surface of it. P_{ab} has length x_{ab}, path velocity v_{ab} and duration t_{ab}. Let us imagine a copy of this situation, with a path segment P_{cd} from the center c of another 3-sphere to a vertex d on its surface. P_{cd} has the same length as P_{ab}, $x_{cd} = x_{ab}$, and thus the two spheres have the same surface areas in 3-space. P_{cd} has path velocity v_{cd} and duration t_{cd}.

Recalling the discussion following (30), in special relativity the distance, and so the radius, in Minkowskian spacetime is given by the proper time. So the radius in the first case is given by $r' = t_{ab}$ and in the second, $r = t_{cd}$. We have:

$$r' = \frac{1}{v_{ab}} x_{ab} \qquad r = \frac{1}{v_{cd}} x_{cd} \tag{46}$$

So,

$$\frac{r}{r'} = \frac{v_{ab}}{v_{cd}} \tag{47}$$

or,

$$r - r' = \left(\frac{v_{ab}}{v_{cd}} - 1\right) r' \tag{48}$$

Now from (37) we know that the ratio of two path velocities is the rest mass μ. If the path velocities are equal, this gives a mass of 1 and there is no shrinking or stretching of space. This is the case for what is called "empty space" in general relativity. The quantity $v_{ab}/v_{cd} - 1 = \mu - 1$ is thus the extra mass in the second sphere of radius r, over and above that in the case where the space is empty. Actually, it is the extra mass *linear density*. It is multiplied in (48) by the radius $r^{'}$ of the first sphere, thus giving the total extra mass which needs to be added to the first sphere to yield the second sphere. It is what Feynman denoted m_{inside} . So we have:

$$r - r^{'} = m_{inside} \tag{49}$$

which is equivalent to what is described in the Feynman quote above given an appropriate choice of units.

9. Quantum Mechanics

We now wish to show how we can recover non-relativistic quantum mechanics from our discrete non-Hausdorff approach to motion. We start by taking the classical limit $v_R \ll 1$ of the theory of relative motion discussed in section 7.

Consider the situation of figure 14. In the limit $v_R \ll 1$, the points c and p are much closer together than either point is to point a. Now from (30) we know that the distance in special relativity is equal to the proper *time*. So in the classical approximation, t_{ap} corresponds to the classical distance B would travel were it at rest relative to A, and t_{ac} is the classical distance it actually travels. The ratio of t_{ac} to t_{ap} is given by (26), which can be written:

$$\frac{t_{ac}}{t_{ap}} = (1 - {v_R}^2)^{-\frac{1}{2}} \tag{50}$$

Since $v_R \ll 1$, we can use the binomial approximation $(1 + x)^n \approx 1 + nx$ for $x \ll 1$ to yield:

$$\frac{t_{ac}}{t_{ap}} \approx 1 + \frac{1}{2}{v_R}^2 \tag{51}$$

or,

$$t_{ac} - t_{ap} = \frac{1}{2}{v_R}^2 t_{ap} = \frac{1}{2}{v_R}^2 \frac{t_{ap}}{T_{ap}} T_{ap} = \frac{1}{2}{v_R}^2 \frac{v_A}{v_B} T_{ap} = \frac{1}{2}{v_R}^2 m T_{ap} \tag{52}$$

where we used (20), (33) and (37). We also used the classical symbol m for the rest mass.

Now, as discussed above, the quantity $t_{ac} - t_{ap}$ is the change in classical distance, or position, so we denote it by the symbol Δx. The time T_{ap} is the time measured by particle A, which plays the role of the "lab frame" and so corresponds to the classical global time t. Thus we can write (52) as:

$$\Delta x = \frac{1}{2} m {v_R}^2 t \tag{53}$$

We recognize that Δx is just the expression for the classical kinetic energy multiplied by the classical time. But it can also be interpreted as motion along a path with path velocity equal to the classical kinetic energy. So, in the classical approximation, we can replace the theory of relative motion of section 7 by a motion along a discrete path with path velocity $\frac{1}{2}m{v_R}^2$. That is, if we insist on studying motion in classical space and time, we will find that motion is periodic, with a phase shift depending on the kinetic energy and duration of motion as per (53).

Thus far in this paper we have only dealt with path segments of integer length. When gluing the start and end portions of such segments, the difference in path lengths was always an integer, and the difference in durations an integer multiple of the period τ. We now wish to generalize our model to cases where path length differences are not integers. We wish to do so while maintaining the essential idea of motion on a non-Hausdorff manifold.

The velocity v of any 1-dimensional motion can be expressed as a real number of spatial units covered in a unit of time. One of the key ideas of this paper, the discreteness of motion, is that these units can be very small, but not infinitesimally so. One spatial unit is covered in $\tau = 1/v$ (see (3)) units of time, so it is an essential feature of motion along a path that it has a period τ. In the case of integer position coordinates introduced in section 2, this period manifested as the duration of time the position stayed constant before it hopped to the next position along the path. But this was just a specific example of how a path can be periodic. In general, we can represent any path by a periodic function of time $\Psi(t)$ which we call the *wavefunction*. The wavefunction must obey the following for all values of t for it to be periodic:

$$\Psi(t) = \Psi(t + \tau) \tag{54}$$

The position, which in general can be any real number x, at any given time t is then:

$$x = vt + x_0 = \frac{t}{\tau} + x_0 \qquad\qquad x, x_0, v, t, \tau \in \mathbb{R} \tag{55}$$

this being the number of spatial units covered in time t plus the position x_0 at time 0.

If there are multiple paths between the same starting and ending positions, we proceed as in section 3 in gluing the starts and ends of the paths into a non-Hausdorff manifold. We do this by setting up a mapping of spatial units along different paths, as in section 3. In that section, where we used integer spatial coordinates, we always mapped one integer to one integer. For example, the position $x_A = 5$ of one path would be be mapped to, or treated as equivalent to, the position $x_B = 4$ of another path. In the generalized case, the spatial coordinates can be real numbers. Yet, we must still glue one whole spatial unit to one whole spatial unit, which has a duration τ in time and spatial length of 1 unit. For example, we can glue the interval $x_A = [5 \mathinner{..} 6)$ of one path to the interval $x_B = [4.2 \mathinner{..} 5.2)$ of another. In this way we can deal with gluing paths together with non-integer differences in length.

If we glue two identical paths together, the result is, by definition, a path of amplitude equal to 2. So the following must hold for any wavefunction:

$$\Psi(t) + \Psi(t) = 2\Psi(t) \tag{56}$$

In general, whenever we glue together two paths of amplitudes A and B both with the same period τ, with the paths offset in time by Δt, the result must also be a path, with some amplitude C, time offset $\Delta t'$, and the same period τ. Namely we must have:

$$A\Psi(t) + B\Psi(t + \Delta t) = C\Psi(t + \Delta t') \tag{57}$$

We also require that:

$$\Psi(t) + \Psi\left(t + \frac{\tau}{2}\right) = 0 \tag{58}$$

that is,

$$-\Psi(t) = \Psi\left(t + \frac{\tau}{2}\right) \tag{59}$$

To show (58) must hold, consider there exists some non-zero wavefunction $\Phi(t)$:

$$\Phi(t) = \Psi(t) + \Psi\left(t + \frac{\tau}{2}\right) \tag{60}$$

Then we would have:

$$\Phi\left(t + \frac{\tau}{2}\right) = \Psi\left(t + \frac{\tau}{2}\right) + \Psi(t + \tau) = \Phi(t) \tag{61}$$

as well as:

$$\Phi(t + \tau) = \Psi(t + \tau) + \Psi\left(t + \tau + \frac{\tau}{2}\right) = \Phi(t) \tag{62}$$

where we used (54). But (61) means $\Phi(t)$ has a period of $\tau/2$, contradicting the requirement that we always glue paths with period τ to obtain one path with the same period τ. Therefore, (58) must hold.

Now let us find the form of the function $\Psi(t)$. We know (57) must hold for all possible choices of A, B and Δt. So let us choose $A = -1/h$, $B = 1/h$ and $\Delta t = h$ where h is a small positive real number. We use equation (59) to obtain the negative amplitude A. We can then write (57) as:

$$\frac{\Psi(t + h) - \Psi(t)}{h} = C\Psi(t + \Delta t') \tag{63}$$

But we recognize the left hand side of (63), in the limit as h goes to zero, to be the derivative of $\Psi(t)$ with respect to t, so we have:

$$\frac{d\Psi(t)}{dt} = C\Psi(t + \Delta t') \tag{64}$$

for some C and $\Delta t'$. Solving (64) and using (54) and (57) gives:

$$\Psi(t) = Ae^{i2\pi vt} \tag{65}$$

because:

$$\frac{d\Psi(t)}{dt} = \frac{dAe^{i2\pi vt}}{dt} = i2\pi vAe^{i2\pi vt} = 2\pi v\Psi\left(t + \frac{\tau}{4}\right) \tag{66}$$

which matches (64).

Motion still proceeds by discrete hops from one spatial unit of amplitude A to the next one, also of amplitude A. It follows the number of distinct yet equivalent hops from one position to the next is given by A^2, just as in section 6. A^2 is also given by the absolute square of the wavefunction (65):

$$|\Psi(t)|^2 = \left|Ae^{i2\pi vt}\right|^2 = A^2 \tag{67}$$

As discussed in section 6, following an interaction a particle will choose locally (at the vertex) and randomly, with equal probability, one of the paths available to it, in proportion to the square of its amplitude, given by the absolute square of the wavefunction as in (67). The proportionality factor is fixed by the requirement that the probabilities of all the paths available to a particle sum to one, that is, the particle will always choose *some* path to travel on. Thus in (67) we have recovered the Born rule of quantum mechanics.

Now we can combine the wavefunction (65) with the approximation (53). At non-relativistic speeds, motion relative to the lab frame with velocity v_R is equivalent to motion along a path with path velocity $\frac{1}{2}m{v_R}^2$:

$$\Psi(t) = e^{i2\pi\frac{1}{2}m{v_R}^2 t} \tag{68}$$

The non-relativistic equation for constant velocity motion is:

$$x = v_R t + x_0 \tag{69}$$

or:

$$v_R t = x - v_R t_0 \tag{70}$$

where $x_0 = v_R t_0$.

Substituting into (68) gives:

$$\Psi(x, t_0) = e^{i2\pi\frac{1}{2}m v_R (x - v_R t_0)} \tag{71}$$

Now taking the partial derivative with respect to t_0 and the partial second derivative with respect to x gives:

$$i\frac{\partial\Psi}{\partial t_0} = -\frac{1}{\pi m}\frac{\partial^2\Psi}{\partial x^2} \tag{72}$$

which is the Schrödinger equation at the time t_0 and position x with zero potential and in units chosen such that Planck's constant $h = 4$.

We have thus shown that the generalized non-Hausdorff model of motion underlies both the Born rule and the Schrödinger equation of quantum mechanics.

A few words are appropriate here on the topic of the interpretation of the double-slit experiment [3] in quantum mechanics. In the standard Copenhagen interpretation, a particle travelling from a source to a detector, which travel is possible along two different paths, doesn't "have" a position until it is measured at the detector. We cannot talk of "where" the particle "really" is until it is detected.

The solution we propose in this work is that of considering space to be a non-Hausdorff manifold. As in section 3, we can glue the start and end portions of two paths together, but leave the middle portion unglued. That is, space "branches" into two paths. Normally, we view space to be a continuum and treat the barrier in the double-slit experiment as a region of space which cannot be occupied by a particle. In the non-Hausdorff approach, in contrast, there is no "space" where the barrier is as far as the particle is concerned. As the particle can never find itself inside the barrier, we feel it is wrong to nevertheless claim that there is space there, as space is nothing but a set of all points at which the particle may be present.

While the particle is travelling along one of a set of branches of a non-Hausdorff manifold, we say it is in a "superposition" of states, like a particle travelling along "both" paths in the double-slit experiment. The particle itself chose locally and randomly one of these branches after its last interaction. So it itself "knows" "where" (on which path) it is, but no other particle or system knows that.

When the particle has another interaction at the detector, the superposition "collapses". Now, as discussed in section 4, we can embed the path segment travelled by the particle based on its distance travelled, and the other path segments connected to the interaction vertex, and so compute the position of the interaction vertex, and so compute the position of the interaction in a Euclidean space. Thus space—the locations of interactions—is built up as interactions occur between particles travelling along non-Hausdorff manifolds, and this allows a natural interpretation of what it means that a particle "has" no position until a measurement (interaction) makes an embedding of the interaction graph possible.

10. Conclusion

Our approach to motion presented in this paper rests on two key ideas. First, that motion is discrete and proceeds in small but finite hops. Second, that motion occurs on a non-Hausdorff manifold, one that can have "branches" on it. These two ideas taken together provide a natural interpretation for the necessity of keeping track of a separate phase for every possible path a particle can take in quantum mechanics. We have shown the phase is a function of the kinetic energy of a particle in the domain of non-relativistic quantum mechanics. Meanwhile, in the domain of the non-Hausdorff manifold being "smooth" and continuous, we have recovered the theory of general relativity, the "smoothness" and thus the Hausdorffness of space discarding any quantum phenomena such as path interference, which indeed are not a feature of general relativity. The underlying discrete non-Hausdorff model of motion thus is more general than, and lies at a deeper level than, both relativity and quantum mechanics. We hope it will be of use in calculating phenomena which involve both gravity and quantum mechanics, both for experimental verification and for the development of a unified theory of all motion.

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