THE GEOMETRIC COLLATZ CORRESPONDENCE

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ABSTRACT

The Collatz Conjecture, one of the most renowned unsolved problems in mathematics, presents a deceptive simplicity that has perplexed both experts and novices. Distinctive in nature, it leaves many unsure of how to approach its analysis. My exploration into this enigma has unveiled two compelling connections: firstly, a link between Collatz orbits and Pythagorean Triples; secondly, a tie to the problem of tiling a 2D plane. This latter association suggests a potential relationship with Penrose Tilings, which are notable for their non-repetitive plane tiling. This quality, reminiscent of the unpredictable yet non-repeating trajectories of Collatz sequences, provides a novel avenue to probe the conjecture's complexities. To clarify these connections, I introduce a framework that interprets the Collatz Function as a process that maps each integer to a unique point on the complex plane.

In a curious twist, my exploration into the 3D geometric interpretation of the Collatz Function has nudged open a small, yet intriguing door to a potential parallel in the world of physics. A subtle link appears to manifest between the properties of certain objects in this space and the atomic energy spectral series of hydrogen, a fundamental aspect in quantum mechanics. While this connection is in its early stages and the depth of its significance is yet to be fully unveiled, it subtly implies a simple merging where pure mathematics and applied physics might come together.

The findings in this paper have led me to pursue development of a new type of number I call a Cam number, which stands for "complex and massive", indicating that it is a number with properties that on one hand act like a scalar, but on the other hand act as a complex number. Cam numbers can be thought of as having somewhat dual identities which reveal their properties and behavior under iterations of the Collatz Function. This paper serves as a motivator for a pursuit of a theory of Cam numbers.

NOTE: Please excuses some of the formatting issues and lack of rigorous proofs. This paper is meant more so to share these ideas in a relatively structured form.

1 Introduction

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases} \tag{1}$$

The Collatz Conjecture, often dubbed the "3n + 1 conjecture", stands as one of the most notorious unsolved problems in the realm of mathematics. Originating from the musings of Lothar Collatz in 1937, this seemingly simple problem has defied solutions and resisted all attempts at a rigorous proof, all the while captivating the imaginations of amateur and professional mathematicians alike.

The conjecture begins with any positive integer n. If n is even, it is halved ($n/2$), and if odd, it is multiplied by three and incremented by one ($3n + 1$). This process is repeated, with each outcome serving as the input for the next iteration. The conjecture posits that regardless of the starting integer, the sequence will invariably arrive at the number 1, after which it will enter a perpetual loop of $4 \to 2 \to 1$.

My goal in writing this paper is not to prove the conjecture, but to start building a framework in which we can map the behaviour of Collatz orbits into some known areas of study. In fact, in most parts of this paper I'll be assuming the conjecture is indeed true. My thinking is perhaps we can make connections to other areas of mathematics, and even physics, in which we might find clues to the reasons as to *why* it's true. The following topics will be explored in this paper.

- Diophantine Equations
- Pythagorean Triples
- Hydrogen Spectral Series

1.1 Outline of Concepts

1. Establish common definitions for well known and lesser known concepts.
2. Understand how we can view the Collatz Function as a mapping from integers into **Collatz Address Space**, which is addressed similarly to how modern computers address memory.
3. Investigate how we can relate **Stopping Times** to solutions to Diophantine Equations
4. Investigate how we can map each orbit to an equation of a circle that appears in **Collatz Address Space**.
5. Show how each circle equation can be associated with a Pythagorean Triple.
6. Show how **Stopping Circle**$_3$ can be associated with the Golden Ratio.
7. Show how we can produce a series of numbers related to the hydrogen spectral series using integers in **Stopping Class**$_3$.
8. Speculate on the development of a new type of number called a "Cam number" that has properties and behaviors that are revealed through the Collatz mapping.

2 Common Definitions

Before we jump into the connections mentioned in the introduction, we will need to define some terms. If you are familiar with the Collatz Conjecture, you might already know these terms. Even so, revisiting them for a refresher might be beneficial.

Orbit$_n$ ($\mathbf{orbit}_n$)	The sequence of numbers you get when you follow the Collatz rules from a starting number n until you reach the number 1.
Total Stopping Time$_n$ ($\mathbf{Tstop}_n$)	The number of steps or moves needed to get to the number 1 when following the Collatz rules from a starting number.
Stopping Time$_n$ ($\mathbf{stop}_n$)	The number of steps or moves needed to arrive at a number lower than your initial starting number n when following the Collatz rules.

3 Lesser-Discussed Definitions Explored

The definitions below cover ideas that seem less explored. I've found few formal discussions about them outside of my own research. I will mention the definitions here and expand upon them further when needed. These are not the only new concepts I will present, but these serve as a good stepping off point.

Stopping Class$_k$ ($\mathbf{Sclass}_k$)	This term gives us a way to represent stopping times as an object with properties. This will be useful when we want to compare general invariants of stopping behavior. **Stopping Class**$_k$ contains all numbers **Stopping Time** $= k$.
Stopping Orbit$_n$ ($\mathbf{Sorbit}_n$)	The sequence of numbers you get when you follow the Collatz rules from a starting number n until you reach a number less than n.
Stopping Destination$_n$ ($\mathbf{Sdest}_n$)	The first number $< n$ reached when applying the Collatz Rules.
Stopping Point$_n$ ($\mathbf{Spoint}_n$)	If a Collatz Orbit has a stopping time, then it also has a stopping point. A stopping point for a number n is defined as a point (x, y) with the following properties: $\mathbf{x} = \mathbf{Sdest}_n$ - n $\mathbf{y} = \mathbf{Sdest}_n$ The idea of establishing a point to represent the stopping behavior of an orbit is that we can now start to talk about Collatz orbits in terms of their geometry, which will become important later. We will see how we can associate each point with a Pythagorean triple if we first construct a Gaussian integer of the form $x + yi$.

Collatz Address Space (Cspace)	Through the Collatz Function, a positive integer can be mapped to a specific location in **Collatz Address Space**. Technically you can think of it as being similar to $\mathbb{R}^3$. For every $n > 1$, **Stopping Signature**$_n$ describes a point in this space by the mapping (**Stopping Time**$_n$, **Stopping Page**$_n$, **Stopping Offset**$_n$) $\rightarrow$ (x, y, z).
Stopping Modulus$_k$ (Smod$_k$)	As we will see, Collatz **Stopping Classes** have some interesting internal structure that works similar to modular arithmetic. Each **Stopping Class k** has a maximum number of "offsets" that can be occupied (similar to a modulus). We call this maximum number of offsets the **Stopping Modulus**$_k$ where **k** is the **Stopping Class** number.
Stopping Page$_n$ (Spage$_n$)	A **Stopping Page** is analogous to a page of memory in a modern computer. As we'll see, we can think of each positive integer as an argument that gets mapped to a point in the 3D space. Points that have the same **Stopping Page**$_n$ tend to be located roughly in the same geometric area. This value is always ≥ 1.
Stopping Offset$_n$ (Soffset$_n$)	A **Stopping Offset** is analogous to a memory offset. Essentially this is the distance from the lower **Stopping Page** boundary.
Stopping Index$_n$ (**Sindex**$_n$)	**Stopping Index**$_n$ = (**Stopping Modulus**$_k$ $\times$ **Stopping Modulus**$_k$) + **Stopping Offset**$_n$ where k = **Stopping Class**$_n$. This is essentially the index of n in the infinite sequence of members of **Stopping Class**$_k$.
Stopping Signature$_n$ (Ssig$_n$)	It is believed that every natural number n (excluding 1) has a finite **Stopping Time**$_n$. Many numbers n may share the same **Stopping Time** as well as the same **Stopping Page**, therefore I have created a term called **Stopping Signature**$_n$ that allows us to uniquely identify an orbit by it's location in **Collatz Address Space**.The **Stopping Signature**$_n$ of an orbit can be uniquely defined by a tuple of three positive integral numbers. These properties are (**Stopping Time**$_n$, **Stopping Page**$_n$, **Stopping Offset**$_n$).
Signature Sequence$_{(k,m)}$ (**Sseq**$_{(k,m)}$)	Let k be some **Stopping Class** and let m be some valid **Stopping Modulus** of **Stopping Class**$_k$. We'll see that there is a unique sequence of **Signature Signatures** of length k where the **Stopping Time** and **Stopping Offset** remain fixed, with only the **Stopping Page** varying between the elements in **Stopping Orbit**$_n$

4 Building a Geometric Intuition

In this section we will build methods to speak about Collatz orbits in terms of their geometric properties. We can do this by investigating the stopping point $\mathbf{Spoint}_n$ of each orbit that begins with a given number n. We find this point by iteratively applying the Collatz Function.

1. Calculate the **Stopping Orbit**$_n$ for n by applying the Collatz Function until you reach a number $< n$. We call this number the **Stopping Destination** ($\mathbf{Sdest}_n$) of n.
2. Using $\mathbf{Sdest}_n$ and n, compute the **Stopping Point**$_n$ (x, y).

   $\mathbf{x} = \mathbf{Sdest}_n$ - n

   $\mathbf{y} = \mathbf{Sdest}_n$

We will focus on the **Stopping Classes** of odd numbers ≥ 3. Below is table showing the stopping times for the first 16 odd numbers ≥ 3.

n	$\mathbf{Spoint}_n$	$\mathbf{stop}_n$
3	(-1, 2)	6
5	(-1, 4)	3
7	(-2, 5)	11
9	(-2, 7)	3
11	(-1, 10)	8
13	(-3, 10)	3
15	(-5, 10)	11
17	(-4, 13)	3
19	(-8, 11)	6
21	(-5, 16)	3
23	(-3, 20)	8
25	(-6, 19)	3
27	(-4, 23)	96
29	(-7, 22)	3
31	(-8, 23)	91
33	(-8, 25)	3

Table 1: Stopping locations and stopping times for first 16 odd numbers ≥ 3

The **Stopping Destination**$_n$ is essentially the *end state* of applying the Collatz Function for a given positive starting integer n. To make this concrete, let's walk through an example using $\mathbf{n = 19}$. You can then apply this to any number n to compute $\mathbf{Spoint}_n$.

$$\mathbf{Sorbit}_{19} = \begin{bmatrix} 58 & 29 & 88 & 44 & 22 & \mathbf{11} \end{bmatrix}$$

$$\mathbf{Sdest}_{19} = 11$$

$$\mathbf{Spoint}_{19} = (\mathbf{Sdest}_{19} - 19, \mathbf{Sdest}_{19}) = (11 - 19, 11) = (-8, 11)$$

The following page shows a plot of the numbers in Table 1.

4.1 Plot for odd numbers less than or equal to 33

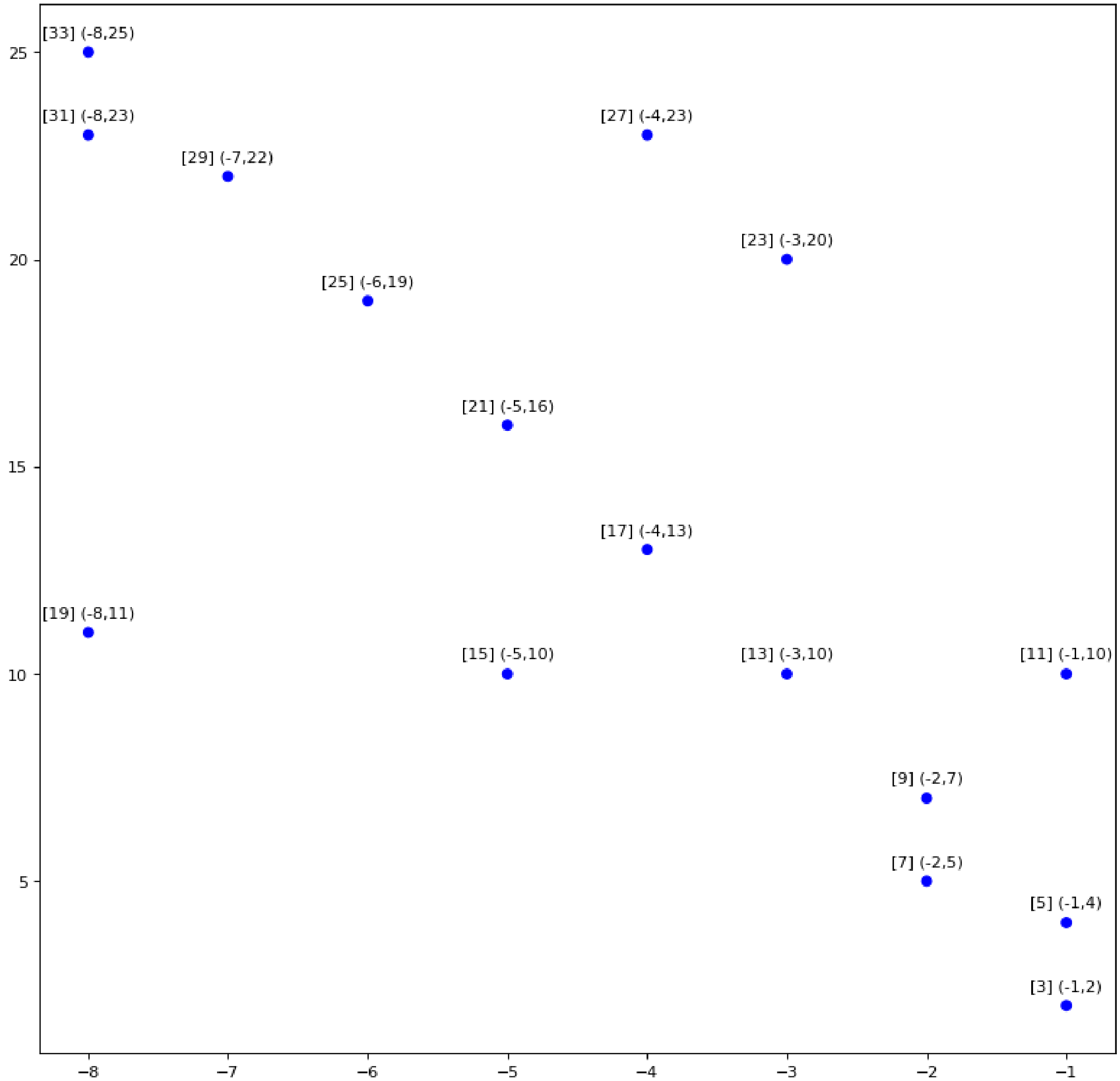

Figure 1: Plot of $\mathbf{Spoint}_n$ for odd numbers ≤ 33.

At first it doesn't seem that the points $\mathbf{Spoint}_n$ have any type of obvious pattern to them. However, if we we look at $\mathbf{Spoint}_n$ for a single stopping time $\mathbf{k}$, we do see some linearity to the points. Let's take the first 8 odd numbers where $\mathbf{stop}_n = 3$, the lowest possible stopping time for odd numbers.

4.2 Data for Orbits with Stopping Time 3

n	$\mathbf{Spoint}_n$	$\mathbf{stop}_n$
5	(-1, 4)	3
9	(-2, 7)	3
13	(-3, 10)	3
17	(-4, 13)	3
21	(-5, 16)	3
25	(-6, 19)	3
29	(-7, 22)	3
33	(-8, 25)	3

Table 2: Numbers where $\mathbf{stop}_n = 3$

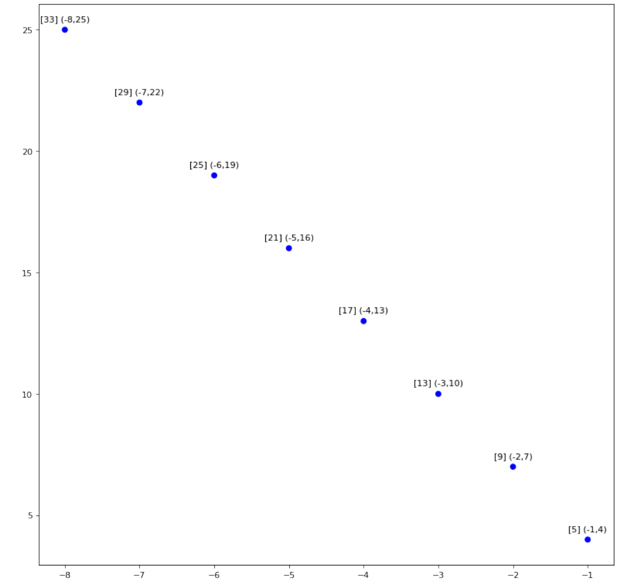

Figure 2: First 8 numbers where $\mathbf{stop}_n = 3$. Points are solutions to the equation $\mathbf{3x + y - 1 = 0}$ with the restriction $x < 0$, $y > 0$, and $|x| + |y| = n$.

We can in fact see that all of these points lie on the on the same line at locations where the coordinates are integers and satisfy the equation $\mathbf{3x + y - 1 = 0}$ $x < 0$, $y > 0$, and $|x| + |y| = n$. This is an interesting result! This may be a clue that ties each **Stopping Class**$_n$ to integer solutions of linear Diophantine equations.

Great! We're starting to see some patterns here! Let's see if these patterns continue to hold for other values of $\mathbf{stop}_n$. Below are the first 8 numbers for $\mathbf{stop}_n = 6$, the next highest allowable stopping time.

4.3 Data for Orbits with Stopping Time 6

$\mathbf{n}$	$\mathbf{Spoint}_n$	$\mathbf{stop}_n$
3	(-1, 2)	6
19	(-8, 11)	6
35	(-15, 20)	6
51	(-22, 29)	6
67	(-29, 38)	6
83	(-36, 47)	6
99	(-43, 56)	6
115	(-50, 65)	6

Table 3: Numbers where $\text{stop}_n = 6$

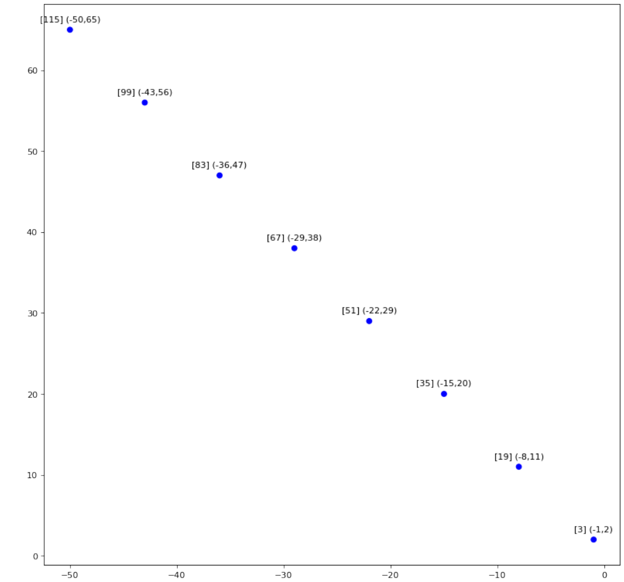

Figure 3: First 8 numbers where $\mathbf{stop}_n = 6$. Points are solutions to the equation $\mathbf{9x + 7y - 5 = 0}$ with the restrictions $x < 0$ and $y > 0$

These points seem to lie on a line as well, this time with slope -$\frac{9}{7}$. And the points seem to be the integer solutions to $\mathbf{9x + 7y - 5 = 0}$ with the restrictions $x < 0$ and $y > 0$.

4.4 Data for Orbits with Stopping Time 8

Let's take a look at one more example where $\mathbf{stop}_n = 8$. This example will serve to motivate our definition of a **Stopping Modulus** and a **Stopping Signature**. Below are the first 8 numbers where $\mathbf{stop}_n = 8$.

n	$\mathbf{Spoint}_n$	$\mathbf{stop}_n$
11	(-1, 10)	8
23	(-3, 20)	8
43	(-6, 37)	8
55	(-8, 47)	8
75	(-11, 64)	8
87	(-13, 74)	8
107	(-16, 91)	8
119	(-18, 101)	8

Table 4: Numbers where $\mathbf{stop}_n = 8$

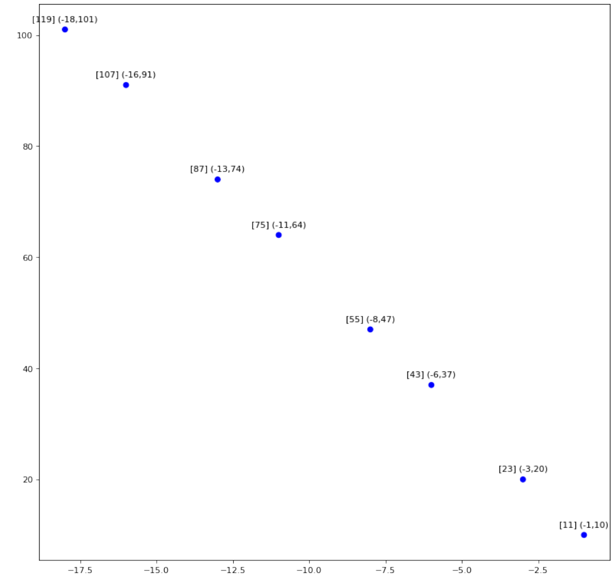

Figure 4: First 8 numbers where $\mathbf{stop}_n = 8$.

The points on the graph do *appear* to fall on the same line, but they actually lie on two separate lines. They also don't seem to be evenly spread out. The points tend to "clump" in groups of **2**. This is where our definition of **Stopping Modulus**$_k$ becomes useful. When **k = 8**, we say **Stopping Modulus**$_8$ $= 2$. This is equivalent to stating **Stopping Class**$_8$ has **Stopping Modulus 2**.

Now we can further classify numbers that belong to **Stopping Class**$_8$ by referencing them by their unique **Stopping Signatures**. See the table below.

$\mathbf{n}$	$\mathbf{Spoint}_n$	$\mathbf{stop}_n$	$\mathbf{Spage}_n$	$\mathbf{Soffset}_n$	$\mathbf{Ssig}_n$
11	(-1, 10)	8	1	0	(8, 1, 0)
23	(-3, 20)	8	1	1	(8, 1, 1)
43	(-6, 37)	8	2	0	(8, 2, 0)
55	(-8, 47)	8	2	1	(8, 2, 1)
75	(-11, 64)	8	3	0	(8, 3, 0)
87	(-13, 74)	8	3	1	(8, 3, 1)
107	(-16, 91)	8	4	0	(8, 4, 0)
119	(-18, 101)	8	4	1	(8, 4, 1)

Table 5: Demonstrations of **Stopping Signatures**$_8$

The **Stopping Points** of **Stopping Class**$_8$ actually lie on two separate lines depending on the value of the **Stopping Offset**. Both lines share the same slope of -$\frac{27}{5}$.

- $\mathbf{27x + 5y - 23 = 0}$ with restrictions $x < 0$, $y > 0$, and $|x| + |y| = n$ when $\mathbf{Soffset}_n$ = 0
- $\mathbf{27x + 5y - 19 = 0}$ with restrictions $x < 0$, $y > 0$, and $|x| + |y| = n$ when $\mathbf{Soffset}_n$ = 1

The fact that there are 2 unique lines on which the **Stopping Points** of **Stopping Class**$_8$ fall seems to be related to the fact that **Stopping Modulus**$_8$ $= 2$. This leads to the following conjecture.

Conjecture 1. *All **Stopping Points** belonging to **Stopping Class**$_k$ fall on one of **Stopping Modulus**$_k$ lines, without exception.*

In the next section, we'll get a better sense of how **Stopping Points** and **Stopping Signatures** relate to each other geometrically by exploring how they map into the Collatz Address Space.

5 Exploring The Collatz Address Space

We've already seen how we can map positive integers $n \geq 3$ from **Stopping Class**$_k$ to **Stopping Points** which are located on the 2D plane. One valid question we might ask is "Does every positive integer get mapped into **Collatz Address Space**? I've written python code that tests the first 10,000,000 numbers $n > 1$, and every number does map to a unique **Stopping Point**. Every even number also maps to a **Stopping Point**. Since every even integer has **Stopping Time** $= 1$, then the following holds true for all even numbers.

Lemma 1. *For all even numbers,* $\mathbf{StoppingPoint}_n = (-\frac{\mathbf{n}}{\mathbf{2}}, \frac{\mathbf{n}}{\mathbf{2}})$

Moreover, if the Collatz Conjecture is true, I assume the following to be true.

Conjecture 2. *Every number* $\mathbf{n} > \mathbf{1}$ *maps to a unique* ***Stopping Point****.*

5.1 Infinite Stopping Classes

In section 4 we explored three **Stopping Classes**: 3, 6, and 8. One might ask the question "How many unique **Stopping Classes** exist?" Since there seem to be an infinite amount of **Stopping Times**, this leads to my next set of conjectures. I'm not sure if they are obvious, which is why they're left as conjectures.

Conjecture 3. *There are an infinite number of unique* ***Stopping Classes****.*

Conjecture 4. ***Stopping Class***$_1$*,* ***Stopping Class***$_3$*, and* ***Stopping Class***$_6$ *are the only* ***Stopping Classes*** *with* ***Stopping Modulus*** $= 1$

Conjecture 5. *For every* $\mathbf{k} \geq \mathbf{8}$*,* ***Stopping Class***$_k$ *has a unique* ***Stopping Modulus*** ≥ 2*.*

Conjecture 6. *For every* $\mathbf{j} \geq \mathbf{8}$*, if* $\mathbf{k} > \mathbf{j}$ *then* ***Stopping Modulus***$_k$ $>$ ***Stopping Modulus***$_j$ *.*

5.2 Understanding the Collatz Address Space via Analogy

In section 3, the definition of **Collatz Address Space** calls out how it is comparable to $\mathbb{R}^3$. In this section I will expand upon this idea and show how **Stopping Points** and **Stopping Signatures** map to this space. We will compare the **Collatz Address Space**, to a more common addressing system - the street addressing system used for United State's mail delivery. The analogy is not exact, but it does help to get a better intuition about what kind of information the **Stopping Signatures** and **Stopping Point** contain.

We will assume you live in the United States, so we can omit the country. Let's say you want to send an address to your friend Alice. Alice lives with her husband Bob at the following address:

137 Prime Drive,
Indianapolis, Indiana 46077

Let's classify parts of this address from most-to-least descriptive and see how it is analogous to the corresponding part of a **Stopping Signature**$_n$ (**Stopping Time**$_n$, **Stopping Page**$_n$, **Stopping Offset**$_n$) and **Stopping Pount**$_n$ **(x, y)**.

- Indiana - This is the state. It narrows down all addresses to a fairly wide geographical area. This is analogous to the **Stopping Time**$_n$
- Indianapolis, 46077 - This is the city and zip code. Again, it narrows down all addresses a bit further than the State level. This is analogous to the **Stopping Page**$_n$. We can think of all n who share **Stopping Time** and **Stopping Page** as living in some sense closer in proximity than two integers that don't share these properties.
- Prime Drive - This is the street. At this level we can start to talk about n even more precisely. This is analogous to the **Stopping Offset**$_n$. Remember that a "city" (**Stopping Class**) can only have so many "streets" (**Stopping Modulus**). An integer is *required* to live on one of these streets. So not only can we talk about n, but we can talk about n's neighbors.
- 137 - This is the house number, and it turns out to be the most precise part of the address. This is analogous to the **Stopping Point**. We can think of an integer as living at the house specified at **Stopping Point (x, y)**.

I hope this section gives the reader a somewhat better way to reason about how **Collatz Address Space** functions. In the next section we will pick a couple integers and visually inspect where they are located in Collatz Space.

5.3 Visualizing Collatz Address Space

Continuing to build off our example in section 4.4, below is a visualization of how the the first 6 entries in Table 5 appear geometrically in **Collatz Address Space**.

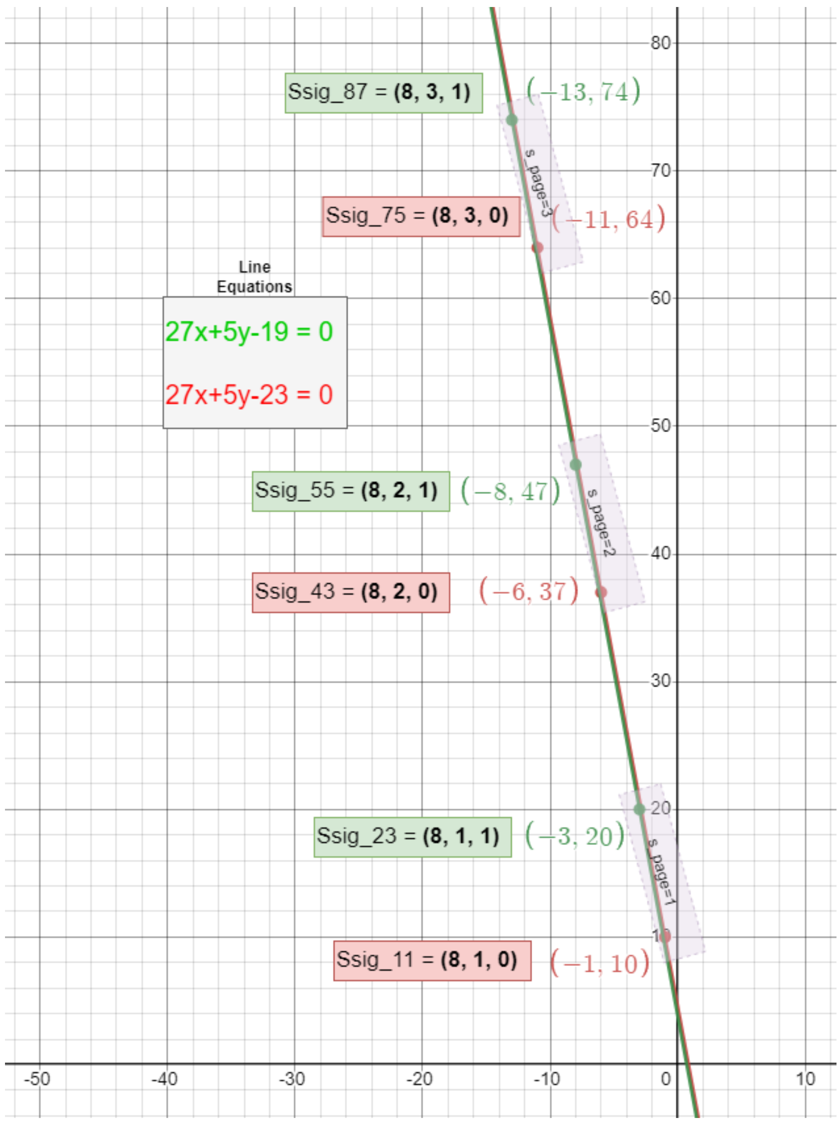

Figure 5: Select **Stopping Points** from **Stopping Class**$_8$

You can start to see how our "postal address" analogy doesn't quite convey exactly what is going on, but it is a good system of comparison.

- You could consider the "state" to be the set of all **Stopping Lines** in a given **Stopping Class**. In this case there are two: the red line and the green line.
- You could consider the "cities" to be be roughly defined by the opacent purple rectangles. These are technically the **Stopping Pages**, and there are an infinite amount of them laid out periodically as you travel up these lines out to infinity. A question of interest is if we could find a rigorous definition for city boundaries.

- The "streets" turn out to be the **Stopping Lines** if you consider them individually. The interesting thing about the streets is they seem to extend forever! There appears to be an infinite number of houses on each street. This is vaguely reminiscent of Hilbert's infinite hotel.

5.4 Slopes of Stopping Lines

Empirical evidence seems to indicate that for all positive integers $j > 1$ having **Stopping Time**$_k$, all $StoppingPoints$ in **Stopping Class**$_k$ fall on lines that have the same slope. Below you'll find a table of information on the first 10 **Stopping Classes**, including the slopes of the lines passing through all points in **Stopping Class**$_k$

$\mathbf{stop_k}$	$\mathbf{Smod}_k$	Slope
1	1	-1
3	1	-3
6	1	$-\frac{9}{7}$
8	2	$-\frac{27}{5}$
11	3	$-\frac{81}{47}$
13	7	$-\frac{243}{13}$
16	12	$-\frac{729}{295}$
19	30	$-\frac{2187}{1909}$
21	85	$-\frac{6561}{1631}$

Table 6: Slopes of lines passing through $\mathbf{StoppingPoints}$ belonging to **Stopping Class**$_k$

There are few a interesting observations to point out from this table.

1. The $\mathbf{stop_k}$ column is a well known sequence: A122437.
2. The $\mathbf{Smod_k}$ column is also a well known sequence: A100982.
3. The numbers appearing in the **slope** column are a combination of well known sequences.
   - The numerators are simply powers of 3 A000244.
   - The denominators seem to be the difference between and the next larger or equal power of 3^n and 2. from this series A063003
   - All terms in the sequence appear to be negative.

It is pretty well known that the sequences in items 1 and 2 are related to the Collatz Conjecture. However, I have not come across any literature stating direct connections between the sequence appearing in item 3. This seems to imply there is some relationship connecting **Stopping Classes** and the gaps between the powers of 2 and 3 to the Collatz Conjecutre.

It's fairly obvious to see how powers of 2 effect the Collatz Conjecture. When a power of two turns up within an orbit, the orbit falls directly to 1 in $log_2(n)$ steps. But powers of 3 show no obvious pattern as far as I've explored.

5.5 Discretizing The Slope

Table 6 shows the sequence made from the slopes for the first 10 **Stopping Classes**. There are well known formulas for the two sequences, but none that are precise. To avoid any heuristic arguments, we want to avoid having to use any rounding functions like *floor*(n) or *ceil*(n). It turns out, there is a way to compute this sequence discretely in the following manner.

1. Let $l =$ the number of digits in the base 2 representation of 3^n.
2. Then $a_n = 3^n - 2^l$

Remarkably, this generates the sequence of denominators of the slopes. It's a clue that there may be a deep connection between the geometry described by **Stopping Points**. This could perhaps open the Collatz Conjecture up to being studied by the beautiful field of algebraic geometry.

In the next section, we'll explore the upper and lower bounds of this slopes of lines through **Stopping Classes**

5.6 Finding an Upper Bound for the value of slopes representing Stopping Classes

According to **Lemma 1**, we can pretty easily see that all stopping points for even numbers must fall on the line $\mathbf{y} = -\mathbf{x}$. In the figure below, you'll see the **Stopping Points** for 2, 4, 6, 8, and 10, and the line **y = -x**.

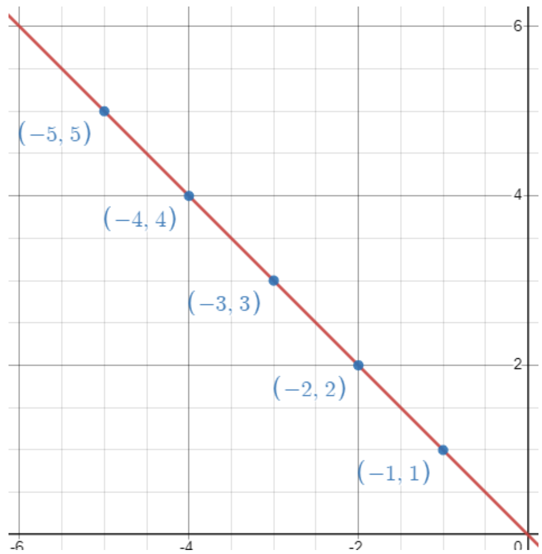

Figure 6: **Stopping Points** for 2, 4, 6, 8, and 10

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In fact, the slope of the line intersecting the **Stopping Points** of even numbers must have the maximum slope allowed for any line representing **Stopping Class**$_k$. Remember, these points are found by computing the **Stopping Destination**$_n$ if it exists, which must be lower than n by definition. The only way to reach a number lower than n (as per the rules of the Collatz Function), is to divide by two. Since you can only divide by two when you encounter an even number, this means **Stopping Destination**$_n$ can only be reached after a "divide by two" operation. This leads to the following observation:

Lemma 2. *The maximum slope for a line representing* ***Stopping Class***$_k$ *is -1 and belongs to the line passing through the* ***Stopping Points*** *of the even numbers.*

5.7 Finding a Lower Bound for the value of slopes representing Stopping Classes

As n increases, intuitively you can think of the the lines as both getting steeper (having a higher negative magnitude value for slope) and moving higher up the y-axis (increasing the value of the y intercept). It's almost like there's a translation of slide + rotate occurring. See Figure 6 for the first 3 **Stopping Point** Lines.

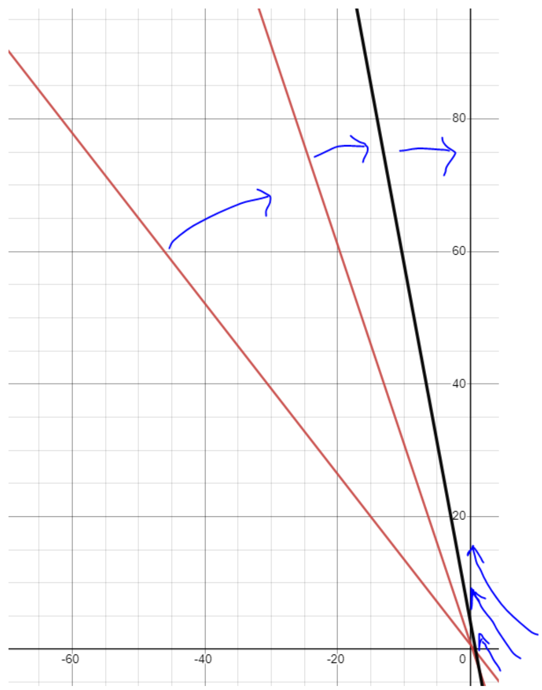

Figure 7: Plot of first 3 **Stopping Points** lines.

If **Conjecture 2** turns out to be true and there are indeed an infinite number of **Stopping Classes**, then it may follow that there are an infinite number of lines that describe their **Stopping Points**! As n grows, we should expect these lines to have decreasing slopes, but higher y-intercepts. As n increases, we're getting closer and closer to mapping $\mathbf{y} = -\mathbf{x}$ onto $\mathbf{x} = \mathbf{0}$ via a rotation about the origin. Perhaps we can employ some tools from calculus or topology to make some definitive statements about the convergence of these two lines.

In the next section we will turn to investigating the **Stopping Points** and see an interesting way we can map each number n to a unique circle on the 2D plane. This is where an unlikely number shows up - the Golden Ratio.

6 Mapping Orbits to Circles

Table 1 shows **Stopping Points** of first 16 odd numbers $\geq n$. I found an interesting property that seems to hold for all **Stopping Point**$_n$ where $n \geq 3$. These points all appear to lie on their own unique circles located in **Collatz Address Space**.

Each circle **Stopping Circle**$_n$ seems to have the following properties.

1. It is centered at point $(\mathbf{x}, \mathbf{y})$ where $|\mathbf{x}| + |\mathbf{y}| = \frac{n}{2}$
2. It intersects 4 other points.
   (0, 0)

   ($\mathbf{SpointX}_n$, $\mathbf{SpointY}_n$)

   (0, $\mathbf{SpointY}_n$)

   ($\mathbf{SpointX}_n$, 0)
3. If you order the sequence of y-intercepts of **Stopping Class**$_n$ by n, you seem to get this sequence: A122437, which is the sequence built from following the trajectory of $2n + 1$ in the $3n + 1$ problem.
4. The circles existing in **Stopping Class**$_k$ seem to have a strong relationship amongst each other. To illustrate, see figure 7 which shows the first 8 circles in **Stopping Class**$_3$.

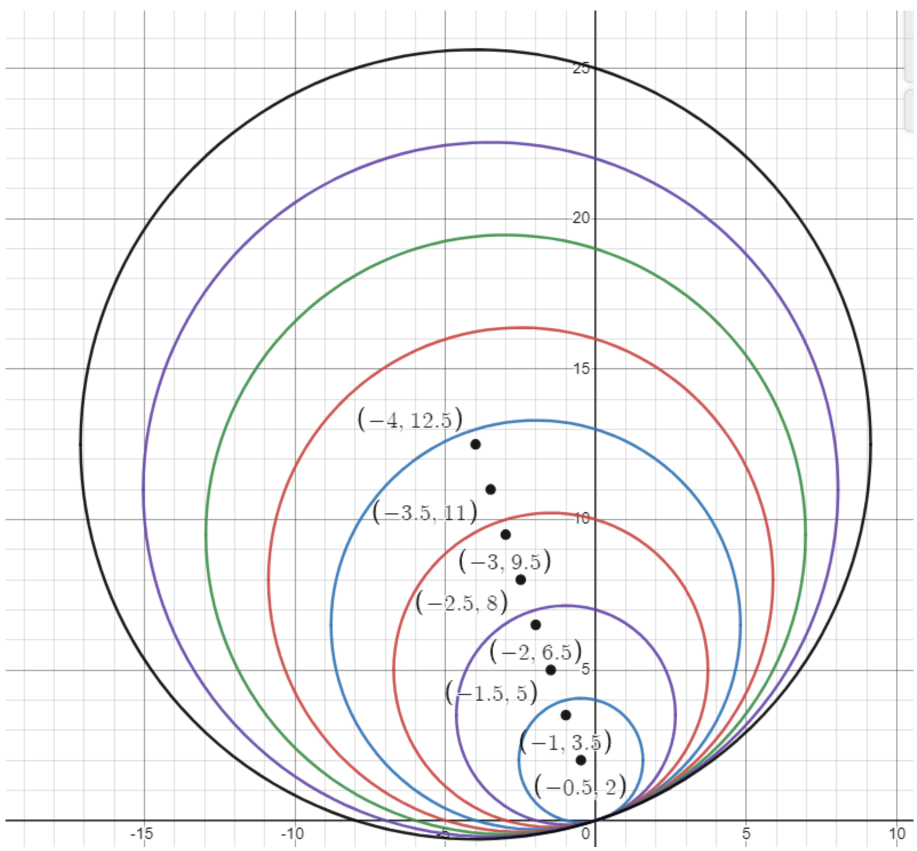

Figure 8: Equations of the first 8 **Stopping Circles** of **Stopping Class**$_3$ listed in table 8.

n	Equation
5	$(x+.5)^2+(y-2)^2=4.25$
9	$(x+1)^2+(y-3.5)^2=13.25$
13	$(x+1.5)^2+(y-5)^2=27.25$
17	$(x+2)^2+(y-6.5)^2=46.25$
21	$(x+2.5)^2+(y-8)^2=70.25$
25	$(x+3)^2+(y-9.5)^2=99.25$
29	$(x+3.5)^2+(y-11)^2=133.25$
33	$(x+5)^2+(y-12.5)^2=172.25$

Table 7: Equations of the circles in figure 7

5. If you parameterize the x coordinate of **Stopping Circle**$_3$, which is the smallest **Stopping Circle** and first odd prime value with a **Stopping Time**, you end up with the equation:

$$x = -.5 + \sqrt{\frac{5}{4}} \times \cos(\theta)$$

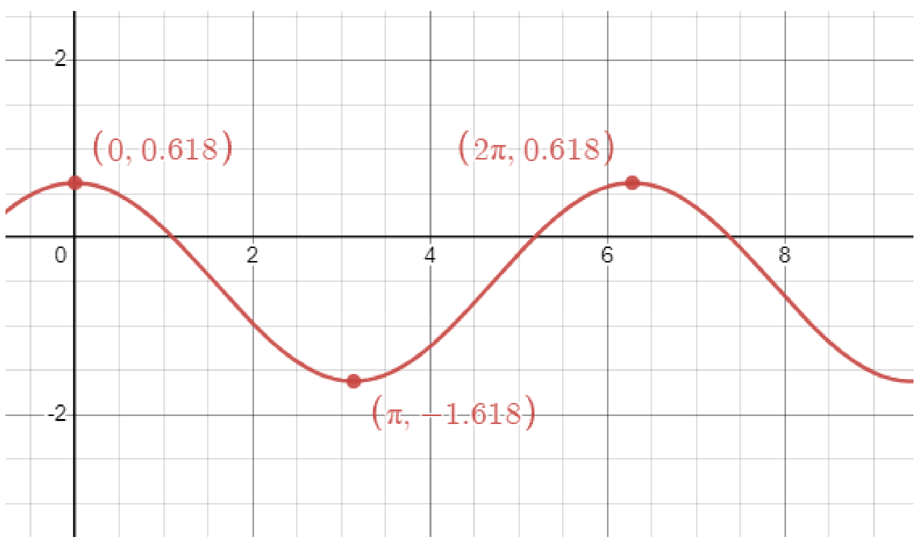

Figure 9: Parameterization of **Stopping Circle**$_3$ listed in table 8.

There are a few remarkable facts about this function related to the Golden Ratio $\phi \approx 1.618...$.

(a) The amplitude of the sin wave produce by this function is equal to $(2 \times \phi) - 1$

(b) The sin wave produced by this function oscillates between between a maximum value of $\phi - 1$ and a minimum value of $-\phi$.

This is such a beautiful connection between two of the most popularly known transcendental numbers. This link seems like it would be a very good explanation for some of the behaviors we see in the Collatz Function. I have my suspicions that we may be able to connect the Collatz Function to Penrose Tilings. Collatz Orbits seem to traverse the number line without hitting a number more than once, similarly to the way Penrose Tilings cover the plane without repeating any patterns.

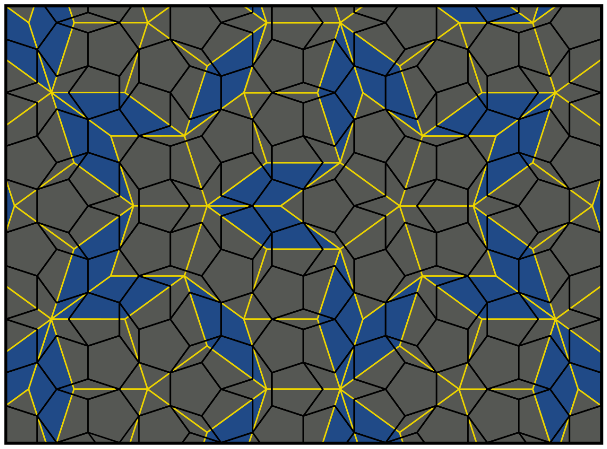

Figure 10: Example of the pentagonal Penrose tiling (P1)

7 Links to Pythagorean Triples and Quantum Mechanics

It is well known that you can create a Pythagorean Triple from a Gaussian integer by squaring it. A Gaussian integer is a complex number **a + bi** where **a** and **b** are integers. If we let $L = (a + bi)^2$ you produce a Pythagorean triple.

$$(L.Re)^2 + (L.Imag)^2 = c^2$$

It's easy to see that we can always construct a Gaussian integer from **Stopping Point**$_n$ by substituting x for a, and y for b. For example, if we take **Stopping Point**$_5 = (-1, 4)$, we can create the Gaussian integer $-\mathbf{1} + \mathbf{4i}$. We can then illustrate the following.

$$(-1 + 4i)^2 = (-15 - 8i)$$
$$(-15)^2 + (8)^2 = (17)^2$$

More generally, there seems to be a direct relationship the Pythagorean triple $a^2 + b^2 = c^2$ and the coordinates of **Stopping Point**$_n$.

$$a = x^2 - y^2$$
$$b = (-2) \times y \times x$$
$$c = x^2 + y^2$$

This is a fascinating relationship that can now be used to explore the **Collatz Address Space** from a more rigorous mathematical perspective. Using the geometry of **Stopping Points**, we can link each orbit to a Pythagorean Triple. Furthermore, we can use these triples as objects of study as we explore **Stopping Classes** and the relationships between them. In the next section we will explore a speculative link between this geometry and quantum mechanics. This requires a brief overview of the hydrogen spectral series.

7.1 Hydrogen Spectral Series and Stopping Class 3

The Hydrogen spectral series is incredibly important within the realm of quantum mechanics. I won't go into details about why this is so crucial to our understanding of quantum mechanics as I'm not an expert on the topic, but there is a great video overview of this topic on YouTube called The Hydrogen Spectral Series.

To give a brief overview, electrons can only exist at certain energy levels within an atom. The energy of an electron orbiting at level k must be a multiple of Planck's constant. Electron's increase their energy by absorbing a photon and moving to a higher energy state, they lose energy by emitting a photon and moving to a lower energy state. Below is a diagram help visually understand what's going on.

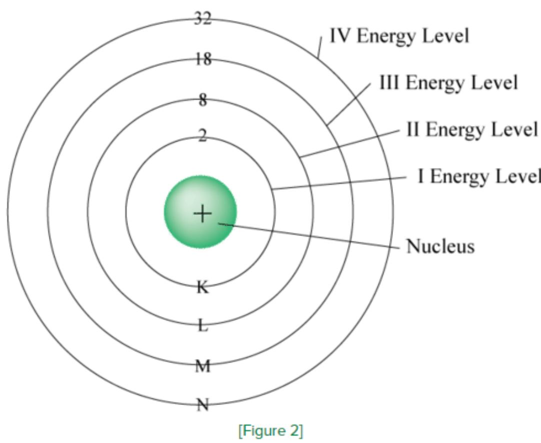

Figure 11: Energy Levels

Electrons in atoms are arranged in shells and subshells. The maximum number of electrons that can be accommodated in each principal energy level (shell) is $a_n = 2n^2$. These are the numbers shown in the diagram.

So when an electron moves from initial energy level n_i to a final lower energy level n_f, it emits a photon with energy ΔE. We can represent ΔE with a formula based on these two energy levels.

$$\Delta E = E_I \times (\frac{1}{{n_i}^2} - \frac{1}{{n_f}^2}) \tag{2}$$

We can ignore E_I for now as it's derived from fundamental physical constants. What we're really interested in is $(\frac{1}{{n_f}^2} - \frac{1}{{n_i}^2})$. We can plug in different initial and final energy levels to a number that represents the energy of the photon emitted when an electron falls to a lower energy state. To get an idea of what these values look like, we will create a table from the following formula.

$$a[n][k] = \frac{1}{(2n+1)^2} - \frac{1}{(2n+1+k)^2} \quad \forall n, k \in \{0, 1, 2, \ldots, n\} \tag{3}$$

According to entry A169603, the **numerators** of of the energies as they appear in the hydrogen spectrum can be represented by the formula for the triangle below.

$$T(n, k) = k \times (4n + k + 2)$$

0;										
0,	**7**;									
0,	11,	**24**;								
0,	15,	32,	**51**;							
0,	19,	40,	63,	**88**;						
0,	23,	48,	75,	104,	**135**;					
0,	27,	56,	87,	120,	155,	**192**;				
0,	31,	64,	99,	136,	175,	216,	**259**;			
0,	35,	72,	111,	152,	195,	240,	287,	**336**;		
0,	39,	80,	123,	168,	215,	264,	315,	368,	**423**;	
0,	43,	88,	135,	184,	235,	288,	343,	400,	459,	**520**;

Table 8: Numerators of $\frac{1}{(2n+1)^2} - \frac{1}{(2n+k+1)^2}$ as they appear in the energies of the hydrogen spectrum.

The diagonal of this table is defined by the series below.

$$a_n = n \times (2 + 5n) = 0, 7, 24, 51, 88...$$

It turns out that we can generate this sequence using the **Stopping Points** and norms of **Stopping Signatures** belonging to **Stopping Class**$_3$, which we will demonstrate in a later section. But first we will take a look at the non-reduced denominators generated by *equation 3*.

0;										
0,	**144**;									
0,	900,	**1225**;								
0,	3136,	3969,	**4900**;							
0,	8100,	9801,	11664,	**13689**;						
0,	17424,	20449,	23716,	27225,	**30976**;					
0,	33124,	38025,	43264,	48841,	54756,	**61009**;				
0,	57600,	65025,	72900,	81225,	90000,	1647243,	**108900**;			
0,	93636,	104329,	115600,	127449,	139876,	152881,	166464,	**180625**;		
0,	144400,	159201,	174724,	190969,	207936,	225625,	244036,	263169,	**283024**;	
0,	213444,	233289,	254016,	275625,	298116,	321489,	345744,	370881,	396900,	**423801**;

Table 9: Denominators of $\frac{1}{(2n+1)^2} - \frac{1}{(2n+k+1)^2}$ as they appear in the energies of the hydrogen spectrum.

Remarkably, the diagonal of this table seems to be related to the squares of pentagonal numbers! A100255. That is, you can generate the diagonal by plugging all odd numbers $n >= 3, 5, 7, 9...$ into the series below.

$$a_n = \frac{1}{4} \times n^2 \times (3n-1)^2$$

As far as I can tell, this is not a well known relation to the hydrogen spectrum series. The closest mentioning of this series I could find is A120073, but it does not specifically call out the relation to pentagonal numbers. It is well known that the Pentagon is related to the Golden Ratio, but I'm not sure if this connection has been made to the energies of the hydrogen spectrum series.

There is also much to be said about the columns of these tables. There seems to be a formula to generate each column. For example **Table 8, Column 1** = $4n+3$. **Table 9's** columns can also be generated in a similar way with a different formula for each column.

In an attempt to show how closely the two tables are related to the geometry described in section 4, I'll show how the diagonals in **Table 8** and **Table 9** can be derived from the **Stopping Points** and **Stopping Signatures** belonging to **Stopping Class**$_3$.

7.2 Geometric Norms of Stopping Class 3

$\mathbf{n}$	$\mathbf{Spoint}_n$	$\mathbf{Ssig}_n$
5	(-1, 4)	(3, 1, 0)
9	(-2, 7)	(3, 2, 0)
13	(-3, 10)	(3, 3, 0)
17	(-4, 13)	(3, 4, 0)
21	(-5, 16)	(3, 5, 0)
25	(-6, 19)	(3, 6, 0)
29	(-7, 22)	(3, 7, 0)
33	(-8, 25)	(3, 8, 0)

Table 10: First 8 numbers where $\mathbf{stop}_n = 3$

Table 10 shows the first 8 numbers in **Stopping Class**$_3$. We will use the $\mathbf{Spoint}_n$ column to generate a number t_n, which will be the Euclidean distance from **Stopping Point**$_n$ to the point $(-\frac{n}{2}, \frac{n}{2})$ and dividing by $\sqrt{2}$. We will then use the $\mathbf{Ssig}_n$ column to generate a number $t`_n$ by simply dividing the norm by $2\sqrt{2}$.

$$t_n = \frac{\sqrt{(x_n + \frac{n}{2})^2 + (y_n - \frac{n}{2})^2}}{\sqrt{2}} \tag{4}$$

$$t'_n = \frac{|Ssig_n|}{2\sqrt{2}} \tag{5}$$

If then compute $t'_n - t_n$ we can see how it appears to generate the diagonal in **Table 8**. See the table below for results.

$\mathbf{n}$	$\mathbf{t}_n$	$\mathbf{t`}_n$	$\mathbf{t`}_n - t_n$
5	1.5	8.5	7
9	2.5	26.5	24
13	3.5	54.5	51
17	4.5	92.5	88
21	5.5	140.5	135
25	6.5	198.5	192
29	7.5	266.5	259
33	8.5	344.5	336

Table 11: $t`_n - t_n$ column is the diagonal of **Table 9**

The Diagonal in **Table 10** can be generated via $a[i] = ((x_n + y_n) \times (x_n + n_i))^2$ where i is the number at index i in the ordered list of all elements in **Stopping Class**$_3$ and (x_n, y_n) are the coordinates of **Stopping Point**$_n$.

At last, I have introduced all of the concepts needed to make a conjecture about the existence of a new kind of number that exists in Collatz Address Space called a **Cam** number.

8 Conclusion - The Pursuit of a theory of Cam Numbers

In this paper I've tried to show how we might be able to examine the Collatz Conjecture from a geometric perspective. Though most of the ideas in this paper a simple conjectures, I've empirically tested most of them up to the first 1 million integers. I can't help but think all of the pieces to either prove or disprove the Collatz Conjecture (and maybe even others) are related to the geometry described by the constructions laid out in this paper.

In conclusion, I believe what will be needed to finally prove or disprove the Collatz Conjecture is a new type of number called a "Cam" number. "Cam" stands for "complex and massive", indicating that the number has properties that are related to both scalars and complex numbers. A full blown theory of Cam numbers should might be able to answer the following questions about the geometry discussed in this paper.

1. Why do **Stopping Classes** take on the specific values we see? 1,3,6,8....
2. Why does every integer > 1 seem to have a **Stopping Point**?.
3. What are the significance of the slopes and y-intercepts of the **Stopping Lines**?.
4. Why do **Stopping Moduli** take on the specific values we see?
5. Why do the **Stopping Lines** have the specific slope values we see?
6. What is the relationship between the n and the members of **Stopping Orbit**$_n$

9 Select References

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