A proof of a conjecture about the Riemann zeta-function at even integers

Hervé G.

September 13, 2023

Recently, Jean-Christophe PAIN have stated the conjecture [1] : for $p \geq 1$, an integer,

$$\frac{1}{\zeta(2p+2)} \int_{-\infty}^{+\infty} \frac{x^{2p} \ln\left(1+\mathrm{e}^{x}\right)}{1+\mathrm{e}^{x}} dx$$

is a rational number.

Proof :

$$\begin{aligned}
J_p &= \int_{-\infty}^{+\infty} \frac{x^{2p} \ln\left(1+\mathrm{e}^{x}\right)}{1+\mathrm{e}^{x}} dx \\
&= \int_{0}^{+\infty} \left(\frac{x^{2p} \ln\left(1+\mathrm{e}^{-x}\right)}{1+\mathrm{e}^{-x}} + \frac{x^{2p} \ln\left(1+\mathrm{e}^{x}\right)}{1+\mathrm{e}^{x}} \right) dx \\
&\overset{u=\mathrm{e}^{-x}}{=} \int_0^1 \frac{\ln(1+u)\ln^{2p} u}{u(1+u)} du + \int_0^1 \frac{\ln\left(1+\frac{1}{u}\right)\ln^{2p} u}{u\left(1+\frac{1}{u}\right)} du \\
&= \left(\int_0^1 \frac{\ln(1+u)\ln^{2p} u}{u} du - \int_0^1 \frac{\ln(1+u)\ln^{2p} u}{1+u} du \right) + \\
&\left(\int_0^1 \frac{\ln(1+u)\ln^{2p} u}{1+u} du - \int_0^1 \frac{\ln^{2p+1} u}{1+u} du \right) \\
&= \underbrace{\int_0^1 \frac{\ln(1+u)\ln^{2p} u}{u} du}_{\text{IBP}} - \int_0^1 \frac{\ln^{2p+1} u}{1+u} du \\
&= -\frac{1}{2p+1} \int_0^1 \frac{\ln^{2p+1} u}{1+u} du - \int_0^1 \frac{\ln^{2p+1} u}{1+u} du = -\frac{2(p+1)}{2p+1} \int_0^1 \frac{\ln^{2p+1} u}{1+u} du \\
&= -\frac{2(p+1)}{2p+1} \left(\int_0^1 \frac{\ln^{2p+1} u}{1-u} du - \underbrace{\int_0^1 \frac{2u \ln^{2p+1} u}{1-u^2} du}_{z=u^2} \right)
\end{aligned}$$

$$
\begin{aligned}
J_p &= \frac{(p+1)\left(1-2^{2p+1}\right)}{(2p+1)2^{2p}} \int_0^1 \frac{\ln^{2p+1} u}{1-u} du \\
&= \boxed{\frac{(p+1)\left(2^{2p+1}-1\right)(2p+1)!}{(2p+1)2^{2p}} \zeta(2p+2)}
\end{aligned}
$$

NB : I assume, for $r \geq 1$ integer :

$$
\int_0^1 \frac{\ln^r u}{1-u} du = (-1)^r r! \zeta(r+1)
$$

Références

[1] Jean-Christophe PAIN, An integral representation for $\zeta(4)$, https://arxiv.org/pdf/2309.00539