On Bifurcations and Beauty

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Abstract

This paper focuses on two ideas: the beginning focuses on standard and chaotic bifurcations, and the end focuses on beauty through mathematical coincidences. The scope of the bifurcation side is ambitious: relating bifurcation theory not only to the logistic map but also to prime spirals, the Riemann hypothesis, the Lambert W function, the Collatz conjecture, the Mandelbrot set, and music theory. The scope of the beauty side is similar: a proposed sequence that is opposite to the primes in some sense, finite sequences with peculiar properties, Fibonacci-like sequences, trees of primitive Pythagorean triples, Babylonian math, Grimm's conjecture, and Shell sort. Rather than providing rigorous analysis, my goal is to revitalize qualitative mathematics.

Contents

Foreword

1.1 Introduction

Notably, there is almost no mathematical rigor in this work, so virtually every novel idea presented in this paper is a conjecture, a few of which are backed up by empirical evidence. I consider myself a recreational mathematician, and my methods tend to be qualitative; I agree there is some misfortune here, but there is also opportunity. A side effect of the qualitative analysis I do is that many of the statements I make involve very simple math (some might argue vacuous math), and I actively encourage the reader to doubt that any creative ideas have been presented.

I anticipate many of these conjectures will be wrong, so reading this paper with a healthy dose of skepticism is recommended. To me, skepticism is not necessarily pessimistic; it is a tool like rope or breadcrumbs that allows for backtracking when we realize we may have the wrong bearing.

1.2 Terminology

Quotations are used in this paper to coin a novel term, if no surrounding source is given.

1.3 Quick Preview

Since this paper is long, I recommend the following three sections to gauge relevance:

(2.3.4) A Different Formula for The Riemann Zeta Function, the only rigor-ish part.
(3.4.1) The 12-Tone Octave, for those interested in music theory.
(4.1) "Hyperstructured Numbers", for those interested in primes.

Bifurcations

Bifurcations are a mechanism by which one outcome splits into two outcomes. Bifurcations without a "chaotic bifurcation" qualifier, in the context of this work, refer to bifurcations that have two equally probable outcomes after the split.

2.1 The Logistic Map

The logistic map is defined by the recurrence relation: $x_{n+1} = r \cdot x_n \cdot (1 - x_n)$. After several iterations, values produce a graph that looks like:

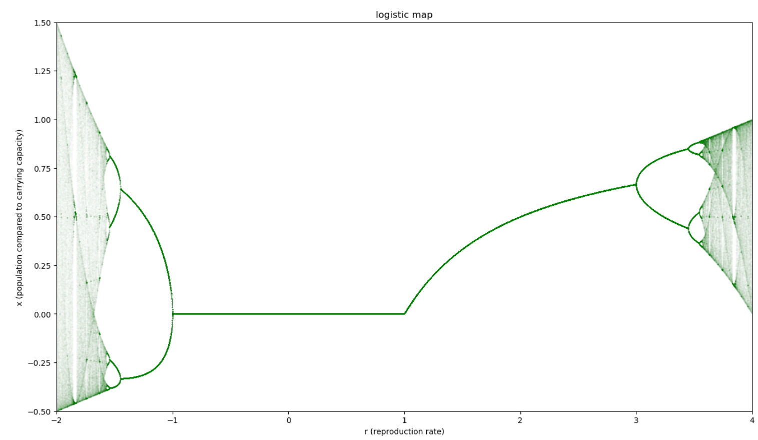

Figure 2.1: The logistic map, defined by the recurrence relation $x_{n+1} = r \cdot x_n \cdot (1 - x_n)$ as it evolves over time.

The logistic map is one of the simplest and best-known examples of period-doubling bifurcation, and it is useful because it can be used in population modeling, where r is the reproduction rate and x_n is the population at a given time.

A point I want to make about the logistic map is that it has a confusing definition where the convention is to place r (the reproduction rate) along the x-axis, and x (the population) along the y-axis, so it is best to use the terms horizontal and vertical axis to avoid ambiguity.

The logistic map has six important equations that I am aware of, and I will cover them in pairs. The simplest two are the equations that define two thirds of all values with symmetry around $r = 1$, and their equations are $x = 0$ and $x = \frac{r-1}{r}$:

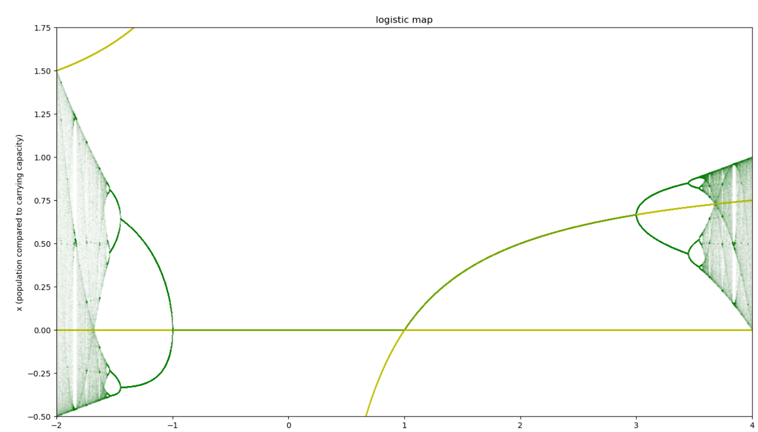

Figure 2.2: The logistic map in green with the initial curves $x = 0$ and $x = \frac{r-1}{r}$ in yellow.

The next two are the exact solutions for the first bifurcations, which are valid between $r = [1-\sqrt{6}, -1]$ and $r = [3, 1+\sqrt{6}]$, and their equations are $x = \frac{(r+1)\pm\sqrt{(r+1)(r-3)}}{2r}$, which can notably be rewritten as a quadratic equation with $a = r$, $b = r+1$, $c = 1 + \frac{1}{r}$:

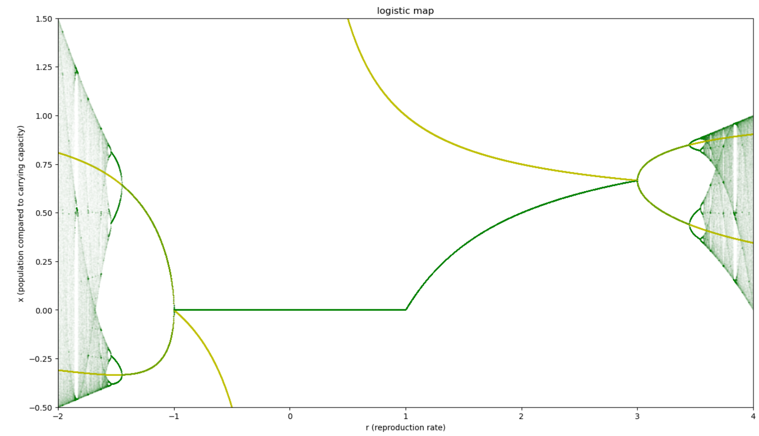

Figure 2.3: The logistic map in green with the first bifurcation $x = \frac{(r+1)\pm\sqrt{(r+1)(r-3)}}{2r}$ in yellow.

After the first bifurcation, you have more period-doubling bifurcations that split from 2 to 4 to 8 to 16 and so on. Some geometric ratios approximating how frequently these bifurcations occur are the Feigenbaum constants (i.e. 4.669201... and 2.502907...). Eventually these splits get so small that they cannot be seen, hitting what is known as either the accumulation point or the onset of chaos, depending on the literature. From this point onward, it is more meaningful to talk about the logistic map in terms of statistical likelihood, except for sparse pockets of order known as islands of stability.

The final two equations are the min and max values after the onset of chaos. These equations are valid between $r = [-2, -1.56994567]$ and $r = [3.56994567, 4]$ and are defined by the linear $x = \frac{r}{4}$ and cubic $x = r \cdot \frac{r}{4} \cdot (1 - \frac{r}{4})$:

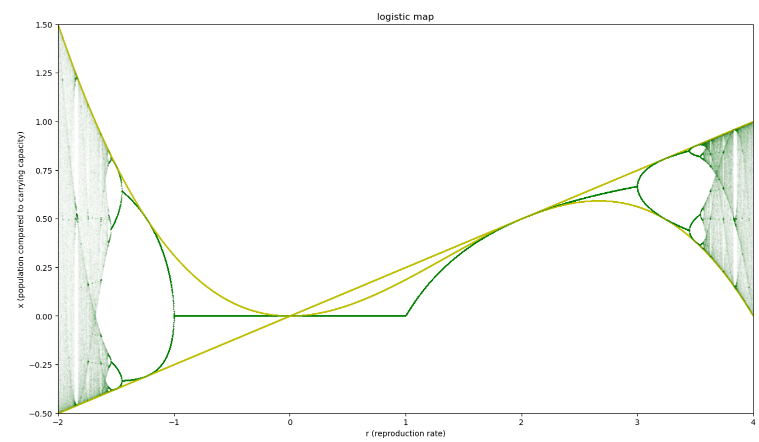

Figure 2.4: The logistic map in green with the linear boundary $x = \frac{r}{4}$ and cubic boundary $x = r \cdot \frac{r}{4} \cdot (1 - \frac{r}{4})$ in yellow.

Beyond the interval $[-2, +4]$, the logistic map almost always tends to negative infinity, with the exception of a Cantor set outside the interval $[1 - \sqrt{6}, 1 + \sqrt{6}]$.

Additionally, the logistic map has known solutions for 4 cases: $r = -2$[1]$, 0, +2$[2]$, +4$[3], with $r = 0$ being trivial.

The following picture is a conjecture based on the logistic map:

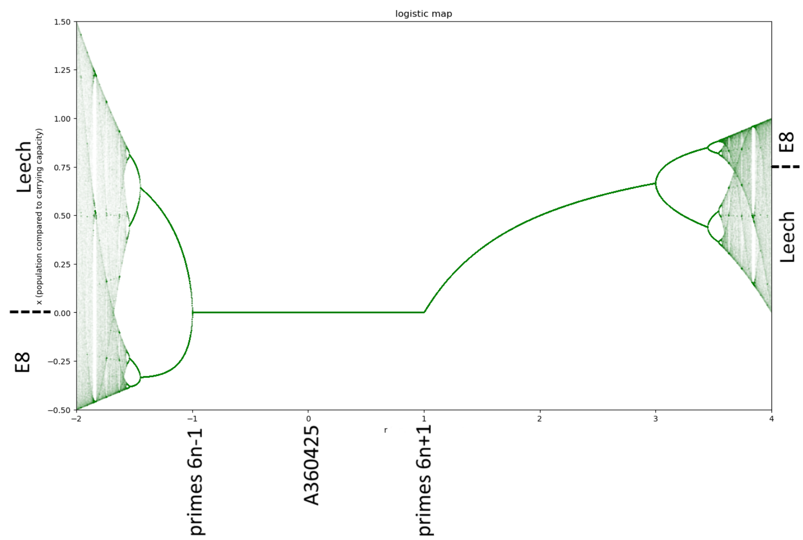

Figure 2.5: Conjectures about the logistic map regarding primes and "hyperstructured numbers" (4.1) on the horizontal axis and the E8 and Leech lattices on the vertical axis.

Where I essentially conjecture the logistic map is related to both prime numbers and "hyperstructured numbers" (4.1) on the horizontal axis, and E8 and Leech lattices on the vertical axis. The rationale for the primes is that they are almost always congruent to ± 1 mod 6, whereas "hyperstructured numbers" are conjectured to be almost always congruent to 0 mod 6. With respect to the E8 and Leech lattices at $r = -2$ and $r = +4$, there are precise solutions at those two locations on the logistic map. Finally, the E8 lattice is 8-dimensional and the Leech lattice is 24-dimensional, and the population range is split by $x = 0$ and $x = \frac{r-1}{r}$ at a 1:3 ratio.

2.1.1 The "Ideal Bifurcation"

Most attention on the logistic map is focused on the positive region r in $[0, 4]$, but the logistic map is well-defined for the region $[-2, +4]$. It is because of this I think one important detail on the negative side is glossed over: the bifurcation at $r = -1$ is unusual because it has a line bifurcate exactly 90 degrees into two curves:

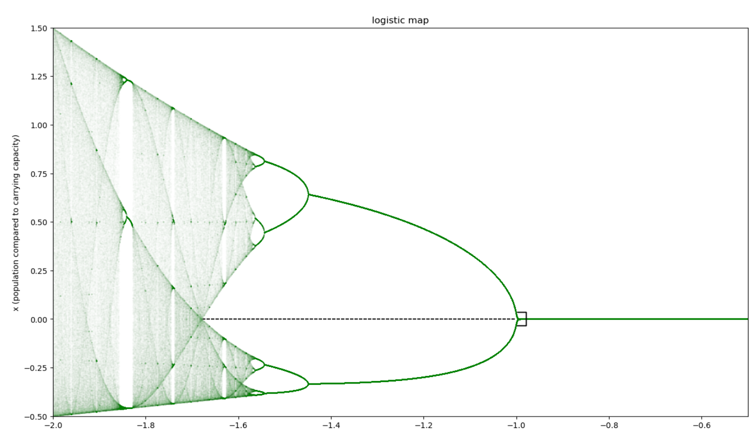

Figure 2.6: The "ideal bifurcation" at $r = -1$. If enough precision is allowed, the angles are 90 degrees in infinitesimal terms. A dashed (imaginary) line shows the path if the bifurcation never happened.

I have not seen this among other bifurcations in the logistic map. At first this case might seem like a lucky coincidence without consequence, but I believe it is somewhat likely that because of this fact we get the formula for the first bifurcation:

$$x = \frac{(r+1) \pm \sqrt{(r+1)(r-3)}}{2r}$$

Where notably the $(r + 1)$ term can be taken as the location of this particular bifurcation along the negative axis, with the $(r - 3)$ term representing the symmetric side of the bifurcation along the positive axis. Thus, I am arguing that the negative side of the logistic map has the most fundamental bifurcation in the whole dynamical system. This may imply the primes congruent to -1 mod 6 are more primitive in some sense, which may relate to "serendipitous numbers" (4.2.7).

Based on the bifurcation at $r = -1$, I define an "ideal bifurcation" as a function that splits into two orthogonal functions after a pole, where you can also consider an imaginary line corresponding to the original function if the bifurcation had not happened. An "ideal bifurcation" should have all four curves at 90 degree angles in some geometric sense.

Another distinguishing factor is that information should be split 50:50, such that both orthogonal functions have the same density. This 50:50 split can be seen by the alternating nature of the logistic map after the first bifurcation. When you plug in a value, it jumps from the first solution to the second solution in an alternating fashion such that neither case is more probable.

2.1.2 "Singularity Points"

Yoshimoto and Nishizawa (1991) introduced an important value in the logistic map: "the only real root of the equation $\lambda^3 - 2\lambda^2 - 4\lambda - 8 = 0$.... [where] two chaotic bands have merged into a single band"[4], which has the solution $\lambda = 2 \cdot \frac{1+\sqrt[3]{19-3\sqrt{33}}+\sqrt[3]{19+3\sqrt{33}}}{3}$ or $r = 3.678573510428322...$ or twice the tribonacci constant:

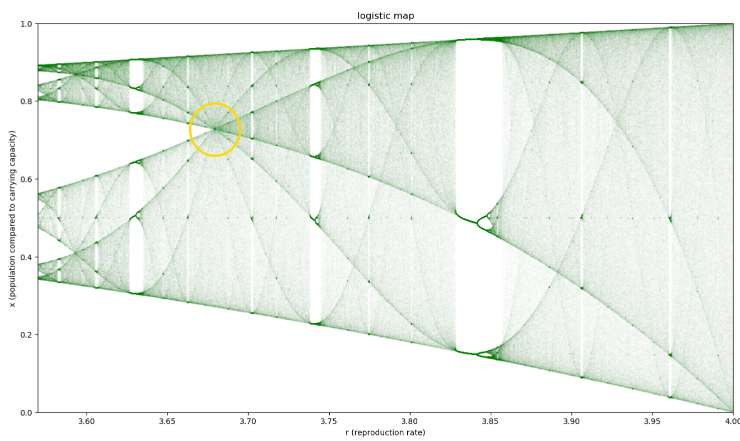

Figure 2.7: The most prominent "singularity point" at r=3.678573510428322...

If you plug this point into $x = \frac{r-1}{r}$ you can also get the vertical position of the point, and it has the interesting property that it is an unstable equilibrium (a 1-cycle) if the exact r and x values are used.

Looking at the graph of the logistic map between the onset of chaos and this point, an infinite number of similar points can be found, doubling similarly to the bifurcation points in the logistic map:

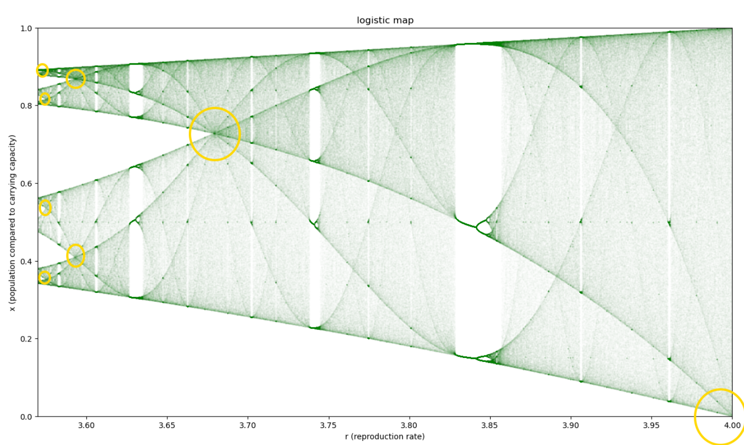

Figure 2.8: The most visible "singularity points" on the positive axis. They are harder to see as they get closer to the onset of chaos. There is also a point on the bottom right.

I call these points "singularity points", and they should be the location where a bifurcation recombines as if it had never bifurcated, which is a median case in some sense. For this reason, bifurcation points and "singularity points" should have a 1:1 correspondence, at least within the logistic map.

Here is a table of a few "singularity points" I chose to document, as well as their corresponding bifurcation points:

Bifurcation	r	x values	x values	r	"Singularity"
0th	1	0	0	4	0th
1st	3	$\frac{2}{3}$	.728155	3.6785735	1st
2nd	$1+\sqrt{6}$	.439960, .849937	.409580, .868771	3.5925721	2nd
3rd	3.5440903	.363290, .523594, .819785, .884049	.352763, .536272, .816204, .888997	3.5748049	3rd
4th	3.5644072	.346764, .374767, .490608, .554272, .807405, .835200, .880603, .890787	.344025, .375844, .485503, .558654, .805871, .837701, .880460, .891996	3.5709859	4th
5th	3.5687594	.343409, .347946, .367903, .379693, .478338, .503750, .549942, .560899, .804681, .809679, .829916, .840537, .878954, .883289, .890515, .892140	.342753, .347897, .366729, .380244, .476568, .505792, .549568, .562028, .804263, .809945, .829132, .841341, .878805, .883770, .890581, .892422	3.5701684	5th

Where notably, the x values are listed in sorted order rather than their cycling order. Additionally, the r values asymptotically approach the onset of chaos (i.e. 3.56994567) on both the bifurcation side and the singularity side.

In the same way bifurcation points cycle in predictable orders, "singularity points" should be no different, cycling in an order that is non-chaotic (albeit an unstable equilibrium).

2.1.3 Evens and Odds within Chaos

"Singularity points" are not only interesting because they can be used to create geometric ratios like the Feigenbaum constants, they are also useful because of qualitative properties. One of these properties is the "tendrils" that spread out from a "singularity point" in multiple directions. These curves that connect to the "singularity points" have very peculiar properties that will be covered here.

The "tendrils" are highlighted here:

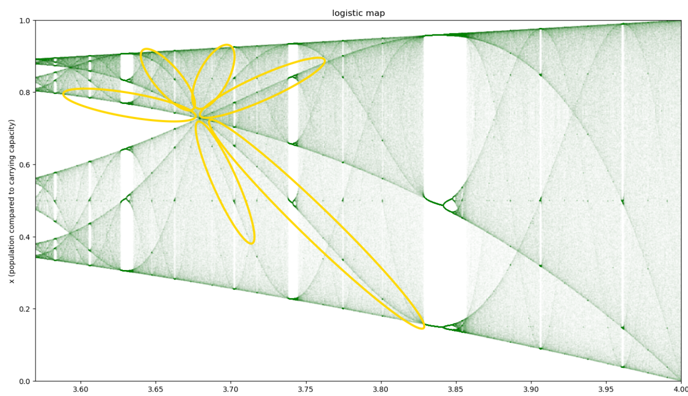

Figure 2.9: Some of the "tendrils" circled. A "tendril" refers to the entire visible curve, not just the part circled.

Where notably there should be an infinite number of "tendrils". "Tendrils" also have geometric properties like their angles, but the most useful qualitative property of the "tendril" is the maximal or minimal point at the end of its upwards or downwards trajectory, shown here:

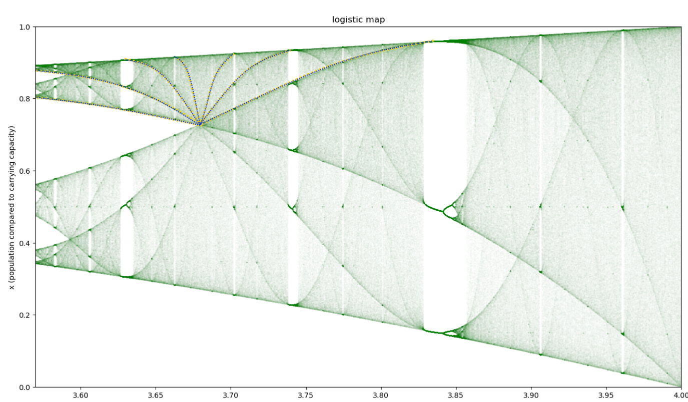

Figure 2.10: The paths the upper "tendrils" take and the points they reach a maximal height.

At first, it might seem like there is no information given by these "tendrils", but if you check the cycle length at the maximal (or minimal) point you find a curious trend:

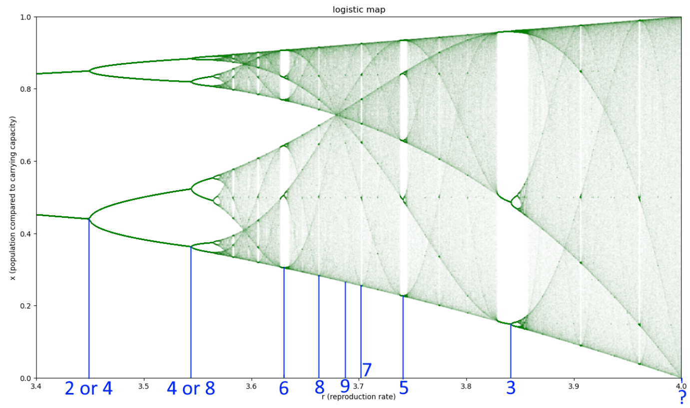

Figure 2.11: The cycle length at the minimal point reached by the most visible "tendrils". Note where evens and odds are relative to the "singularity point".

Where, notably, the left-hand side seems to list even integers (i.e. 2, 4, 6, 8...), and the right-hand side seems to list odd integers (i.e. 1, 3, 5, 7, 9...), although how to formalize this concept is not immediately clear.

This leads to a clever redefinition where instead of going to the absolute maximal or minimal point, we instead get the "tendril" as close as possible to the envelope defined by the two curves that represent the extrema after the onset of chaos, $x = \frac{r}{4}$ and $x = r \cdot \frac{r}{4} \cdot (1 - \frac{r}{4})$.

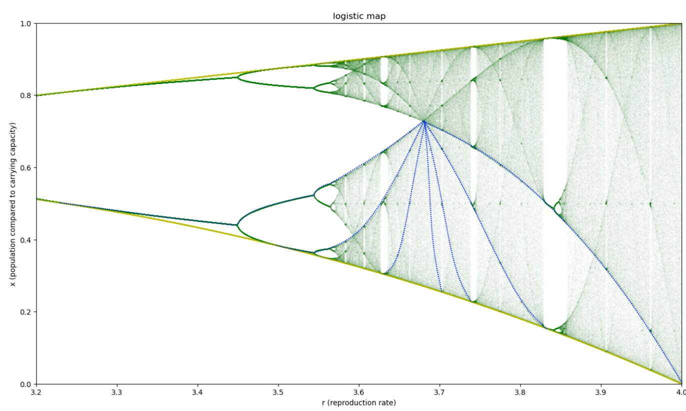

Figure 2.12: The paths of the "tendrils" when continued until reaching the extrema of the envelope $x = r \cdot \frac{r}{4} \cdot (1 - \frac{r}{4})$.

This leads to the following points:

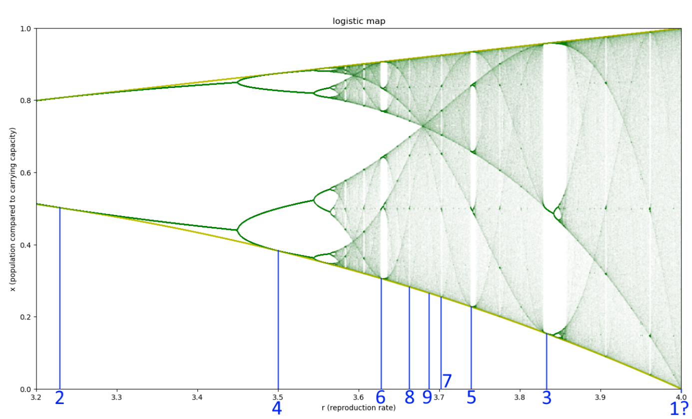

Figure 2.13: The cycle length of visible "tendrils" when they reach the extrema of $x = r \cdot \frac{r}{4} \cdot (1 - \frac{r}{4})$. The evens and odds are more clearly defined.

This leads me to the definition for this table, where: "singular" refers to the canonical odds and evens; "extrema" refers to the first intersection with the min/max envelope; and "non-singular" refers to the point that is double or half the cycle length of the "singular" column:

Even	"Singular"	"Extrema"	"Non-singular"	"Singular"	"Extrema"	"Non-singular"	Odd
2	3	2	1	4	∞	-2	1
4	$1+\sqrt{6}$	$1+\sqrt{5}$	3	$1+\sqrt{8}$	3.8318740	3.8463508	3
6	3.6265531	3.6275575	3.6387789	3.7381723	3.7389149	3.7440164	5
8	3.6621089	3.6621925	3.6625272	3.7016407	3.7017691	3.7024588	7
10	3.6729992	3.6730082	3.6730590	3.6871968	3.6872161	3.6872999	9
12	3.6766342	3.6766352	3.6766405	3.6817160	3.6817186	3.6817284	11

Where if the $(4, \infty, -2)$ triple is correct, then it is arguably the most interesting, since values in that range diverge to $-\infty$, which is outside the envelope of $x = \frac{r}{4}$ and $x = r \cdot \frac{r}{4} \cdot (1 - \frac{r}{4})$ for finite r.

Additionally, geometric ratios can be constructed like:

12) $(\frac{3.6766342-3.6766352}{3.6766352-3.6766405}) \approx \frac{10}{53}$
11) $(\frac{3.6817160-3.6817186}{3.6817186-3.6817284}) \approx \frac{13}{49}$

The code that generates the table (except cycles 1,2,4) can be found here, although the individual calls are still manual:

```
import numpy as np
import math

def logistic_map_core(r):
    reps = 4000
    numtoplot = 256

    lims = np.zeros(reps)
    lims[0] = .5
    for i in range(reps-1):
        lims[i+1] = r*(lims[i])*(1 - lims[i])
        if abs(lims[i+1]) > 3/2:
            break
    return lims[reps-numtoplot:]

def logistic_cycle_inflection(below, above):
    epsilon = 1e-8
    epsilon2 = 1e-9

    below_evaluation = max(logistic_map_core(below)) - below/4
    above_evaluation = max(logistic_map_core(above)) - above/4
    while below + epsilon < above:
        middle = below + (above-below)/2
        below2 = middle - epsilon2
        below2_evaluation = max(logistic_map_core(below2)) - below2/4
        above2 = middle + epsilon2
        above2_evaluation = max(logistic_map_core(above2)) - above2/4
        if above2_evaluation == below2_evaluation:
            if above_evaluation < below_evaluation:
                above = above2
                above_evaluation = above2_evaluation
            else:
                below = below2
                below_evaluation = below2_evaluation
        elif above2_evaluation < below2_evaluation:
            above = above2
            above_evaluation = above2_evaluation
        else:
            below = below2
            below_evaluation = below2_evaluation
    return below if below_evaluation > above_evaluation else above

def logistic_cycle(r, c):
    epsilon = 1e-8
```

```python
    additive_rounds = 2000
    converging_rounds = 2000

    lims = np.zeros(2*c+1)
    lims[0] = .5
    for a in range(additive_rounds):
        for i in range(c):
            lims[i+1] = r*(lims[i])*(1 - lims[i])
        lims[0] = (lims[0] + lims[c])/2
    for a in range(converging_rounds):
        for i in range(2*c):
            lims[i+1] = r*(lims[i])*(1 - lims[i])
        lims[0] = lims[2*c]
    cycle_within_margin = True
    for i in range(c):
        if abs(lims[i] - lims[i+c]) > epsilon:
            cycle_within_margin = False
    return cycle_within_margin

def logistic_cycle_edge(below, above, c):
    epsilon = 1e-8

    if below > above:
        (below, above) = (above, below)
    below_cycles = logistic_cycle(below, c)
    above_cycles = logistic_cycle(above, c)

    if above_cycles and below_cycles:
        return "Only one input should cycle (not two)"
    if not above_cycles and not below_cycles:
        return "One input should cycle (not zero)"

    while above - below > epsilon:
        middle = (above + below)/2
        middle_cycles = logistic_cycle(middle, c)
        if middle_cycles == below_cycles:
            below = middle
        else:
            above = middle
    return below if below_cycles else above

def truncate(number, digits) -> float:
    nbDecimals = len(str(number).split('.')[1])
    if nbDecimals <= digits:
        return number
    stepper = 10.0 ** digits
    return math.trunc(stepper * number) / stepper

def logistic_cycle_statistics(below, middle, above, c):
    lower = logistic_cycle_edge(below, middle, c)
    upper = logistic_cycle_edge(middle, above, c)
    extrema = logistic_cycle_inflection(lower, upper)
    lower_string = "%.7f" % truncate(lower, 7)
    upper_string = "%.7f" % truncate(upper, 7)
    extrema_string = "%.7f" % truncate(extrema, 7)
    return (lower_string, extrema_string, upper_string)

print(logistic_cycle_statistics(3.82, 3.832, 3.86, 3)) # finds details of the 3-cycle
```

This code can certainly be improved, but it should at least provide a few digits of accuracy. I would like to note that I tend to dislike calculations that involve precision and accuracy, so it is best to manually verify on your own, beyond my due diligence.

2.1.4 The Invisible Curve and Unusual Forms

It is easy to notice if something is present but difficult to notice if it is absent. The logistic map's "tendrils" give us an example of this behavior, where there is an invisible curve that corresponds to the points $(+4, +1)$ and $(-2, -\frac{1}{2})$ formed from a missing "tendril".

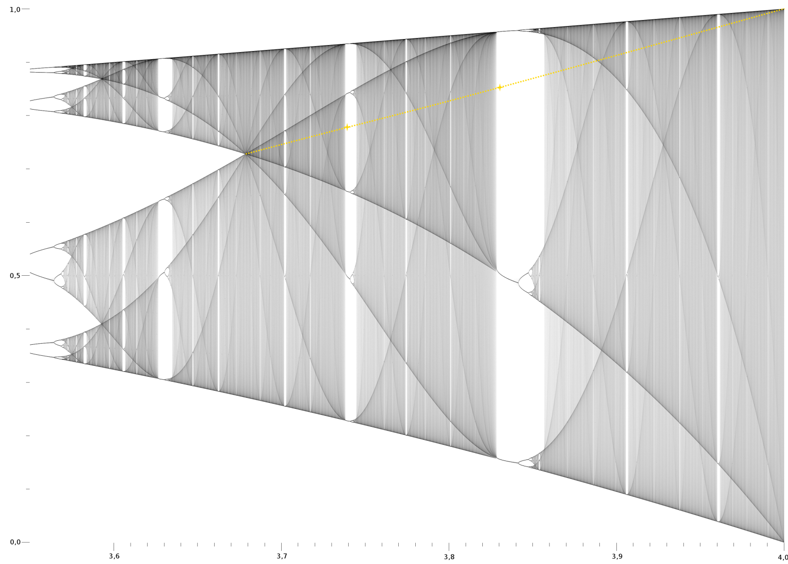

Figure 2.14: An approximation of the missing "tendril" in yellow. Modified from a high-resolution graph[5].

Upon further inspection, there is a clear reason why this curve is missing: all "tendrils" in the logistic map must pass through a "singularity point", and the "singularity points" exist at $(+4, 0)$ and $(-2, +\frac{3}{2})$, not $(+4, +1)$ and $(-2, -\frac{1}{2})$. However, that raises an even deeper question about the graph, since all other "singularity points" have two bifurcated sides recombine, and the only candidate for the second side is a distance of 6 away along the horizontal axis.

Moreover, every "singularity point" should be paired with a bifurcation point, and this implies that the 135 degree bend at $(+1, 0)$ is in some sense a bifurcation that corresponds to two half "singularity points" at $(+4, 0)$ and $(-2, +\frac{3}{2})$. This is unusual for several reasons, one of which being bifurcations are normally rotated 180 degrees from each other. It also heavily implies that $(+4, 0)$ and $(-2, +\frac{3}{2})$ are somehow equivalent, presumably in the sense of modular arithmetic.

At first glance, it may appear that the line would be difficult to deduce, but I believe that an infinite number of points are known, where the r values correspond to the "extrema" of a k-cycle of an island of stability, and the x value being the intercept with additional phantom lines:

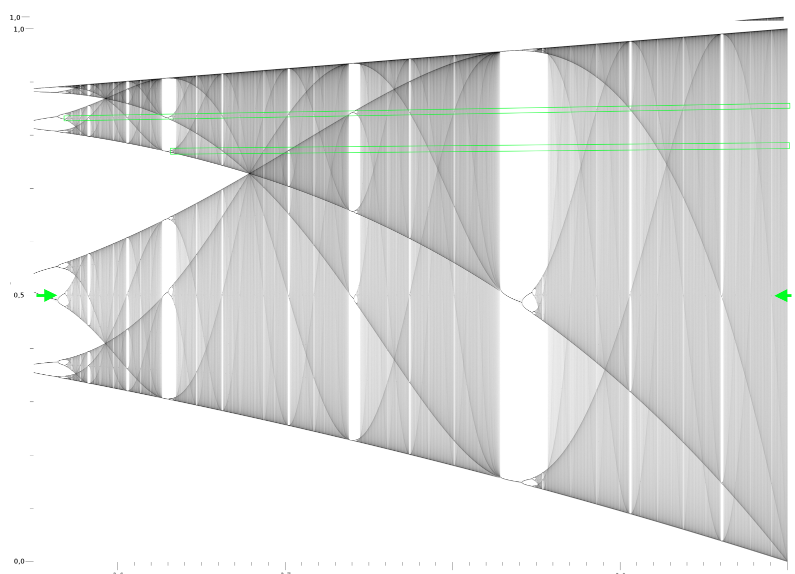

Figure 2.15: Faint and distorted curves that pass through many k-cycles. Modified from a high-resolution graph[5].

Using these lines, it may be possible to deduce the exact curve and use that to learn even more about the logistic map.

2.1.5 Medians and Quartiles

The last aspect of the logistic map I want to highlight is the median. Initially, when you look at the median, it appear incredibly boring:

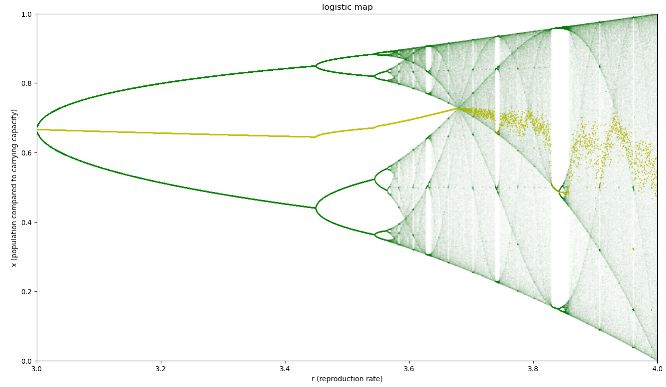

Figure 2.16: The median of the logistic map, which becomes chaotic after the "singularity point" at $r = 3.678573510428322...$

But the interesting information is on the micro scale, not the macro scale. If you zoom in, the median of the logistic map can be a series of curves connected in an unusual way.

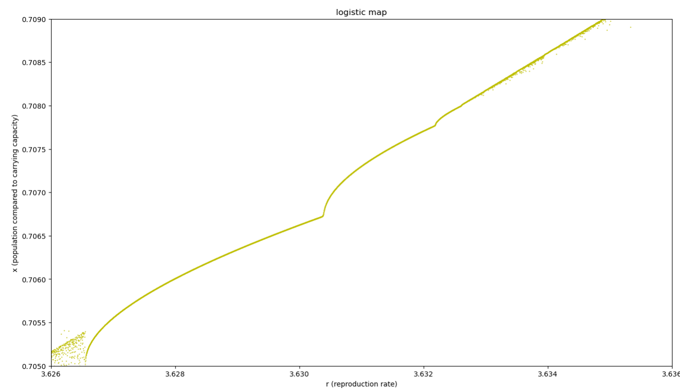

Figure 2.17: The median of the logistic map at a micro scale around the 6-cycle at $r = 3.6265531...$

Similarly, the quartiles (and more generally fractions of the form $\frac{n}{2^k}$ can be constructed such that they travel through the "singularity points" and each of them seem to have this same structure.

After some thought, I have decided that a likely mechanism for bifurcations could be the Lambert W Function (2.4.1), and this could explain why these different curves seem to be present along the median line.

Similarly, it is worth mentioning that the average values are interesting as well[6], where there seem to be averages arbitrarily close to zero as r approaches 4.

2.2 Prime Spirals

The Ulam spiral is a method of visualizing primes that are part of quadratic equations. It starts with a number, usually 1, and draws numbered boxes clockwise or counterclockwise around the original box on a square grid, highlighting prime numbers at the end.

1165	1032	**907**	790	681	580	**487**	402	325	324	323	322	321	320	319	318	**317**	316	315	314	**313**	312	**311**	310	309	308	**307**	380	**461**	550	**647**	752	865	986	1115
1166	**1033**	908	791	682	581	488	403	326	**257**	256	255	254	253	252	**251**	250	249	248	247	246	245	244	243	242	**241**	306	**379**	460	549	646	**751**	864	985	1114
1167	1034	909	792	**683**	582	489	404	327	258	**197**	196	195	194	**193**	192	**191**	190	189	188	187	186	185	184	183	240	305	378	459	548	645	750	**863**	984	1113
1168	1035	910	793	684	583	490	405	328	259	198	145	144	143	142	141	140	**139**	138	**137**	136	135	134	133	182	**239**	304	377	458	**547**	644	749	862	**983**	1112
1169	1036	**911**	794	685	584	**491**	406	329	260	**199**	146	**101**	100	99	98	**97**	96	95	94	93	92	91	132	**181**	238	303	376	**457**	546	**643**	748	861	982	1111
1170	1037	912	795	686	585	492	407	330	261	200	147	102	65	64	63	62	**61**	60	**59**	58	57	90	**131**	180	237	302	375	456	545	642	747	860	981	1110
1171	1038	913	796	687	586	493	408	**331**	262	201	148	**103**	66	**37**	36	35	34	33	32	**31**	56	**89**	130	**179**	236	301	374	455	544	**641**	746	**859**	980	**1109**
1172	**1039**	914	**797**	688	**587**	494	**409**	332	**263**	202	**149**	104	**67**	38	**17**	16	15	14	**13**	30	55	88	129	178	235	300	**373**	454	543	640	745	858	979	1108
1173	1040	915	798	689	588	495	410	333	264	203	150	105	68	39	18	**5**	4	**3**	12	**29**	54	87	128	177	234	299	372	453	542	639	744	**857**	978	1107
1174	1041	916	799	690	589	496	411	334	265	204	**151**	106	69	40	**19**	6	1	**2**	**11**	28	**53**	86	**127**	176	**233**	298	371	452	**541**	638	**743**	856	**977**	1106
1175	1042	917	800	**691**	590	497	412	335	266	205	152	**107**	70	**41**	20	**7**	8	9	10	27	52	85	126	175	232	297	370	451	540	637	742	855	976	1105
1176	1043	918	801	692	591	498	413	336	267	206	153	108	**71**	42	21	22	**23**	24	25	26	51	84	125	174	231	296	369	450	539	636	741	854	975	1104
1177	1044	**919**	802	693	592	**499**	414	**337**	268	207	154	**109**	72	**43**	44	45	46	**47**	48	49	50	**83**	124	**173**	230	295	368	**449**	538	635	740	**853**	974	**1103**
1178	1045	920	803	694	**593**	500	415	338	**269**	208	155	110	**73**	74	75	76	77	78	**79**	80	81	82	123	172	**229**	294	**367**	448	537	634	**739**	852	973	1102
1179	1046	921	804	695	594	501	416	339	270	209	156	111	112	**113**	114	115	116	117	118	119	120	121	122	171	228	**293**	366	447	536	633	738	851	972	1101
1180	1047	922	805	696	595	502	417	340	**271**	210	**157**	158	159	160	161	162	**163**	164	165	166	**167**	168	169	170	**227**	292	365	446	535	632	737	850	**971**	1100
1181	1048	923	806	697	596	**503**	418	341	272	**211**	212	213	214	215	216	217	218	219	220	221	222	**223**	224	225	226	291	364	445	534	**631**	736	849	970	1099
1182	**1049**	924	807	698	597	504	**419**	342	273	274	275	276	**277**	278	279	280	**281**	282	**283**	284	285	286	287	288	289	290	363	444	533	630	735	848	969	1098
1183	1050	925	808	699	598	505	420	343	344	345	346	**347**	348	**349**	350	351	352	**353**	354	355	356	357	358	**359**	360	361	362	**443**	532	629	734	847	968	**1097**

Figure 2.18: The Ulam spiral centered around 1. Primes are highlighted after enumerating 1, 2, 3... in a square spiral.

The Sacks spiral is essentially the same thing, but it uses continuous numbers with the canonical spiral:

$$radius = \sqrt{n}, \quad angle = 2\pi\sqrt{n}$$

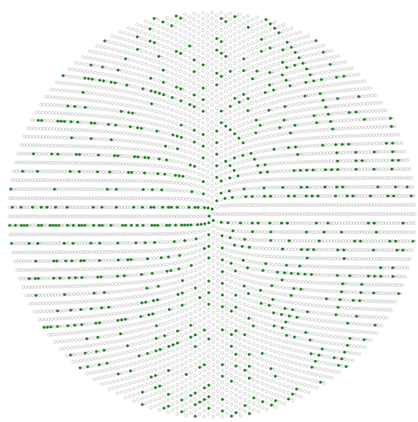

Figure 2.19: Sacks spiral (implicitly centered at zero).

While these graphs are useful, they tend to have a definition that starts from 1 or 0, and it is better to accept both positive and negative offsets (and still highlight negative primes).

For more examples, see the lucky numbers of Euler (4.2.2), the class 1 numbers (4.2.1), and the "lucky numbers of Martin" (4.2.5).

2.2.1 The "Quantized Bifurcation"

As can be seen with the logistic map, not all bifurcations have the same properties. A good example of this is the Sacks spiral, which can be used to visualize the square numbers bifurcating into the triangular numbers:

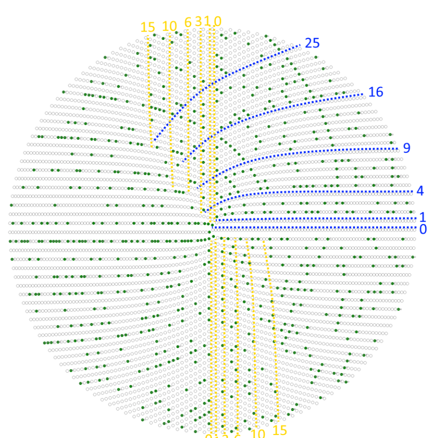

Figure 2.20: The Sacks spiral, featuring a real axis with square numbers and imaginary axis with triangular numbers.

I call this a "quantized bifurcation" because there are different quanta that aggregate to a bifurcation. The quanta for square numbers has a positive skew, whereas the quanta for triangular numbers has both a positive and negative skew (that likely cancels resulting in no bias).

Notably, this likely has to do with the duality of even and odd polygonal numbers, which I will discuss alongside "quadratic polynomial sieves" (2.2.3).

2.2.2 "1D Möbius Strips"

Another detail that I have not seen discussed about the Ulam spiral is the presence of a Möbius strip-like even set, as opposed to typical quadratic equations that tend to be odd in nature. These Möbius strip-like lines are visible when creating an Ulam spiral starting with even integers (e.g. 0 and 30):

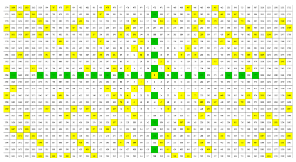

Figure 2.21: Ulam spiral centered around 0.

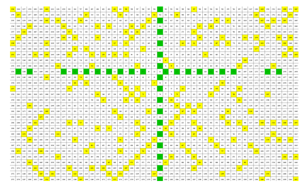

Figure 2.22: Ulam spiral centered around 30.

At first, these may look like two lines going vertically and horizontally, but the quadratic equations both have 90 degree bends. The polynomials that produce the four curves can be seen below, with primes bolded:

Equation	$n=-9$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
$16n^2+2n-1$	**1277**	1007	**769**	**563**	**389**	247	**137**	**59**	**13**	-1	**17**	**67**	**149**	**263**	**409**	**587**	**797**	**1039**	1313
$16n^2+2n+29$	**1307**	1037	799	**593**	**419**	**277**	**167**	**89**	**43**	**29**	**47**	**97**	**179**	**293**	**439**	**617**	**827**	**1069**	1343
$16n^2+6n+1$	1243	**977**	**743**	**541**	371	**233**	**127**	**53**	**11**	1	**23**	77	**163**	**281**	**431**	**613**	**827**	**1073**	1351
$16n^2+6n+31$	1273	1007	**773**	**571**	**401**	**263**	**157**	**83**	**41**	**31**	**53**	**107**	**193**	**311**	**461**	**643**	**857**	**1103**	**1381**

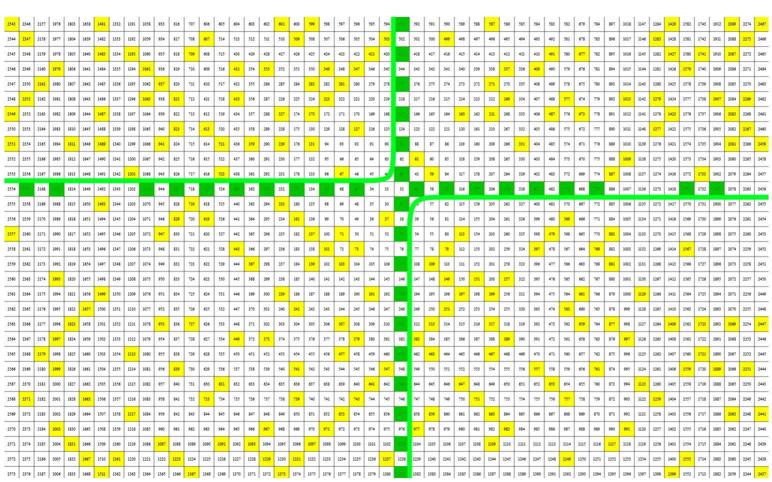

Figure 2.23: Conjectured "1D Möbius Strip" derived from Ulam spiral centered around 30.

This leads to my prediction that these are best characterized as a single Möbius strip-like object that meets at infinity.

An astute reader might note that Möbius strips are 2D surfaces, whereas the curve being shown is 1D (and thus cannot be a surface) and fits in a 2D plane, this means it is different. This got me to think about perhaps the most mundane fact in this entire paper in the context of "1D Möbius Strips" and Klein bottles:

a	$a+a$	$a \cdot a$	a^a	$\frac{a}{a}$	$a-a$
0	$0+0=\mathbf{0}$	$0\cdot 0=\mathbf{0}$	0^0	$\frac{0}{0}$	$0-0=\mathbf{0}$
1	$1+1=2$	$1\cdot 1=\mathbf{1}$	$1^1=\mathbf{1}$	$\frac{1}{1}=\mathbf{1}$	$1-1=0$
2	$2+2=\mathbf{4}$	$2\cdot 2=\mathbf{4}$	$2^2=\mathbf{4}$	$\frac{2}{2}=1$	$2-2=0$

Where, notably, each of the three cases has three out of five operators with a consistent output. Zero fulfills all additive cases and multiplication; one fulfills all multiplicative cases and exponentiation; two fulfills addition, multiplication, and exponentiation. It also make me want to try to define $\frac{0}{0}=1$, the multiplicative identity, and $0^0=0$, the additive identity, then use those definitions to define division by zero, although it will probably take a lot of effort to assign meaning to these undefined operations, and they may vary, e.g. for infinitesimal numbers. If these two definitions were used, I find it notable the column sums would become $6 - \{0, 1, 1, 3, 6\}$, related to triangular numbers through $6 - \frac{Fibonacci(n)\cdot(Fibonacci(n)+1)}{2}$.

This makes me wonder if a "1D Möbius Strip" might be more primitive in some sense. That being said, there are plenty of geometric shapes (e.g. Möbius strip, Boy's surface, Klein bottle), and it is difficult to argue any one could be more primitive. Part of the reason I want to define the concept of the "1D Möbius Strip" is to get a relation between even and odd numbers, where evens essentially have twice the information. If there is a path forward, I think relating the curve to the Riemann sphere is a good first step.

2.2.3 The "Quadratic Polynomial Sieve"

This section is my oldest research from ten years ago. It covers something I used to call "repeated boolean masks" but now prefer to categorize as the "quadratic polynomial sieve", based on the class 1 numbers (4.2.1) and lucky numbers of Euler (4.2.2). The concept is related to the Ulam spiral, and I probably would have called it a quadratic sieve if that name was still available, so perhaps it relates to quadratic sieves.

The fundamentals are about counting the number of prime factors in quadratic sequences of the form $n^2+n+\{3, 5, 11, 17, 41\}$ for a "quadratic polynomial sieve" using the lucky numbers of Euler. For example, the number of prime factors of the sequence $n^2 + n + 41$ is shown by:

	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	1	1	2	1	1	1	1	2	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	2	1	1	1	1
2	2	1	2	1	1	2	1	2	1	2	1	1	1	1	2	1	1	1	1	1	2	1	2	1	1	1	1	2	1	1	1	1	1	1	1	2	1	1	1	2
2	2	1	1	2	2	1	1	2	1	1	1	1	1	2	1	2	1	2	1	1	2	1	1	1	2	1	1	1	1	1	1	1	2	1	1	1	2	1	2	2
2	2	1	1	1	1	1	2	1	2	2	1	1	1	1	2	1	1	1	1	1	2	2	2	2	1	1	2	1	1	1	1	1	1	1	1	1	1	2	1	1
2	2	1	2	2	2	1	1	1	2	1	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	1	2	2	1	2	1	1	2	1	

Where the first forty 1's indicate n^2+n+41 is a good prime-generating polynomial that accurately predicts the primality of its first 40 terms. After that, each row relates to even polygonal numbers. For instance, the second row has composites (with ≥ 2 prime factors) at square indices (i.e. 0, 1, 4, 9, 16, 25, 36...) colored cyan; the third row has composites at generalized hexagonal indices (i.e. 0, 1, 3, 6, 10, 15, 21, 28, 36...) colored magenta in addition to a few square numbers continuing from the prior row; the fourth row has composites at generalized octagonal indices (i.e. 0, 1, 5, 8, 16, 21, 33, 40...) colored yellow in addition to stragglers from previous rows; and so on.

Similarly, a "quadratic polynomial sieve" using class 1 numbers can be constructed by counting the number of prime factors in quadratic sequences of the form $4n^2 + \{11, 19, 43, 67, 163\}$. For example, the number of prime factors of the sequence $4n^2 + 163$ is shown by:

																					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	1	2	1	1	1	2	1	1	1	1	2	1	1	1	1	1	2	1	1	1	1	1	1	2	1	1	1	1	1	1	1	2	1	1	1	1
2	2	2	1	2	2	1	2	1	1	1	1	2	1	2	2	1	1	1	1	1	1	2	1	1	2	2	1	1	1	1	1	1	1	1	2	1	2	1	1	2
2	2	1	1	2	1	1	2	1	2	2	1	1	2	1	1	2	1	2	1	1	1	1	2	1	1	1	2	1	2	1	1	1	1	2	1	2	1	2	1	1
2	2	1	1	1	2	2	1	1	2	2	1	1	2	2	1	1	1	2	2																					

Where the first twenty 1's indicate $4n^2+163$ is a good prime-generating polynomial that accurately predicts the primality of its first 20 terms. After that, each row relates to odd polygonal numbers. For instance, the second row has composites (with ≥ 2 prime factors) at triangular indices (i.e. 0, 1, 3, 6, 10, 15, 21, 28, 36...) colored cyan; the third row has composites at generalized pentagonal indices (i.e. 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40...) colored magenta in addition to a few triangular numbers continuing from the prior row; the fourth row has composites at generalized heptagonal indices (i.e. 0, 1, 4, 7, 13, 18, 27, 34...) colored yellow in addition to stragglers from previous rows; and so on.

Both of these methods can be used to sieve essentially any number, with the caveat that the lucky number of Euler variant can be applied to n^2+n+x with exceptions for the class 1 numbers; and the caveat that the class 1 variant can be applied to $4n^2+(4x-1)$ with exceptions for the lucky numbers of Euler. Doing so results in many more false positives where a number is thought to be prime but may not actually be prime. A simple example of this is applying the Euler variant "quadratic polynomial sieve" to $n^2 + n + 16$, which cannot contain a prime, so a large amount of false positives exist where a prime is anticipated but the true answer is actually composite.

I chose to truncate each grid at a point where the patterns got more complicated (which happens after about six rows in the lucky number of Euler variant and after about four rows in the class 1 variant). From this point onward, additional strategies must be incorporated beyond polygonal numbers, and there is less precision with respect to counting the exact number of prime factors (although the strategy never seems to fail at anticipating if a number is composite). The newer, more optimized code can be found here[7], whereas the older code that tried to account for the most patterns can be found here[8].

With respect to what these sieves might represent, I believe they can be thought of as representing "top-down number theory", where positional information mostly does not matter and behaviors typically emerge from quasicrystal-like structure.

2.3 The Riemann Hypothesis

The Riemann hypothesis is a prediction about the location of nontrivial zeroes of the Riemann zeta function. It predicts that all nontrivial zeroes to the Riemann zeta function have a real part of $\frac{1}{2}$ and form a critical line.

The functional equation is:

$$zeta(s) = 2^s \cdot \pi^{s-1} \cdot sin(\frac{\pi s}{2}) \cdot gamma(1-s) \cdot zeta(1-s)$$

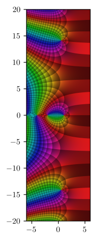

Figure 2.24: The Riemann zeta function modified from Wikipedia[9]. It features the pole at $s = 1$, trivial zeroes at negative even integers, and non-trivial zeroes at $s = \frac{1}{2} \pm 14.13472i$.

Where the evaluation of the Riemann zeta function at a point relates to its evaluation at a symmetric point reflected across $(+\frac{1}{2}, 0)$, which explains why all known zeta zeroes are symmetric across the horizontal axis.

There are two important values around the point $(+\frac{1}{2}, 0)$: $(0, 0)$, a point that is only defined by its limiting behavior resulting in $zeta(0) = -\frac{1}{2}$, and $(+1, 0)$, the simple pole that diverges to infinity because of a division by zero in the gamma function. Another interesting note is that the equation has trivial zeroes at negative even integers where the sine term results in multiplication by zero. For positive even values, the poles of the sine function and the gamma function cancel, resulting in standard evaluation. Thus, the point $(0, 0)$ is special because it is like a saddle point between the trivial zeta zeroes and their cancellation.

2.3.1 The Critical Strip as A Bifurcation

The Riemann zeta function has features that make me believe there are properties of a bifurcation embedded in the zeta zeroes: e.g. the discussed points $(0, 0)$, which I believe is an "imaginary pole", and $(+1, 0)$, which is a simple pole.

If you look at the behavior around the simple pole at $s = 1$, the trivial zeroes would naturally continue onward into the even integers, but due to cancellation with the gamma function, I believe they are evicted by $zeta'(0)$ and must instead travel elsewhere. Since the real plane has no existing options, the values must travel into the complex plane instead.

I believe the Riemann zeta function is a particularly ideal bifurcation, even among other known bifurcations. At the point of bifurcation $(+\frac{1}{2}, 0)$, it travels orthogonally at 90 degrees, and this geometry is true globally, unlike other curved bifurcations. Additionally, it does perfect information splitting, resulting in an exact 50:50 split of information between the positive and negative imaginary axes.

Additionally, it has the property that a dashed line can be constructed along the positive even integers, plotting the path of the bifurcation as if it had never happened (corresponding to the cancellation between the sine and gamma functions).

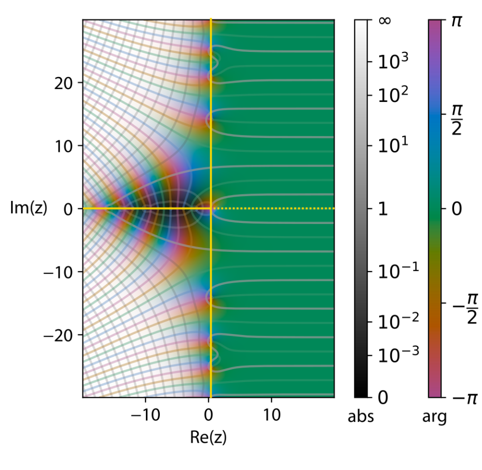

Figure 2.25: The Riemann zeta function modified from Wikipedia[10]. A bifurcation in yellow is superimposed.

This could explain why the positive even integers have such elegant valuations (i.e. $zeta(2k)$), whereas the odd integers do not. The argument would essentially be that the positive even integers are "imaginary zeta zeroes" in some sense.

Additionally, the zeta zeroes are a sparse set of points rather than a continuous line, so I think it is best to label the Riemann zeta zeroes as a "sparse bifurcation", which may indicate there is a more ideal form of the zeta function. This "sparse bifurcation" would generally imply that 0% of the bifurcation is accounted for, which may explain why the Riemann hypothesis gives a square root bound on the distribution of primes, rather than a more optimistic bound like the Montgomery's pair correlation conjecture might imply (or even an exact formula for primes).

2.3.2 A Circle of Bifurcations

The Riemann zeta function does not have anything resembling a closed circuit of zeta zeroes, even if wrapping at infinity is allowed. This is not necessarily a flaw, but I believe that important details are missing.

At the same time, if the Riemann zeta function were to bifurcate beyond $(+\frac{1}{2}, 0)$, the bifurcation might travel outside of the critical strip (and we know zeta zeroes only exist among the trivial zeroes or within the critical strip). Because of this, I propose that the critical strip does bifurcate, but not within a finite distance. Instead, it needs to accumulate (in an infinite sense) before bifurcating at infinity. My hypothesis for the mechanism is here:

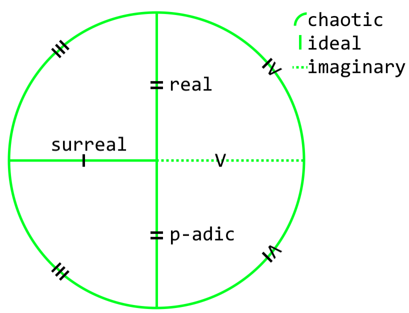

Figure 2.26: A conjectured circle of bifurcations corresponding to the Riemann zeta function around $\frac{1}{2} + 0i$.

Notably, I use three classes of bifurcations 1) standard, 2) imaginary, and 3) chaotic, where standard and imaginary have been discussed, but not chaotic.

By conjecture, the lines and curves have different properties: both a type and a direction. This results in a "type II" curve being seen as real numbers or p-adic numbers depending on the direction. Meanwhile you can view the "type I" curve as surreal numbers (which have properties of infinite, infinitesimal, and real numbers). The "type V" is imaginary, and thus has a distance that is best interpreted as imaginary, which is ironic because it happens to be along the positive real axis.

I believe that for this system, there is the least energy in some sense along the negative axis, a moderate amount in the middle, and a critically large amount along the positive axis. I believe the difference for chaotic bifurcations is that they do not result in a 50:50 information split and they also regularly shed or gain energy through an infinite number of smaller bifurcations. In this sense, the "type III" and "type IV" bifurcations are the most mysterious.

It is also worth noting that the Lee-Yang theorem states that the zeroes of the ferromagnetic Ising model lie on a unit circle, which for a problem often tied to the Riemann hypothesis is worth including. A simpler analogy would be the Pappus chain, which can be a finite circle or an infinite line depending on perspective.

2.3.3 The Riemann Zeta Function for Real Integer Inputs

As mundane as the real axis of the zeta function is, I think it can be enlightening and provide a roadmap for new number systems.

Among the non-positive integers, I mark these as interesting:

zeta(0): $\frac{-1}{2}$
zeta(-1): $\frac{-1}{12}$
zeta(-3): $\frac{1}{120}$
zeta(-5): $\frac{-1}{252}$
zeta(-7): $\frac{1}{240}$
zeta(-9): $\frac{-1}{132}$
zeta(-13): $\frac{-1}{12}$

Where all other values should have a numerator greater than 1.

And among the non-negative integers, I mark these as interesting:

zeta(0): $\frac{-1}{2}$
zeta(2): $\frac{\pi^2}{6}$
zeta(4): $\frac{\pi^4}{90}$
zeta(6): $\frac{\pi^6}{945}$
zeta(8): $\frac{\pi^8}{9450}$
zeta(10): $\frac{\pi^{10}}{93555}$
zeta(14): $\frac{2\pi^{14}}{18243225}$

Where all other values should have a numerator greater than 2, ignoring the π terms.

Due to the symmetry of the Riemann zeta function across $(+\frac{1}{2}, 0)$, it is not surprising that the two lists are similar, but there are some qualitative features I would like to highlight.

For the negative side you have:

zeta(0): $|zeta(-2+2) \cdot (2) \cdot (2-1)^2| = 1$
zeta(-1): $|zeta(-3+2) \cdot (3) \cdot (3-1)^2| = 1$
zeta(-3): $|zeta(-5+2) \cdot (5) \cdot (5-1)^2| = \frac{2}{3}$
zeta(-5): $|zeta(-7+2) \cdot (7) \cdot (7-1)^2| = 1$
zeta(-7): $|zeta(-9+2) \cdot (9) \cdot (9-1)^2| = \frac{12}{5}$
zeta(-9): $|zeta(-11+2) \cdot (11) \cdot (11-1)^2| = \frac{25}{3}$
zeta(-13): $|zeta(-15+2) \cdot (15) \cdot (15-1)^2| = 245$

Where notably the first four terms are very reminiscent of:

$1! + 1 = 2$
$2! + 1 = 3$
$(\frac{8}{3})! + 1 \approx 5$
$3! + 1 = 7$

Additionally, if we correlate $zeta(0)$ with 2, $zeta(-1)$ with 3, $zeta(-3)$ with 5, and $zeta(-5)$ with 7, it implies the critical line of the Riemann hypothesis occurs at $\frac{3}{2}$. Additionally, it means that 5 corresponds to a minimum at $zeta(-3)$ in some sense.

Taking this a step further, you can look at:

zeta(0): $\frac{-1}{2} \cdot 2 = -1$
zeta(-1): $\frac{-1}{12} \cdot 2 = \frac{-1}{6}$ (3rd triangular number, 2nd hexagonal number)
zeta(-3): $\frac{1}{120} \cdot 2 = \frac{1}{60}$ (3rd 21-gonal number)
zeta(-5): $\frac{-1}{252} \cdot 2 = \frac{-1}{126}$ (9th pentatope number)
zeta(-7): $\frac{1}{240} \cdot 2 = \frac{1}{120}$ (15th triangular number, 8th hexagonal number)
zeta(-9): $\frac{-1}{132} \cdot 2 = \frac{-1}{66}$ (11th triangular number, 6th hexagonal number)
zeta(-13): $\frac{-1}{12} \cdot 2 = \frac{-1}{6}$ (3rd triangular number, 2nd hexagonal number)

I think almost everyone would agree a direct causality looks extremely improbable, but I want to highlight something interesting about pentatope numbers to relate it to the concept of 2, 3, 7 being offset factorial numbers: pentatope numbers contain two thirds pentagonal numbers and one third centered pentagonal numbers, which is eerily reminiscent of the fact that $(\frac{8}{3})! + 1 \approx 5$.

Looking instead at the positive side, I have far fewer ideas of what they could mean, but I do want to highlight one curious aspect where:

zeta(0): $\frac{-1}{2} \div \pi^0 \cdot \frac{1}{2} \cdot (0+1)!! = \frac{-1}{4}$
zeta(2): $\frac{\pi^2}{6} \div \pi^2 \cdot \frac{1}{2} \cdot (2+1)!! = \frac{1}{4}$
zeta(4): $\frac{\pi^4}{90} \div \pi^4 \cdot \frac{1}{2} \cdot (4+1)!! = \frac{1}{12}$
zeta(6): $\frac{\pi^6}{945} \div \pi^6 \cdot \frac{1}{2} \cdot (6+1)!! = \frac{1}{18}$
zeta(8): $\frac{\pi^8}{9450} \div \pi^8 \cdot \frac{1}{2} \cdot (8+1)!! = \frac{1}{20}$
zeta(10): $\frac{\pi^{10}}{93555} \div \pi^{10} \cdot \frac{1}{2} \cdot (10+1)!! = \frac{1}{18}$
zeta(14): $\frac{2\pi^{14}}{18243225} \div \pi^{14} \cdot \frac{1}{2} \cdot (14+1)!! = \frac{1}{9}$

Where the double factorial is defined recursively like the factorial:

$$n!! = n \cdot (n-2)!!$$

Notably, a separate value is exceptional here, $zeta(8)$, which notably does not correspond symmetrically with $zeta(-3)$. An exploration further into the double factorial might produce the insight that:

$$n!! \sim \{\sqrt{\pi n} \cdot (\frac{n}{e})^{\frac{n}{2}} \text{for even n}, \sqrt{2n} \cdot (\frac{n}{e})^{\frac{n}{2}} \text{for odd n}\}[11]$$

This is an asymptotic based on Stirling's approximation which can lead to the insight:

$$\lceil \frac{2^n}{zeta(n)} - (2^n - 1) \rceil$$

Which generates:

-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
∞	2	∞	1	∞	2	∞	1	∞	2	∞	1	∞	3	∞	-6	∞	16	∞	-6	-2	-1	0	0	0	0	0

Where the terms fall into a 4-cycle on the negative side and a 1-cycle on the positive side.

Like before, $zeta(-3)$ is exceptional with a value of 16, and the error term along the positive axis is very small for the ceiling function.

Positive integers basically just correspond to $\frac{2^n}{2^n-1} \sim zeta(n)$ with an error term roughly equal to $\frac{n}{2}$ zeroes of precision (possibly due to $\pi^2 \approx 10$). Additionally, it raises a question about what the numerator looks like along the negative axis.

I am going to break my usual convention and just provide a link in the language of Wolfram Alpha[12]:

$$Table[Boole[frac(log2(abs(Numerator[\frac{2^{2n-3}}{zeta(2n-3)}]) \pm 1)) \neq 0]], n = -1000 \text{ to } 0$$

Where, at the time of this writing, Wolfram Alpha automatically groups 30 terms on each line, which highlights the curious behavior mod 30. More accurately, since half of the terms are zeta zeroes, you can observe values corresponding to $(\{1, 25, 37, 49\} - 2)$ mod 60, when inputted into the zeta function, satisfy the boolean statement much less often, resulting in a column of primarily 0's.

While this query is unusual, I think it might provide some insight into the Riemann zeta function. It essentially looks at how often the numerators of $\frac{2^{2n-3}}{zeta(2n-3)}$ are within 1 of a power of 2.

2.3.4 A Different Formula for The Riemann Zeta Function

The functional equation of the Riemann zeta function is:

$$zeta(s) = 2^s \cdot \pi^{s-1} \cdot sin(\frac{\pi s}{2}) \cdot gamma(1-s) \cdot zeta(1-s)$$

Which is the source of the partial reflection symmetry across the point $(+\frac{1}{2}, 0)$ that I discussed earlier. This equation is valid for the entire critical strip except the simple pole at $s = 1$ (and, if limiting behavior is not included, the "imaginary pole" at $s = 0$).

From here I will derive a different formula for the Riemann zeta function.

Multiply each side of the functional equation by $gamma(s)$:

$$zeta(s) \cdot gamma(s) = 2^s \cdot \pi^{s-1} \cdot sin(\frac{\pi s}{2}) \cdot gamma(s) \cdot gamma(1-s) \cdot zeta(1-s)$$

Gamma Reflection Identity, $gamma(s) \cdot gamma(1-s) = csc(\pi s) \cdot \pi$:

$$zeta(s) \cdot gamma(s) = 2^s \cdot \pi^{s-1} \cdot sin(\frac{\pi s}{2}) \cdot csc(\pi s) \cdot \pi \cdot zeta(1-s)$$

Combine trigonometry, turning sine and cosecant into secant:

$$zeta(s) \cdot gamma(s) = 2^s \cdot \pi^{s-1} \cdot \frac{sec(\frac{\pi s}{2})}{2} \cdot \pi \cdot zeta(1-s)$$

Group π and 2 terms:

$$zeta(s) \cdot gamma(s) = 2^{s-1} \cdot \pi^s \cdot sec(\frac{\pi s}{2}) \cdot zeta(1-s)$$

Move $gamma(s)$ to the right side:

$$zeta(s) = 2^{s-1} \cdot \pi^s \cdot sec(\frac{\pi s}{2}) \cdot \frac{zeta(1-s)}{gamma(s)}$$

At this juncture, we must combine this newest equation with the functional equation from the very beginning. Multiply the original critical strip identity by this new equation:

$$zeta(s)^2 = (2^s \cdot \pi^{s-1} \cdot sin(\frac{\pi s}{2}) \cdot gamma(1-s) \cdot zeta(1-s)) \cdot (2^{s-1} \cdot \pi^s \cdot sec(\frac{\pi s}{2}) \cdot \frac{zeta(1-s)}{gamma(s)})$$

Group π and 2 terms:

$$zeta(s)^2 = (2 \cdot \pi)^{2s-1} \cdot sin(\frac{\pi s}{2}) \cdot gamma(1-s) \cdot zeta(1-s) \cdot sec(\frac{\pi s}{2}) \cdot \frac{zeta(1-s)}{gamma(s)}$$

Group $zeta(1-s)$ terms:

$$zeta(s)^2 = zeta(1-s)^2 \cdot (2 \cdot \pi)^{2s-1} \cdot sin(\frac{\pi s}{2}) \cdot gamma(1-s) \cdot sec(\frac{\pi s}{2}) \cdot \frac{1}{gamma(s)}$$

Group trigonometry and gamma:

$$zeta(s)^2 = zeta(1-s)^2 \cdot (2 \cdot \pi)^{2s-1} \cdot sin(\frac{\pi s}{2}) \cdot sec(\frac{\pi s}{2}) \cdot \frac{gamma(1-s)}{gamma(s)}$$

Combine trigonometry, turning sine and secant into tangent:

$$zeta(s)^2 = zeta(1-s)^2 \cdot (2 \cdot \pi)^{2s-1} \cdot tan(\frac{\pi s}{2}) \cdot \frac{gamma(1-s)}{gamma(s)}$$

Move $zeta(1-s)$ terms to the left side:

$$\boldsymbol{(\frac{zeta(s)}{zeta(1-s)})^2 = (2 \cdot \pi)^{2s-1} \cdot tan(\frac{\pi s}{2}) \cdot \frac{gamma(1-s)}{gamma(s)}}$$

Finally, you have the equation I worked to derive. There may be a simpler way to derive it, but that was how I found it.

After all that work, you may wonder: What is so special about this equation? There is no obvious answer to that, but one thing you might see and try to do (indeed, as I did without thinking) is take the square root of both sides.

Next, take the square root of both sides and convert the above equation into two similar equations and plot their error:

$+)$ $zeta(s) \neq +zeta(1-s) \cdot (2 \cdot \pi)^{s-\frac{1}{2}} \cdot \sqrt{tan(\frac{\pi s}{2}) \cdot \frac{gamma(1-s)}{gamma(s)}}$

$-)$ $zeta(s) \neq -zeta(1-s) \cdot (2 \cdot \pi)^{s-\frac{1}{2}} \cdot \sqrt{tan(\frac{\pi s}{2}) \cdot \frac{gamma(1-s)}{gamma(s)}}$

Which have relative error terms of:

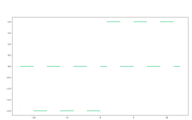

Figure 2.27: Relative error, +

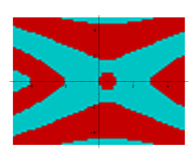

Figure 2.28: $\frac{zeta(s)}{zeta(1-s) \cdot (2\pi)^{s-\frac{1}{2}} \sqrt{tan(\frac{\pi s}{2}) \frac{gamma(1-s)}{gamma(s)}}}$

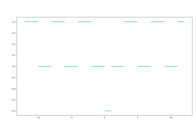

Figure 2.29: Relative error, −

Notably, the relative error is zero about half of the time, essentially forming a square wave of wavelength 4.

For convenience I will now define a helper function "seon", which is an even-odd sign function:

$$seon(n) = -((\frac{1}{2} - \frac{sign(n)}{2}) \cdot sign(sine(\pi + \frac{n\pi}{2})) + (\frac{1}{2} + \frac{sign(n)}{2}) \cdot sign(sine(\frac{-\pi}{2} + \frac{n\pi}{2})))$$

There may be a much better derivation of this, but that was the first one I thought of and there was no obvious-to-me way to simplify it. Notably, this version is probably not graceful with complex inputs, but at this stage I am not sure what the output should look like for complex inputs (ideally the form would make the next equation exact).

Notably, I personally view this "seon" function as similar to the evens and odds surrounding "singularity points" in the logistic map (2.1.3), because left of zero has bands that start and end on even integers, and right of zero has bands that start and end on odd integers.

Combining the two, the following statement might be true for real values s (but not complex values s yet):

$$zeta(s) = seon(s) \cdot zeta(1-s) \cdot (2 \cdot \pi)^{s-\frac{1}{2}} \cdot \sqrt{tan(\frac{\pi s}{2}) \cdot \frac{gamma(1-s)}{gamma(s)}}$$

Notably, graphing has some imprecision, but it seems to fluctuate above and below zero, with the least numerical stability around integers. This is why I joke this is the only rigor-ish part of the paper. The steps until taking the square root to yield this form are based on rigor, but the steps involving the derivation of the "seon" function are empirical. Even if this form is incorrect with real inputs, I believe exploring why it could ultimately be incorrect would still have value.

2.3.5 The Possibility of A Product Formula for The Riemann Zeta Function

Originally, I was trying to form a new equation for the Riemann zeta function using log rules and the functional equation, but I went back to an old idea: what if the intended route to the Riemann hypothesis is not to use the version derived from Riemann, but instead to backtrack and use products from the beginning?

This could explain why the trivial zeroes grow so slowly whereas the nontrivial zeroes grow much faster.

One equation that is only given a cursory note in the derivation of the Riemann zeta function is:

$$zeta(s) = \prod_{n=1}^{\infty} \frac{1}{1-prime(n)^{-s}} = \prod_{n=1}^{\infty} \frac{prime(n)^s}{prime(n)^s - 1}$$

The problem with this equation is that it assumes knowledge of the primes already. Ideally, the equation should not rely directly on primes.

I am now going to consider seven ingredients I think could be relevant to a derivation of a product formula of the zeta function:

Prime testing with Wilson's theorem: $\lfloor \frac{n! \bmod (n+1)}{n} \rfloor \cdot (n-1) + 2$, noting the anomaly of $A061006(4) = 2$[13]
Stirling's approximation: $n! \sim \sqrt{2\pi n} \cdot (\frac{n}{e})^n$
Infinity factorial using $zeta'(0)$: $\infty!$ "=" $\sqrt{2\pi}$
Normal distribution: $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$, with a mean of 1 (from $zeta(\infty)$)
Schoenfeld's prime-counting function bound[14]: $|\pi(n) - li(n)| < \frac{1}{8\pi}\sqrt{n} \cdot log_e(n), n \geq 2657$
Modified Merten's third theorem[15]: $e^\gamma = \lim_{n\to+\infty} \frac{1}{log_e(prime(n))} \prod_{k=1}^{n} \frac{prime(k)}{prime(k)-1}$
A number system unlike the real numbers, p-adic numbers, and opposing the surreal numbers (musical and chaotic).

And according to my theory, the resulting formula would most suitably be placed along the critical strip, and thus its properties would be more imaginary than real.

I also believe this behavior relates to the unusual substructure in the median of the logistic map (2.1.5), with the mechanism likely being the Lambert W function. This could also relate to "hyperstructured numbers" (4.1), since they may asymptotically include all prime factors (and thus can be divided by a primorial without losing a prime factor, in some sense).

The ultimate goal is to derive two new equations for the Riemann zeta function that relate to product rules, and then stitch together four total equations. The idea is that the Riemann zeta function's integral is not defined as nicely as the derivative, so in the same spirit of analytically continuing to $s \leq 1$, I want to analytically continue to $derivative(zeta, n), n < 0$ (allowing any arbitrary integral). By having four equations, I believe the system can be constrained in such a way that the whole function becomes well-defined.

Related, I have been wondering if the resulting product formula would be identical to the summation formula. My natural inclination is that the poles would be inverted, such that $s = 0$ is ill-defined, which would mean the functional equations would be dissimilar in some sense. I am trying to reconcile this with the principal of uniqueness of the analytical continuation. My best guess is that if the equations somehow manage to be different, it is because the integrals were not well defined for the zeta function in the first place, thus time can be reversed in some sense and the same calculations can be performed to a different effect.

Upon thinking about this even more, I think my precise meaning may be to cut the Riemann zeta function at $s = 1$, creating a seam, and then stitch positive infinity to negative infinity with the Riemann sphere and analytically continue. Maybe this is an inconsistent idea, but perhaps it will lead to something better. The original Riemann zeta function essentially has a seam at infinity, so I think creating a seam at $s = 1$ is permissible.

2.4 "Bifurcation Artifact" of Functions

While I have conjectured bifurcations appear in a lot of places, the logistic map is the only example in this paper recognized in the academic community. Still, another interesting place to look for bifurcations is within functions.

I consider a "bifurcation artifact" to be a behavior within a function that is best described by the principles of bifurcation theory.

2.4.1 The Lambert W Function

The Lambert W function, sometimes called the product logarithm, is a multivalued function that resolves the question: $W(x) \cdot e^{W(x)} = x$ for which values? where x can be any complex number. For real inputs, it is only necessary to work with two branches W_0 and W_{-1}, which can be seen here:

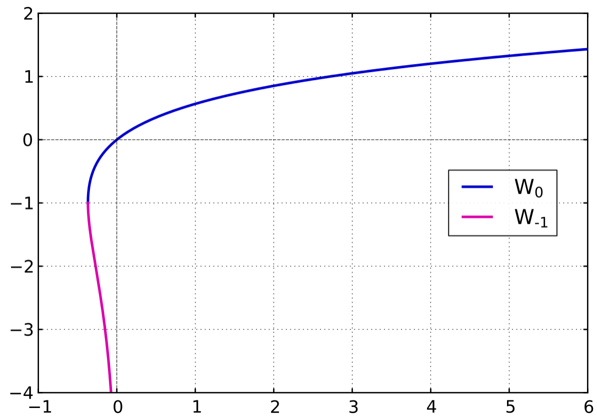

Figure 2.30: The main two branches of the Lambert W function from Wikipedia[16].

But for complex inputs there are branches corresponding to all integers, not just W_0 and W_{-1}.

Based on qualitative details, I feel like this relates to bifurcations. One possible example of this is the fine details in the median of the logistic map (2.1.5).

An important point here is that the two branches essentially have a 1:1 correspondence that would account for a 50:50 split of information along each of W_0 and W_{-1}, which could explain the behavior for "ideal bifurcations". "Chaotic bifurcations", on the other hand, might be explained by using more than two branches.

There is a possibility the Lambert W function may also relate to the "seon" function, which is inspired by the quote: "the imaginary part of W[n,x+i y].... is like that of the multivalued complex logarithm function except that the spacing between sheets is not constant *and the connection of the principal sheet is different*"[17] (emphasis mine), where notably the "seon" function has an unusual seam between the even and odd numbers where one section is of half the width.

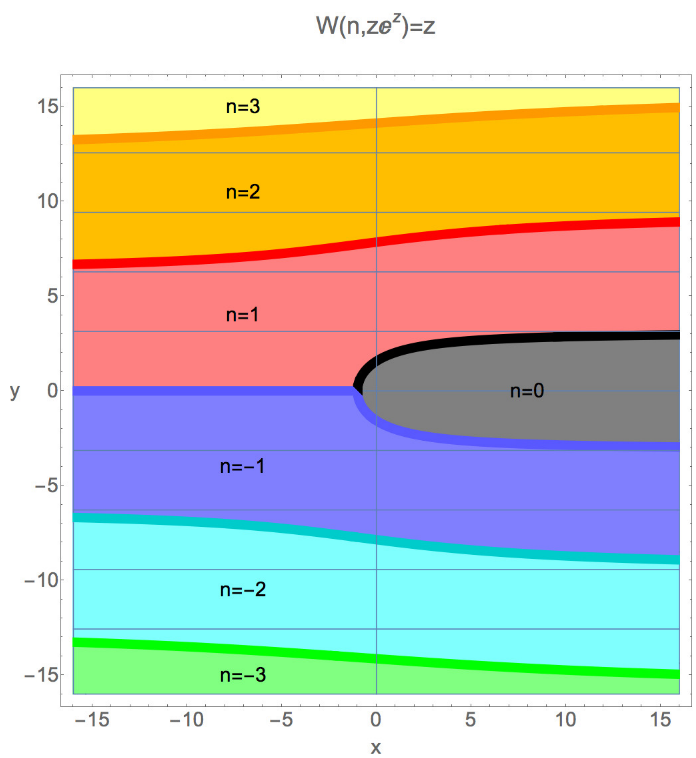

Figure 2.31: Inverted Lambert W function where $W(n, (x + iy) \cdot e^{x+iy}) = x + iy$, from Wikipedia[18].

2.4.2 $cos^{12}(x) + i \cdot sin^{12}(x)$

If you look at the equation $cos^{12}(x) + i \cdot sin^{12}(x)$ it might seem utterly unremarkable. My goal is to convince you it has an interesting quality.

Looking at equations like this, there are two forms:

$cos^{(2k+1)}(x) + i \cdot sin^{(2k+1)}(x)$, which repeats after the range $[0, 2\pi]$.
$cos^{(2k)}(x) + i \cdot sin^{(2k)}(x)$, which repeats after the range $[0, \pi]$.

Thus values that have even exponents seem to have interesting behavior where they seem to create an internal redundancy that can best be seen in the roots of the equation. For the case of 12 we get:

$x_{01} \approx +0.785398 + 0.807445i + 2\pi n$, solution known
$x_{02} \approx +0.785398 - 0.201599i + 2\pi n$, solution known
$x_{03} \approx +0.785398 - 1.362520i + 2\pi n$
$x_{04} \approx +0.785398 - 0.540208i + 2\pi n$
$x_{05} \approx +0.785398 + 0.353475i + 2\pi n$
$x_{06} \approx +0.785398 + 0.065637i + 2\pi n$
$x_{07} \approx -0.785398 - 0.807445i + 2\pi n$, solution known
$x_{08} \approx -0.785398 + 0.201599i + 2\pi n$, solution known
$x_{09} \approx -0.785398 + 1.362520i + 2\pi n$
$x_{10} \approx -0.785398 + 0.540208i + 2\pi n$
$x_{11} \approx -0.785398 - 0.353475i + 2\pi n$
$x_{12} \approx -0.785398 - 0.065637i + 2\pi n$
$x_{13} \approx +2.356194 - 0.807445i + 2\pi n$, solution known
$x_{14} \approx +2.356194 + 0.201599i + 2\pi n$, solution known
$x_{15} \approx +2.356194 + 1.362520i + 2\pi n$
$x_{16} \approx +2.356194 + 0.540208i + 2\pi n$
$x_{17} \approx +2.356194 - 0.353475i + 2\pi n$
$x_{18} \approx +2.356194 - 0.065637i + 2\pi n$
$x_{19} \approx -2.356194 + 0.807445i + 2\pi n$, solution known
$x_{20} \approx -2.356194 - 0.201599i + 2\pi n$, solution known
$x_{21} \approx -2.356194 - 1.362520i + 2\pi n$
$x_{22} \approx -2.356194 - 0.540208i + 2\pi n$
$x_{23} \approx -2.356194 + 0.353475i + 2\pi n$
$x_{24} \approx -2.356194 + 0.065637i + 2\pi n$

Which can easily be seen in Wolfram Alpha[12].

I believe that the "bifurcation artifact" here is that the real values are always $\pm x$ with two possible values of x, in this case ± 0.7853981633 and ± 2.3561944901.

Additionally, I find it unusual that Wolfram Alpha cannot solve four particular imaginary values, since it has 16 equations and 4 values to infer, and I suspect there is something particularly interesting to be learned here about finding roots of equations. It seems like using this kind of symmetry will help with solving more complex problems.

2.4.3 $cos^{28}(x) + i \cdot sin^{28}(x)$

A similar equation I am interested in is the 28th power version: $cos^{28}(x) + i \cdot sin^{28}(x)$.

This equation has 56 roots (8 known). Additionally, it has 48 equations to solve 12 variables.

While I am not going to write out each root of $cos^{28}(x)+i \cdot sin^{28}(x)$, I will mention that part of the reason I am interested in it has to do with what I call the "lucky numbers of Martin" (4.2.5), which seems to have a relation to divisors of 12 and 28.

"Chaotic Bifurcations"

I define a "chaotic bifurcation" as a bifurcation that splits into two outcomes with unequal probabilities.

3.1 The Collatz Conjecture

The Collatz conjecture is a prediction about whether a particular mathematical process eventually ends by reaching 1. There are only two rules for computing a Collatz (or hailstone) sequence of a positive integer n:

> If n is even, divide by two.
> If n is odd, multiply by three and add one.

Where the process ends when you reach 1. If you continue after reaching 1, you endlessly cycle in a $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$ loop. The conjecture states that all positive integers eventually lead to 1, without blowing up to infinity or cycling in a different loop.

The Collatz conjecture also features a Collatz constant i.e. $\frac{1+\sqrt{\frac{7}{3}}}{2}$, which is tied to how many iterations are needed to reach 1[19].

3.1.1 Local Bifurcations

The Collatz conjecture can be considered in reverse as a tree structure. The tree only requires two rules as before:

> If $n \equiv 4 \bmod 6$, connect n to two branches $2n$ and $\frac{n-1}{3}$.
> Otherwise, connect n to $2n$.

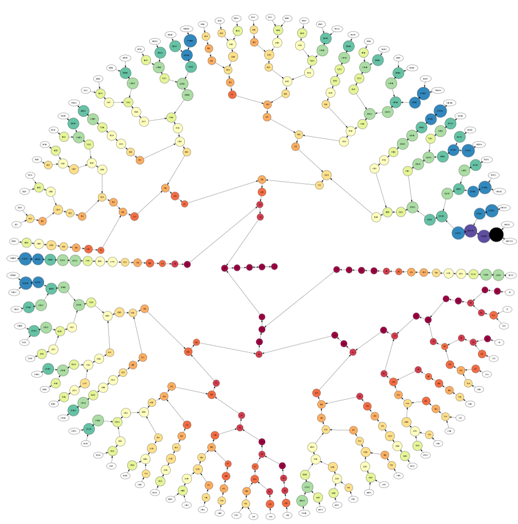

Figure 3.1: Numbers with depth less than 20 in the Collatz tree from Wikipedia[20].

The above definition is the mechanism by which I relate the Collatz conjecture to bifurcation theory. I consider this to be a "chaotic bifurcation" because the integers do not have a 50:50 split along each branch.

The naive way to determine the probability a given branch is taken is to compute across all integers from 1 to n[21], although I have had more success sampling pseudorandom numbers in a large range. For the case of the first bifurcation at $n = 16$ (into 5 and 32), I conjecture the probability might be explained by $P_{even} = 1 - P_{odd} = (\frac{5}{32})^{\frac{3}{2}} \approx 0.061763...$, where

P_{even} is the probability a randomly chosen number will visit from the even branch. The empirical code I used to make this conjecture can be found in the following section.

While empirical methods can be used to try to infer the probability a pseudorandom number will go through a given branch, these methods are far from perfect. I have not seen a proof of convergence, so it is possible the probability of a node branching cannot be determined in any meaningful sense, although I have seen people try to compute these probabilities and they seem to converge.

The issue with this pseudorandom methodology is sampling data is less effective than ideal. Naively sampling works well for the bifurcation at $n = 16$ since almost all values need to travel through that node to reach 1, but getting a lot of sample data for a branch that has a low probability of being visited is difficult.

Still, the method of looking at the Collatz tree from the perspective of conditional probabilities can be alluring. One interesting observation is that paths to $3k$ (i.e. 3, 6, 9, 12, 15...) must be visited almost never (i.e. 0% of the time for each), but since they can still be visited it makes sense to refer to the probabilities as infinitesimals, which I believe is a compelling case for relating the Collatz conjecture to surreal numbers. Because multiples of three can be removed from the graph in some sense, papers like Kleinnijenhuis and Kleinnijenhuis and Aydogan (2008)[22] generate a tree structure that essentially relies on bases 18, 54, and 288 to remove uninteresting branches.

Most branching probabilities seem small for the even path, and because of this an otherwise chaotic graph does have likely and unlikely outcomes, which might be useful for finding underlying structure.

Another useful strategy might be to use graph theory alongside knowledge surrounding values separated by a fixed distance. The observation is that the hailstone sequence of the pair $(8n + 4, 8n + 5)$ are essentially the same, so those two nodes are definitely not isolated from each other. You can increase the distance to nodes separated by a distance of 2 and greater to try to reduce the number of isolated nodes further and further.

3.1.2 Global Bifurcations

Although it is useful to look at local properties of the Collatz tree, I believe "chaotic bifurcations" are usually caused by global phenomenon. I wrote some code that tries to capture this phenomenon. Originally it was intended to look at cumulative probabilities of each branch, but since I accidentally used $1 - P_{odd}$, I stumbled on a relatively orderly graph, since the more probable case takes up a smaller percentage of the graph. I am not entirely certain this phenomenon is surprising, but I suspect it relates to underlying order in the Collatz conjecture.

The code can be seen here:

```python
import svgwrite
import math
import sympy
from random import randrange, uniform

center = (4000, 4000)
canvas_size = (2*center[0], 2*center[1])

class CollatzMetrics:
    n = 0
    times_visited = 0 # randomly chosen test points not counted
    times_visited_by_odd_path = 0

    def __init__(self, n):
        self.n = n

def step(n):
    return n//2 if n % 2 == 0 else 3*n + 1

visited_nodes = {}
def augmented_step(n):
    was_even = (n % 2 == 0)
    n = step(n)
    if not n in visited_nodes.keys():
        visited_nodes[n] = CollatzMetrics(n)
    metrics = visited_nodes[n]
    metrics.times_visited += 1
    if not was_even:
        metrics.times_visited_by_odd_path += 1
    return n

def simulate(n):
```

```python
    while (n != 1):
        n = augmented_step(n)

def print_node(number, radius, angle):
    position = (center[0] + 20*radius*math.cos(angle), center[1] + 20*radius*math.sin(angle))
    text_position = (position[0] - 10, position[1] + 5)
    if abs(position[0] - center[0]) > center[0] or abs(position[1] - center[1]) > center[1]:
        return
    canvas.add(canvas.circle(center=position, r=15, fill='green'))
    canvas.add(canvas.text(number, insert=text_position, fill='black'))

def lerp(a, b, t):
    return a + (b-a) * t

def print_tree(node=8,min_probability=0, max_probability=1):
    # base case
    if not node in visited_nodes:
        return

    # rendering step
    print_node(node, 1.1*node, 2*math.pi*max_probability)

    # recursive cases (tree traversal)
    if node % 6 == 4:
        if (node - 1)//3 % 3 == 0:
            print_tree((node - 1)//3, min_probability, min_probability) # epsilon path 3*2^k
            print_tree(2*node, min_probability, max_probability)
        else:
            odd_probability = visited_nodes[node].times_visited_by_odd_path/visited_nodes[node].times_visited
            mid_probability = lerp(min_probability, max_probability, 1 - odd_probability)
            print_tree((node - 1)//3, min_probability, mid_probability)
            print_tree(2*node, mid_probability, max_probability)
    else:
        print_tree(2*node, min_probability, max_probability)

min_starting_value = 16 + 1
max_starting_value = int(2**128) - 1
test_cases = 10000
for test_case in range(test_cases):
    test_value = randrange(min_starting_value, max_starting_value)
    simulate(test_value)

canvas = svgwrite.Drawing('Collatz_Quasi_Cumulative_Probabilities.svg', profile='tiny', size=canvas_size)
print_tree()
canvas.save()

print(1 - visited_nodes[16].times_visited_by_odd_path/visited_nodes[16].times_visited)
```

Which results in the following graph:

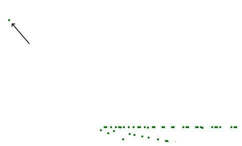

Figure 3.2: Zoomed-out view showing stray diagonal.

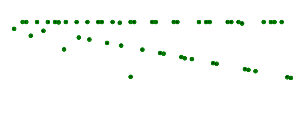

Figure 3.3: Zoomed-in view with better detail.

The portion of this graph I find the weirdest is the diagonal travelling up and left, which includes $\{85, 151, 170, 227\} = \{5 \cdot 17, prime(6^2), 2 \cdot 5 \cdot 17, prime(7^2)\}$, where $2 \cdot 5 \cdot 17$ is reminiscent of both the incremented exponents within the Fermat primes and also the negative cycles in the Collatz conjecture (using -2, -5, -17). The precise angle of the diagonal itself is difficult to measure due to fluctuation, but I suspect it may be 4 radians, relating to $log_e(\frac{4}{\pi})$, sometimes known as the "alternating Euler constant"[15].

While it is naive to call any graph of the Collatz conjecture orderly, the lines are somewhat orderly compared to other visualizations I have seen.

3.2 "Chaotic Quantized Bifurcations"

Like "quantized bifurcations" that result in 50:50 splits of outcomes, I believe there can be "chaotic quantized bifurcations" that result in uneven outcomes.

3.2.1 Centered Hexagon Prime Spirals

Typically the Ulam spiral is generalized to a hexagonal grid with an identical strategy: counting clockwise or counterclockwise from a central cell.

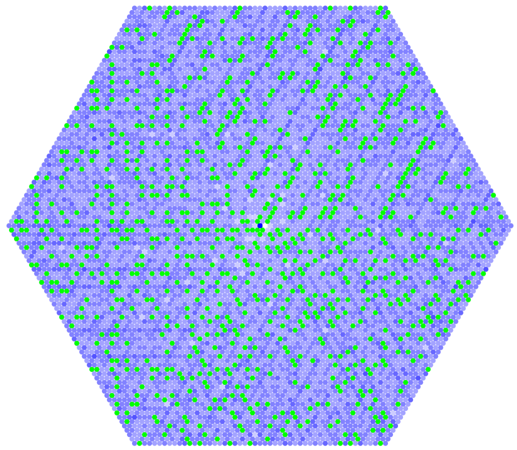

Figure 3.4: Modified image of the hexagon prime spiral from Wikipedia[23].

While this method highlights some trends, I do not consider it as visually remarkable as the Ulam spiral, and I believe this is due to a flaw.

I later discovered a double hexagon spiral that has interesting properties:

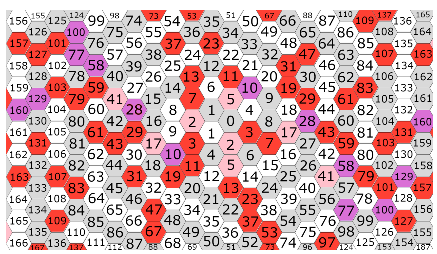

Figure 3.5: A prime spiral using the double hexagon (gray and white spirals). Primes are in red; cumulative sums of primes in purple; both conditions in pink.

Where the highlighted elements are primes in red and sums of primes in purple (and pink when both conditions are met). Where notably the sums of primes have an interesting diagonal. Studying this diagonal, the sequence:

2, 8, 15, 27, 40, 58, 77, 101...

Can be turned into $\frac{3n^2-5n+3+(-1)^{1+n}}{2}$ where notably the eight elements in $n=[2,9]$ are within $(3-\lfloor\frac{n-1}{2}\rfloor)$ of the sum of the first n primes. Interestingly, if you try to get a similar formula for the first n primes, you get seven terms accurately for $\frac{6n-9-(-1)^n}{2}$, which is correct for terms $n=[3,9]$. The reason the summation version gets an additional term correct has to do with how the estimation of primes essentially looks at all primes congruent to 1 or 5 mod 6, which results in an estimate {-1, +1, 5, 7, 11, 13, 17, 19, 23}, and when the summation happens for the first two terms (i.e. -1 and $+1$), they cancel resulting in an additional valid term, with proper modification.

This means there are two competing metrics:

Type	Equation	Below	Exact	Above
Primes	$\frac{6n-9-(-1)^n}{2}$	underestimates $[1,2]$	$[3,9]$	underestimates $[10,\infty)$
Sum of Primes	$\frac{3n^2-5n+3+(-1)^{1+n}}{2}+(3-\lfloor\frac{n-1}{2}\rfloor)$	overestimates $[1,1]$	$[2,9]$	underestimates $[10,\infty)$

Taking this a step further, a continuous diagram can be made, which I conjecture is a precursor to the "imaginary quantized bifurcation" based on the Sacks spiral:

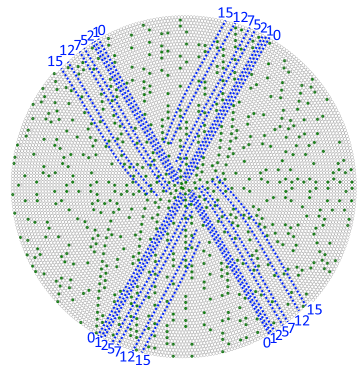

Figure 3.6: Conjectured precursor to "imaginary quantized bifurcation" derived from two copies of $radius = \frac{\sqrt{24n-3}-3}{12}$ that creates four counterclockwise sets of generalized pentagonal number (i.e. 0, 1, 2, 5, 7, 12, 15...) composed of composites.

Which notably has diagonal lines that seem to be statistically significant to the eye, like the Sacks spiral. There are several interesting prime-generating polynomials:

$6n^2 + 5, n = 0...4$
$6n^2 + 7, n = 0...6$
$6n^2 + 13, n = 0...12$
$6n^2 + 17, n = 0...16$
$6n^2 + 6n + 2, n = 0...0$
$6n^2 + 6n + 5, n = 0...3$
$6n^2 + 6n + 3, n = 0...0$
$6n^2 + 6n + 7, n = 0...4$
$6n^2 + 6n + 11, n = 0...8$
$6n^2 + 6n + 17, n = 0...14$
$6n^2 + 6n + 31, n = 0...28$

And I conjecture these two classes relate to the Fibonacci and Lucas numbers:

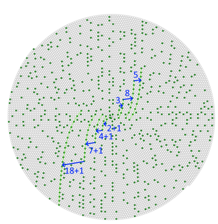

Figure 3.7: Fibonacci and Lucas connections to prime diagonals in the conjectured precursor to the "imaginary bifurcation".

However, we are interested primarily in the "chaotic quantized bifurcations", which I conjecture also use the centered hexagonal numbers in a different way. Below, three variants can be seen:

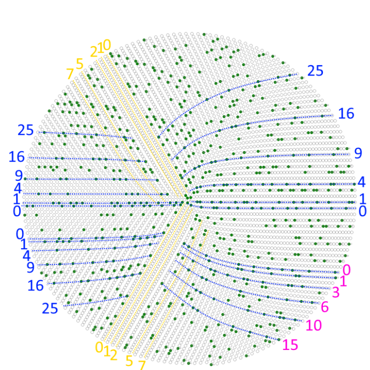

Figure 3.8: $r = \frac{\sqrt{24n-3}-3}{6}$

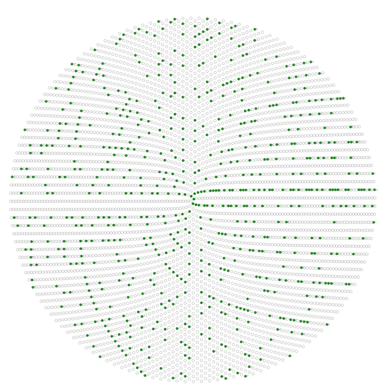

Figure 3.9: $r = \frac{\sqrt{36n-3}-3}{6}$

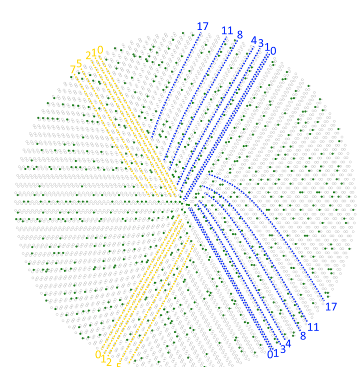

Figure 3.10: $r = \frac{\sqrt{48n-3}-3}{6}$

Where notably the middle of the forms is essentially identical to the original "quantized bifurcation" (due to being degenerate), whereas the outer two are conjectured precursors to the "chaotic quantized bifurcation".

The "chaotic quantized bifurcations" notably use the generalized pentagonal numbers in addition to a new sequence for each. For the left case, tessellated square numbers every 12 can be added (not removed, which is unusual), whereas the right case has A186423[24] or A186424[25], depending on your perspective, removed.

The code for all the graphs can be seen here:

```
import svgwrite
import math
import sympy

center = (2000, 2000)
canvas_size = (2*center[0], 2*center[1])

def print_node(number, radius, angle):
    position = (center[0] + 20*radius*math.cos(angle), center[1] + 20*radius*math.sin(angle))
    text_position = (position[0] - 10, position[1] + 5)

    if sympy.isprime(number):
        canvas.add(canvas.circle(center=position, r=10, fill='green'))
        canvas.add(canvas.text(number, insert=text_position, fill='gray'))
    else:
        canvas.add(canvas.circle(center=position, r=10, fill='none', stroke='black'))

def centered_hexagonal_inverse3(n): return (math.sqrt(24*n - 3) - 3)/6
def centered_hexagonal_inverse_degenerate(n): return (math.sqrt(36*n - 3) - 3)/6
def centered_hexagonal_inverse4(n): return (math.sqrt(48*n - 3) - 3)/6
def centered_hexagonal_inverse5(n): return (math.sqrt(24*n - 3) - 3)/12

def print_sacks_spiral(max_value):
    for n in range(max_value):
        print_node(n, math.sqrt(n), 2*sympy.pi*math.sqrt(n))

def print_imaginary_precursor(max_value):
    for n in range(1,max_value):
        radius = centered_hexagonal_inverse5(n)
        print_node(n, +2*radius, 2*sympy.pi*radius)
        radius = centered_hexagonal_inverse5(n)
        print_node(1, -2*radius, 2*sympy.pi*radius)

def print_chaotic_precursor1(max_value):
    for n in range(1,max_value):
        radius = centered_hexagonal_inverse3(n)
        print_node(n, radius, 2*sympy.pi*radius)

def print_chaotic_precursor2(max_value):
    for n in range(1,max_value):
        radius = centered_hexagonal_inverse4(n)
        print_node(n, radius, 2*sympy.pi*radius)

def print_degenerate_spiral(max_value): #essentially Sacks spiral
    for n in range(1,max_value):
        radius = centered_hexagonal_inverse_degenerate(n)
        print_node(n, radius, 2*sympy.pi*radius)

canvas = svgwrite.Drawing('Prime_Spiral.svg', profile='tiny', size=canvas_size)
print_chaotic_precursor1(5000)
canvas.save()
```

3.3 The Mandelbrot Set

The Mandelbrot set is the group of points on the complex plane that do not diverge under the repeated iteration map $z_{n+1} = {z_n}^2 + c$, where a complex point c with magnitude greater than 2 is guaranteed to blow up to infinity.

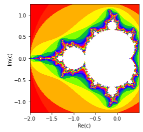

Figure 3.11: The Mandelbrot set in white, with diverging values in different hues to show how slowly they diverge.

The Mandelbrot set has two well-defined conditions, for the 1-bulb (the main cardioid) and the 2-bulb (a perfect disk). All other disks and cardioids, even if they look like perfect circles or cardioids, seem to be stretched.

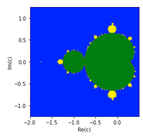

Figure 3.12: The Mandelbrot set with the main cardioid and 2-bulb in green and yellow otherwise; diverging values in blue.

The outer boundary of the main cardioid can be parameterized by $\frac{e^{2\pi ix}}{2} - (\frac{e^{2\pi ix}}{2})^2$ where x is a real number, and testing if a point is within the main cardioid requires checking the inequality $|1 - \sqrt{1-4c}| \leq 1$ where c is the original complex point.

The outer boundary of the 2-bulb can be parameterized by $\frac{e^{2\pi ix}}{4} - 1$ where x is a real number, and testing if a point is within the 2-bulb requires checking the inequality $|c + 1| \leq \frac{1}{4}$ where c is the original complex point again.

If you parameterize the boundaries with rational inputs, you can find attached bulbs. For example, using $x = \frac{1}{3}$ or $x = \frac{2}{3}$ on the main cardioid results in the two 3-bulbs, which spawn out of $-\frac{1}{8} \pm (\frac{\sqrt{3}}{2})^3 i$.

3.3.1 The Mandelbrot Set as An Opposing Representation of The Riemann Zeta Function

I believe the Mandelbrot set is a means of visualizing infinity. Just as we can start with nothing and generate additively, I believe one can also visualize numbers starting from everything and sculpting away subtractively.

I view the Riemann hypothesis as a way of visualizing numbers getting larger over time, where the input to the Riemann zeta function can be thought of as a dimension (usually finite). The Mandelbrot set, alternatively, focuses on numbers that have been iterated indefinitely, so the focus shifts towards numbers that are already large in some sense.

It is based on this thought that I compare the main cardioid of the Mandelbrot set to the Riemann zeta function, so I view the two to be opposites in some sense. Notably, I believe the Riemann zeta function focuses on the additive and small, whereas the Mandelbrot set focuses on the multiplicative and large. The Riemann hypothesis starts as a question about infinite summation; whereas the Mandelbrot set starts as a question about infinite squaring and adjoining.

The thought that we can exceed infinity, or even look at a percentage of infinity, is an unnatural one that most mathematicians recoil against, but I want to suggest that this limitation might be unnecessary.

Consider for a moment Zeno's paradox of Achilles and the tortoise, that Achilles can never run past the tortoise, since Achilles can only halve the distance between them at any moment. We find it hard to grasp the moment where infinity is reached, and we reconcile it with infinite summation or integration, but it would not be a paradox if we only asked when Achilles closes 75% of the distance with the tortoise. From one perspective, we can note we are using a concept of infinity and yet meaningfully derive a notion of 90% or even 110% of infinity. This notion is only weird because it is unnecessary for a model so simple, but that does not mean shedding our limitation in one place does not help us shed that limitation elsewhere.

I believe the nature of infinity is that it fundamentally changes over time. I believe that reaching 110% of infinity is possible with the caveat that the type of data must evolve (e.g. integers into rational numbers). Similarly, I believe it is possible that the nature of bifurcations may correspond to an infinity being reached in some sense, where the underlying properties of the system must evolve into something larger with new emergent properties.

3.3.2 A Cardioid of Bifurcations

Consider the following cardioid of bifurcations:

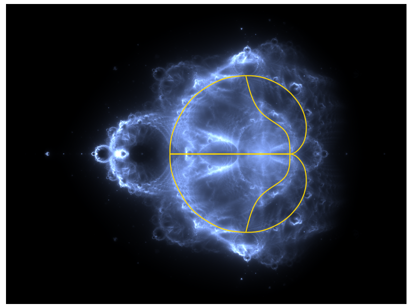

Figure 3.13: A modification of the Buddhabrot set[26] where a conjectured cardioid of bifurcations is superimposed.

Note that the corresponding image for the Riemann zeta function has an imaginary bifurcation on the right, and here it has vanished and been replaced with an "artifact of infinity", where the repeated operations from the definition of the Mandelbrot set remove the imaginary bifurcation altogether. Consider for a moment that cardioids can be created from a fixed point on a circle, with the cardioid defined by the points reflecting the fixed point across all possible tangents. Because of this, you can think about the original circle as:

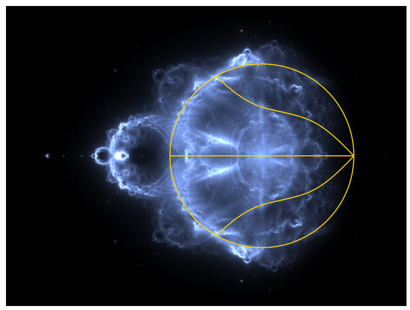

Figure 3.14: A modification of the Buddhabrot set[26] with a conjectured unwrapping of the cardioid of bifurcations.

And the thought process is that this is the natural outcome after being skewed by the concept of infinity.

Looking at internal details of the Mandelbrot set is difficult, because the intention of the Mandelbrot set is to look at the behavior of an infinitely repeated operation, and thus it hides the details of finite elements, as I see it. Arguably, this explains why divergence conditions are colored but not convergence conditions in the Mandelbrot set. Some detail might still be inferable from the Buddhabrot, though, so there is always a path forward, even if it is difficult.

3.3.3 "Mandelbrot Set Recursion"

I believe an unusual extension of the Mandelbrot set is: what if we want to go from infinity backwards? Consider for a moment a concept that makes the Mandelbrot set even more infinite than it already is: take the Mandelbrot set M_0 and consider all points for being included in a new set M_1, if a point only passes through valid points in M_0 it is considered a part of M_1, and this process can be repeated recursively until M_∞. Originally, I thought this set would approach an area of zero, but that was because I missed that the real range approaches $[-1, 0]$ (not just zero). If I had to guess, the leftmost side approaches a cardioid and the rightmost side approaches a comet, although the area could still be zero (with an epsilonic shape) or it could have a vastly different shape.

While this problem may seem difficult at first, I think there is a fair probability the outcome is identical to starting with the main cardioid and the 2-bulb, which should drastically simplify calculations.

I have constructed some versions that I believe show the evolution of this shape over time, and since I tend to avoid thinking about the Mandelbrot set, I will leave this as an exercise for others, if they are interested.

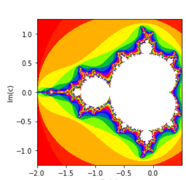

Figure 3.15: Mandelbrot set

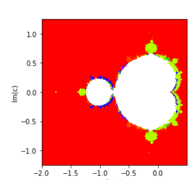

Figure 3.16: 1st Recursion

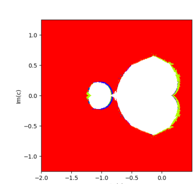

Figure 3.17: 2nd Recursion

I would also like to suggest the possibility that knowing the convex hull[27] on a Riemann sphere of the recursive and original Mandelbrot set (which may include an eighth degree curve[28]) might provide some new insight to the problem.

3.4 Music Theory

Currently, "music theory has no axiomatic foundation in modern mathematics"[29], and my goal is to propose ideas that may help generate music axioms.

3.4.1 The 12-Tone Octave

The Western-style 12-tone octave is often defined by[29]:

Semitone	Transition	Musical Interval	Step Ratio
0	1	unison	
1	$\frac{16}{15}$	semitone	$\frac{16}{15}$
2	$\frac{9}{8}$	major second	$\frac{135}{128}$
3	$\frac{6}{5}$	minor third	$\frac{16}{15}$
4	$\frac{5}{4}$	major third	$\frac{25}{24}$
5	$\frac{4}{3}$	perfect fourth	$\frac{16}{15}$
6	$\frac{45}{32}$	diatonic tritone	$\frac{135}{128}$
7	$\frac{3}{2}$	perfect fifth	$\frac{16}{15}$
8	$\frac{8}{5}$	minor sixth	$\frac{16}{15}$
9	$\frac{5}{3}$	major sixth	$\frac{25}{24}$
10	$\frac{9}{5}$	minor seventh	$\frac{27}{25}$
11	$\frac{15}{8}$	major seventh	$\frac{25}{24}$
12	2	octave	$\frac{16}{15}$

An interesting relationship between music theory and number theory is that the unison, octave, perfect, and major notes can be derived from the 5-smooth numbers[30]:

1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, **24**, 25, **27, 30, 32, 36, 40, 45, 48**, 50, 54, 60, 64, 72, 75, 80, 81, 90, 96...

Where if you take the subsequence in the range $[5^2 - 1, 7^2 - 1]$ except 5^2 and divide by 24, you get:

$\frac{24}{24} = 1$ (unison)
$\frac{27}{24} = \frac{9}{8}$ (major second)
$\frac{30}{24} = \frac{5}{4}$ (major third)
$\frac{32}{24} = \frac{4}{3}$ (perfect fourth)
$\frac{36}{24} = \frac{3}{2}$ (perfect fifth)
$\frac{40}{24} = \frac{5}{3}$ (major sixth)
$\frac{45}{24} = \frac{15}{8}$ (major seventh)
$\frac{48}{24} = 2$ (octave)

Where uni- (i.e. 1) is the prefix of unison and oct- (i.e. 8) is the prefix of octave, so notably this group can be thought of as 1-8, enumerating major notes in the 12-tone octave.

If you look at the minor notes you get:

$\frac{(\frac{\mathbf{128}}{5})}{24} = \frac{16}{15}$ (semitone)
$\frac{(\frac{\mathbf{144}}{5})}{24} = \frac{6}{5}$ (minor third)
$\frac{(\frac{135}{4})}{24} = \frac{45}{32}$ (diatonic tritone)
$\frac{(\frac{192}{5})}{24} = \frac{8}{5}$ (minor sixth)
$\frac{(\frac{\mathbf{216}}{5})}{24} = \frac{9}{5}$ (minor seventh)

Where the terms $128 = 2^7$, $144 = 12^2$, and $216 = 6^3$ may be explained by $128 = (\frac{12}{3!})^{(3!+1)}$, $144 = (\frac{12}{1!})^{(1!+1)}$, $216 = (\frac{12}{2!})^{(2!+1)}$. Using this theory, the other two might be $192 = (\frac{12}{x!})^{(x!+1)}$ and $(13^2 - \frac{1}{4}) = (\frac{12}{x!})^{(x!+1)}$, possibly approximations of $\sqrt{\pi}$ and $\frac{3}{2}$, where the latter would occur at the diatonic tritone. This would result in the following values:

Semitone, s_0	$(s_0 - 2) \bmod 12, s_{-2}$	n	$(\frac{12}{n!})^{(n!+1)}$	Transition	Interval
1	**11**	**3**	$2^7 = \frac{\mathbf{16}^2}{2}$	$\frac{(\frac{2^7}{5})}{24} = \frac{16}{15}$	semitone
3	**1**	**1**	$\mathbf{12}^2$	$\frac{(\frac{12^2}{5})}{24} = \frac{6}{5}$	minor third
6	**4**	$\approx \boldsymbol{\frac{3}{2}}$	$\approx \mathbf{13}^2 - \frac{1}{4}$	$\frac{(\frac{13^2 - \frac{1}{4}}{5})}{24} = \frac{45}{32}$	diatonic tritone
8	**6**	$\approx \boldsymbol{\sqrt{\pi}}$	$\approx \mathbf{14}^2 - 2^2$	$\frac{(\frac{14^2 - 2^2}{5})}{24} = \frac{8}{5}$	minor sixth
10	**8**	**2**	$6^3 = \mathbf{15}^2 - 3^2$	$\frac{(\frac{6^3}{5})}{24} = \frac{9}{5}$	minor seventh

Which hints at a slight misalignment compared to the major notes, with the central value arguably being the minor sixth.

The numbers 1, 4, 6, 8, 11 (corresponding to positive semitone offsets relative to the major second) specifically remind me of an offset problem from the class 1 numbers (4.2.1), specifically the table:

$$\begin{array}{lllllll}
Class1(1) + Class1(3) - 7 + 0 & = & 1 + 3 - 7 + 0 & = & -3 & = & Lucas(-3) + 1 \\
Class1(2) + Class1(4) - 7 - 1 & = & 2 + 7 - 7 - 1 & = & 1 & = & \boldsymbol{Lucas(1)} \\
Class1(3) + Class1(5) - 7 + 0 & = & 3 + 11 - 7 + 0 & = & 7 & = & \boldsymbol{Lucas(4)} \\
Class1(4) + Class1(6) - 7 - 1 & = & 7 + 19 - 7 - 1 & = & 18 & = & \boldsymbol{Lucas(6)} \\
Class1(5) + Class1(7) - 7 + 0 & = & 11 + 43 - 7 + 0 & = & 47 & = & \boldsymbol{Lucas(8)} \\
Class1(6) + Class1(8) - 7 - 1 & = & 19 + 67 - 7 - 1 & = & 78 & = & Lucas(9) + 2 \quad = \quad 3^4 - 3 \\
Class1(7) + Class1(9) - 7 + 0 & = & 43 + 163 - 7 + 0 & = & 199 & = & \boldsymbol{Lucas(11)}
\end{array}$$

Where the indices of the nearest Lucas numbers are: -3, 1, 4, 6, 8, 9, 11 (with -3 and 9 inexact), which correspond to 1, 4, 6, 8, 11 for exact values and might explain why the semitone is an anomaly. The two imprecise Lucas numbers would both correspond to the major seventh.

You can do something similar with another property of class 1 numbers (based on an analog in the "compressor numbers" (4.2.4)):

$1 = Fibonacci(\mathbf{1}) = Fibonacci(\mathbf{2})$
$\frac{2+3}{2} = 2.5 = Fibonacci(\mathbf{3}) - \frac{1}{2} = Fibonacci(\mathbf{4}) + \frac{1}{2}$
$7 = Fibonacci(\mathbf{5}) + 2 = Fibonacci(\mathbf{6}) - 1$
$\frac{11+19}{2} = 15 = Fibonacci(\mathbf{7}) + 2$
$\frac{43+67}{2} = 55 = Fibonacci(\mathbf{10})$
$163 = Fibonacci(\mathbf{12}) + 19$

Where the indices 1, 2, 3, 4, 5, 6, 7, 10, 12 are used to index into the 5-smooth numbers $\frac{FiveSmooth(24-k)}{24}$, which results in the major notes from the octave backwards through the major second, with the perfect fourth and the major sixth listed twice (and notably the $\frac{3}{2}$ and $\sqrt{\pi}$ notes are directly above and below these values).

The most interesting side effect of that interpretation is that major and minor notes have different directions of enumeration, where major notes are essentially counting backwards. This could explain why major and minor notes sound so different.

3.4.2 Different Possibilities for Music Intervals

I believe the unequal split between major and minor notes is characteristic of "chaotic bifurcations". If you observe the ratio between major and minor notes in the standard octave, it resembles $\frac{7}{5} \approx \sqrt{2}$. Here, the "chaotic bifurcation" may create two "entangled chaotic twins" (4.4.3), e.g. in the context of major and minor notes.

If you try to find music intervals like the octave that approximate the ratio of $\sqrt{2} : 1$ major to minor notes you might stumble upon:

Notes	Major	Minor	Conjectured Range	Conjectured Max Prime
5	3	2		
7	4	3		
12	7	5	24-48	5
17	10	7		
29	17	12	144-288	13
41	24	17		
70	41	29	840-1680	31
99	58	41		
169	99	70	2520-5040	71
239	140	99		
408	239	169	3360-6720	167

Where I would advocate for five music intervals from this list: the 12, 29, 70, 169, and 408 tone music intervals.

Thus, for a 29-tone interval, the goal would be to look at the 13-smooth numbers between 144 and 288 and select 17 major notes.

With respect to the 29-tone interval, I believe there is some compelling evidence of its existence through equal temperaments in music, specifically 29 Equal Division of the Octave (EDO): "[which] tunes the 7th, 11th, and 13th harmonics flat, by roughly the same amount. This means intervals such as 7:5, 11:7, 13:11, etc., are all matched extremely well in 29 EDO"[31], although I believe a better analogy would be the tonal language of 31 EDO[32].

It is also worth noting the number of primes I conjecture for each of the five music intervals is $\{5, 13, 31, 71, 167\} = \{3^0, 3^2, 3^3, 67, 163\} + 4$, which are based on powers of three and class 1 numbers. Additionally, the exponents $\{0, 2, 3\}$ match the exact forms of $(12 + k)^2 - k^2$ from the analysis of minor notes and $(\frac{12}{n!})^{(n!+1)}$.

I believe this relates to one of the most contrived mathematical coincidences in this paper:

$$2 \cdot 3^0 = 7 - 5 = 1 + 1$$
$$2 \cdot 3^1 = 7 - 1 = 5 + 1$$
$$2 \cdot 3^2 = 5^2 - 7$$
$$2 \cdot 3^3 = 7^2 + 5$$

Which has little depth, but I find it interesting.

3.4.3 Transformational Theory

My favorite known paradigm for representing music theory is transformational theory. Transformational theory shifts the focus from musical objects to the relationships between musical objects. The idea is to look at transformations (an unformalized distance metric) and functions within the structure of music[33].

Shifting a note up an octave changes the energy of the system, but the transformation is merely one of doubling; whereas shifting from C to C# results in a more complicated change (which becomes apparent when playing the notes simultaneously).

I would like to propose transformational theory is a good basis for research surrounding music theory and suggest the notions of infinites (timbres) and infinitesimals (silences) can be added, as conjecture.

Infinitesimals (silences) could be included through a zero operator, which simulates a sound wave stopping (leaving a vacuum effect). While there would only be one notion of a zero operator, there could be multiple types of silence (and therefore infinitesimals) corresponding to the silence after various notes. While this means there is only one possible silence type following a note, a short note can be played directly preceding silence to create a different effect.

The presence of infinitesimals would suggest the existence of infinites. It is natural to think infinite would mean infinitely loud in terms of amplitude, but here infinite refers to the wavelength. Thus, the different infinites would essentially be timbres of white noise that can be overlaid on a note.

My current belief is that music theory can be thought of as the theory of the continuous yet finite. It might be a way of separating one channel of data into several channels (e.g. 23^2).

Unfortunately, I do not know as much music theory as I should, so these are my suggestions from the outside looking in.

3.4.4 Different Music Intervals

While writing this paper, I completely forgot about the pitfall of Western music, which siphons life from other branches of music theory. There is always more than one model, and music should be no different.

The flaw of Western music theory is that it somehow manages to be both stuck in the past (overemphasizing classical music from the 1800s and earlier) and a one-sided story (with origins focusing near-exclusively on white men)[34]. After reflecting on it further, it is clear that a lot of the problems in music theory arise from how reductive it is: focusing largely on primeness and a cyclic group with twelve notes (both of which are archaic and outdated concepts). This makes a bias of mine from this work more visible: I regularly refer to three particular elements in math (i.e. standard, chaotic, and imaginary). From this, it is clear my bias is towards what I perceive as standard (arguably integers and primeness), rather than chaotic (perhaps real numbers) or imaginary (perhaps rational numbers). The key takeaway is to consider as many viewpoints as time allows.

One particular data point I completely ignored (as far as I am concerned), is music theory using 5, 7, and 9 tones[31] as well as 24 equal temperament from Arabic music.

The Western 12-tone octave is heavily tied into primality, and I forgot that music can be composed out of other concepts.

A very old idea I had was a concern over not being able to find the notes between the notes on a 12-tone piano. I may have thought producing a more complex octave would create these notes, but I realize now I may have been searching in the wrong place entirely.

If the standard 12-tone octave uses primes, then I think it is natural to wonder what the opposite of primes might use to construct music. I personally view the "hyperstructured numbers" (4.1) to oppose primes, but I did not try to relate it to music theory.

Then I remembered an early note I made about the "hyperstructured numbers": they are identical to the 5-smooth numbers in the range $[1, 24]$. The Western 12-tone octave can essentially be explained by the 5-smooth numbers between $5^2 - 1$ and $7^2 - 1$, so if "hyperstructured numbers" are a foil, it can be expected that a similar phenomenon would happen in this range.

The numbers in this range are:

24, 28, 30, 36, 40, 42, 48

Which would mean six, presumably major, notes in some sense without finding the complementary ones.

With respect to finding the complementary notes, I decided to let holonomic sequences do the work for me and just take the sequence mod 1 to see if new residues emerged, and they did, resulting in two new residues: $\frac{17}{12}$ and $\frac{23}{12}$ (after adding 1 again). This results in the table:

Semitone	Transition	Musical Interval	Step Ratio	Step Increment
1	1	unison		
2	$\frac{7}{6}$	"2"	$\frac{7}{6}$	$\frac{2}{12}$
3	$\frac{5}{4}$	major third	$\frac{15}{14}$	$\frac{1}{12}$
4	$\frac{17}{12}$	"4"	$\frac{17}{15}$	$\frac{2}{12}$
5	$\frac{3}{2}$	perfect fifth	$\frac{18}{17}$	$\frac{1}{12}$
6	$\frac{5}{3}$	major sixth	$\frac{10}{9}$	$\frac{2}{12}$
7	$\frac{7}{4}$	"7"	$\frac{21}{20}$	$\frac{1}{12}$
8	$\frac{23}{12}$	"8"	$\frac{23}{21}$	$\frac{2}{12}$
9	2	octave	$\frac{24}{23}$	$\frac{1}{12}$

This is known as the octatonic scale.

While all of the ratios are different; they are essentially four long gaps interleaved with four short gaps of ratio roughly $\frac{9}{4}$.

Alternatively, this can reduced to two notes in an equal temperament that uses an interval of $\sqrt[4]{2}$ instead of 2 and has notes with fundamental frequencies 1 and $\sqrt[6]{2}$.

Theory aside, what does it sound like? Here is an interactive keyboard using Scale Workshop[35]. Music is subjective, but I personally enjoy it.

There is also nothing that says these notes cannot coexist with the original ones from the western octave, and if you do so the number of unique notes goes from eight and twelve individually to sixteen combined. Perhaps thoughts like this will lead to more robust theories of music in the future.

Semitone	Transition	Musical Interval	Step Ratio	Step Increment
0	1	unison		
1	$\frac{16}{15}$	semitone	$\frac{16}{15}$	$\frac{1}{15}$
2	$\frac{10}{9}$	"quasi-major second"	$\frac{25}{24}$	$\frac{2}{45}$
3	$\frac{7}{6}$	"2"	$\frac{21}{20}$	$\frac{1}{18}$
4	$\frac{6}{5}$	minor third	$\frac{36}{35}$	$\frac{1}{30}$
5	$\frac{5}{4}$	major third	$\frac{25}{24}$	$\frac{1}{20}$
6	$\frac{4}{3}$	perfect fourth	$\frac{16}{15}$	$\frac{1}{12}$
7	$\frac{7}{5}$	"quasi-diatonic tritone"	$\frac{21}{20}$	$\frac{1}{15}$
8	$\frac{36}{25}$	"quasi-4"	$\frac{36}{35}$	$\frac{1}{25}$
9	$\frac{3}{2}$	perfect fifth	$\frac{25}{24}$	$\frac{3}{50}$
10	$\frac{8}{5}$	minor sixth	$\frac{16}{15}$	$\frac{1}{10}$
11	$\frac{5}{3}$	major sixth	$\frac{25}{24}$	$\frac{1}{15}$
12	$\frac{7}{4}$	"7"	$\frac{21}{20}$	$\frac{1}{12}$
13	$\frac{9}{5}$	minor seventh	$\frac{36}{35}$	$\frac{1}{20}$
14	$\frac{15}{8}$	major seventh	$\frac{25}{24}$	$\frac{3}{40}$
15	$\frac{27}{14}$	"quasi-8"	$\frac{36}{35}$	$\frac{3}{56}$
16	2	octave	$\frac{28}{27}$	$\frac{1}{14}$

It is notable that the numerators of the step ratios are 25 five times, 36 four times, 16 three times, 21 three times, and 28 once. The numerator and denominator differ by 1, which is why the adjustments for the "quasi-major second" and "quasi-diatonic tritone" were made. I then adjusted for the "quasi-4" and "quasi-8" because their step ratios were unique.

Like before, an interactive keyboard can be found online here. If I had just naively combined the 12-tone octave and new 8-tone equal temperament, it becomes this version.

Notably, the numerators of the step ratios essentially match consecutive square numbers (i.e. 16, 25, 36) and triangular numbers (i.e. 15, 21, 28, 36).

I think the failure to adhere to simple increments (not just ratios) results in the sound feeling incorrect. While I cannot pinpoint the exact flaw, I suspect it has to do with a conflict between just intonation and equal temperaments. Equal temperaments seem like a statistically sound strategy that fails to adhere to ideals in the same way just intonation tries to.

My personal goal, if this ever becomes used, is to create music that uses the 12-tone and 8-tone music intervals simultaneously in different ears (with shared tones played in both ears).

This leads to the question: what would the full musical group look like? My conjecture would be the following table:

Western-style notes	Additional notes	Total notes
12	2^2	2^4
29	7	6^2
70	11	3^4
13^2	3^3	14^2
408	11^2	23^2

Where I would highlight that most of the right two columns use prime factors 2, 3, 7, and 11 (class 1 numbers), with the exception of 23, which may explain why $Z[e^{2\pi in}]$ fails as a unique factorization domain at $n = 23$. This could also explain why primes become particularly chaotic after 23, when the primes cannot be predicted by greedily taking the next term congruent to 1 or 5 mod 6.

I also find it notable that the ratios of $\frac{11^2}{3^3} = 4.481481...$ and $\frac{3^3}{11} = 2.454545...$ somewhat resemble the first and second Feigenbaum constants, 4.669201... and 2.502907..., although the difference is larger than most would permit for comparison. Looking at some ratios involving this idea:

$$\frac{\frac{14}{3} - \frac{11^2}{3^3}}{4.669201 - \frac{14}{3}} \approx 73 = prime(14 \cdot \tfrac{3}{2}) \approx 24 \cdot 3 = hyperstructured(15) \cdot 3$$

$$\frac{\frac{15}{6} - \frac{3^3}{11}}{2.502907 - \frac{15}{6}} \approx \frac{47}{3} = \frac{prime(15)}{3} \approx 16 = hyperstructured(14 \cdot \tfrac{6}{7})$$

There is not an obvious connection, but I believe it is fruitful territory, and it is somewhat reminiscent of the {2, 3, 6, 7} used in my attempt to create precursors to the third tree of primitive Pythagorean triples (4.4.3), developed later.

The idea for adding further layers beyond 16 tones is essentially the following:

1) Consider a line going from integers to rationals to reals.
2) Create two "entangled chaotic twins" that are complementary, these are the reals.
3) Construct a complementary set that is "imaginary", these are the rational.
4) Overconstrain the system to find the only matching "standard" notes that are are successors to the "entangled chaotic twins" and "imaginary", these are the integers.
5) Keep the integers and discard the rest.
6) Repeat steps 2-5 two additional times (three total).
7) Condense the system, in a way that may relate to 53 EDO, an extended Pythagorean tuning.
8) Construct a complementary set that is "imaginary", these are the rational.
9) Overconstrain the system to find the only matching "entangled chaotic twins" that match the "standard" and "imaginary", these are the reals.
10) Condense the system a final time.
11) Fill in the missing notes

Where the 16-tone music interval would be the first iteration of step 4.

Notably, this would only attempt to explain a small subset of music intervals, because my model would ideally be replaced with a more inclusive one. There are plenty of microtonal and xenharmonic music systems that my model does not explain.

Other Mathematical Beauty

The following subjects are ones that occasionally seem to intersect with my theories involving bifurcation. The common theme is mathematical coincidence, but ideas can relate in surprising ways.

4.1 "Hyperstructured Numbers"

The "hyperstructured numbers" are[36]:

1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 28, 30, 36, 40, 42, 48, 54, 56, 60, 72, 80, 84, 90, 105, 108, 120, 140, 144...

The motivation for the "hyperstructured numbers" is best visualized by counting zeroes in multiplication tables of n, reduced mod n. An example for a 12×12 multiplication table:

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	4	6	8	10	0	2	4	6	8	10	0
3	3	6	9	0	3	6	9	0	3	6	9	0
4	4	8	0	4	8	0	4	8	0	4	8	0
5	5	10	3	8	1	6	11	4	9	2	7	0
6	6	0	6	0	6	0	6	0	6	0	6	0
7	7	2	9	4	11	6	1	8	3	10	5	0
8	8	4	0	8	4	0	8	4	0	8	4	0
9	9	6	3	0	9	6	3	0	9	6	3	0
10	10	8	6	4	2	0	10	8	6	4	2	0
11	11	10	9	8	7	6	5	4	3	2	1	0
12	0	0	0	0	0	0	0	0	0	0	0	0

Which is *almost* a multiplication table, except that numbers ≥ 12 are the remainders after division by 12.

The goal is to count the number of zeroes for different sizes of multiplication tables, and more zeroes means a number is more likely to be a "hyperstructured number". For the case of 12, 40 out of 144 values are zeroes, which is why 12 happens to be a "hyperstructured number". This inefficient computation yields: $A018804(12) = 40$[37], and "hyperstructured numbers" are merely the record values in this sequence.

Summarizing this as code, the inefficient way to compute if a number is "hyperstructured" is as follows:

```python
import sympy
def A018804_inefficient(m, r=0):
  r = sympy.Mod(r, m)
  result = 0
  for a in range(m):
    for b in range(m):
      result += 1 if sympy.Mod(a*b, m) == r else 0
  return result
def A360425_inefficient(n, r=0):
  # using r == 0 results in values that are exactly the "hyperstructured numbers" A360425.
  # using r != 0 results in values that are almost {|r|*prime(n)}.
  record = 0
  for m in range(1, n):
    value = A018804_inefficient(m, r)
    if value > record:
      record = value
      print(m, end=", ")
A360425_inefficient(100, 0)
```

The reason I have coded this inefficiently is as follows: imagine you were counting something other than zeroes?

Consider the output for values other than zero, where negative numbers are treated as *table size plus r*:

-2) 1, 2, 4, 6, 8, 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 106, 118, 122, 134, 142, 146, 158, 166, 178, 194, 202, 206...
-1) 1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127...
0) 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 28, 30, 36, 40, 42, 48, 54, 56, 60, 72, 80, 84, 90, 105, 108, 120, 140, 144...
1) 1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127...
2) 1, 2, 4, 6, 8, 10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 106, 118, 122, 134, 142, 146, 158, 166, 178, 194, 202, 206...
3) 1, 3, 7, 9, 15, 21, 27, 33, 39, 51, 57, 69, 87, 93, 111, 123, 129, 141, 159, 177, 183, 201, 213, 219, 237, 249, 267, 291, 303...
4) 1, 2, 4, 8, 12, 14, 16, 20, 26, 28, 38, 44, 52, 68, 76, 92, 116, 124, 148, 164, 172, 188, 212, 236, 244, 268, 284, 292, 316...
5) 1, 3, 5, 11, 13, 15, 23, 25, 35, 55, 65, 85, 95, 115, 125, 145, 155, 185, 205, 215, 235, 265, 295, 305, 335, 355, 365, 395...

Can you see the pattern?

Aside from the "hyperstructured numbers" at row 0, the other rows seem to contain the primes multiplied by their row number with a finite number of exceptions in each row.

This helps explain why "hyperstructured numbers" after 5 are composite, with a lot of prime factors. In fact, it seems like the n-th "hyperstructured number" is divisible by any positive integer as n approaches infinity.

For example: consider the table of "hyperstructured numbers" not divisible by d (I conjecture there are a finite number of exceptions for each):

2) 1, 3, 5, 9, 15, 105
3) 1, 2, 4, 5, 8, 10, 16, 20, 28, 40, 56, 80, 140, 280
4) 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 42, 54, 90, 105, 210, 270, 330, 630, 990, 1890, 2310, 6930
5) 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 28, 36, 42, 48, 54, 56, 72, 84, 108, 144, 168, 252, 288, 336, 504, 1008, 3024, 3276
6) 1, 2, 3, 4, 5, 8, 9, 10, 15, 16, 20, 28, 40, 56, 80, 105, 140, 280
7) 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 30, 36, 40, 48, 54, 60, 72, 80, 90, 108, 120, 144 ... 23760, 102960, 308880
8) 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 28, 30, 36, 42, 54, 60, 84, 90, 105, 108, 140, 180 ... 15315300, 45945900, 58198140
9) 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 28, 30, 40, 42, 48, 56, 60, 80, 84, 105, 120 ... 40840800, 81681600, 1551950400
10) 1, 2, 3, 4, 5, 6, 8, 9, 12, 15, 16, 18, 24, 28, 36, 42, 48, 54, 56, 72, 84, 105, 108, 144, 168, 252, 288 ... 1008, 3024, 3276

Which explains why "hyperstructured numbers" after 3276 seem to always have at least one trailing zero.

If you are still unconvinced "hyperstructured numbers" relate to primes, here is some additional evidence:

The first 17 primes divided by the first 17 "hyperstructured numbers" are:

$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{11}{5}, \frac{13}{6}, \frac{17}{8}, \frac{19}{9}, \frac{23}{10}, \frac{29}{12}, \frac{31}{15}, \frac{37}{16}, \frac{41}{18}, \frac{43}{20}, \frac{47}{24}, \frac{53}{28}, \frac{59}{30}$

Notably the terms at indices $[1, 8]$, 11, 15, and 17 are essentially a ratio of 2 (with an error of at most one on the numerator). These indices are essentially a combination of n and $19 - 2^n$. It is worth mentioning the first eight essentially correspond to the "serendipitous numbers" (4.2.7), whereas the latter may correspond to:

$$CumulativeSum(\{prime(3), prime(11), prime(15), prime(17)\}) = \{5, 36, 83, 142\} = \{3^2 - 4, 6^2 + 0, 9^2 + 2, 12^2 - 2\}$$

Another interesting detail lies in a conjectured approximation of the n-th "hyperstructured number":

$$n \sim c \cdot log_e(\sqrt{h_n})^{\frac{7}{4}}$$

Where $c = 6$ is a good early estimate, but seems to fail to fluctuate around its target after 1847 elements. The estimate of primes is similar:

$$p_n \sim n \cdot log_e(n)$$

With the fundamental difference being one uses the natural logarithm of the index and the other uses the natural logarithm of the function's output. This seems to indicate the two are opposite in some sense.

Now that some of the properties of "hyperstructured numbers" have been shown, here is a more efficient implementation (correctness conjectured) that can compute "hyperstructured numbers" below 10^{50} in less than a minute through a breadth-first search:

```
from sympy import factorint, primerange
from math import prod, log
import heapq

def A018804(m): return prod(p**(e-1)*((p-1)*e+p) for p, e in factorint(m).items())

prime = list(primerange(0,10000))
```

```
class hyperstructured_node:
  value = 0
  prime_index = 0
  value_of_current_child_node = 0
  max_value_of_child_node = 0
  def __init__(self, value):
    self.value = value
    current_prime_index = 0
    min_missing_prime = 2
    while True:
      if self.value % prime[current_prime_index] != 0:
        min_missing_prime = prime[current_prime_index]
        break
      current_prime_index += 1
    self.max_value_of_child_node = (7/2)*min_missing_prime*self.value
    self.value_of_current_child_node = self.value*2
  def __lt__(self, other):
    return self.value_of_current_child_node < other.value_of_current_child_node
  def increment(self):
    global prime
    self.prime_index += 1
    self.value_of_current_child_node = self.value*prime[self.prime_index]

def A360425_efficient(n):
  global prime
  hyperstructured = [1]
  hyperstructured_nodes = []
  heapq.heappush(hyperstructured_nodes, hyperstructured_node(1))
  record = 1
  while True:
    node = heapq.heappop(hyperstructured_nodes)
    if node.value_of_current_child_node > n:
      break
    value = A018804(node.value_of_current_child_node)
    if value > record:
      hyperstructured.append(node.value_of_current_child_node)
      heapq.heappush(hyperstructured_nodes, hyperstructured_node(node.value_of_current_child_node))
      record = value
    node.increment()
    if node.value_of_current_child_node <= node.max_value_of_child_node:
      heapq.heappush(hyperstructured_nodes, node)
  return hyperstructured
print(A360425_efficient(1E20))
```

I have been considering what the "hyperstructured numbers" represent, and I think within the realm of my theories they are related to imaginary concepts, whereas the primes are standard concepts (non-chaotic). This raises the question of what would be chaotic, and I think the answer relates to (or is) zeta zeroes (using the coefficient of $\frac{5}{3}$ from turbulence[38] in some way).

With respect to suggestions about the sequence itself, I think it would be interesting (and perhaps difficult) to look at the exponents in the prime factorizations of "hyperstructured numbers" as they evolve over time. It seems like "hyperstructured numbers" are divisible by most prime powers below their max prime, but this theory is too informal (e.g. for 105).

Another aspect worth considering is whether "hyperstructured numbers" relate to the density of practical numbers relative to the primes (which uses a coefficient of 1.33607...[39]). This makes me wonder if a primitive additive set of integers can be constructed, like the primes are multiplicative primitive.

The last aspect worth considering is whether a rearrangement with duplicates could be more canonical than this sequence. A standard way of enumerating values is in increasing order, but since the "hyperstructured numbers" seem to have 2+ neighbors within a prime factor multiple, it may make sense to enumerate in a more graph theoretic order.

4.2 Seven Sequences

This section defines seven finite sequences with peculiar properties. The class 1 and 2 numbers, lucky numbers of Euler, and "Ibrishimova numbers" exist in the On-Line Encyclopedia of Integer Sequences, but the other three are new.

4.2.1 Class 1 Numbers

The class 1 numbers are[40]:

1, 2, 3, 7, 11, 19, 43, 67, 163

Which represent values n for which the imaginary quadratic field $Q(\sqrt{-n})$ has unique factorizations.

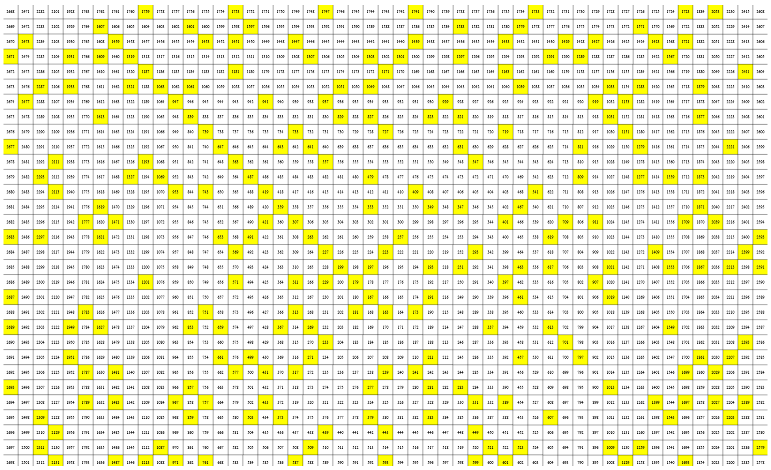

Figure 4.1: Ulam spiral centered on 163, emphasizing the prime diagonal $4n^2 + 163$.

The class 1 numbers can be written using the primes $2 = 1! + 1$, $3 = 2! + 1$, $7 = 3! + 1$:

$163 = 4 \cdot 2 \cdot 3 \cdot 7 - 5$
$67 = 4 \cdot 2 \cdot 3 \cdot 3 - 5$
$43 = 4 \cdot 2 \cdot 2 \cdot 3 - 5$
$19 = 4 \cdot 2 \cdot 3 - 5$
$11 = 4 \cdot 2 \cdot 2 - 5$
$7 = 4 \cdot 3 - 5$
$3 = 4 \cdot 2 - 5$
$2 = 7 - 5$
$1 = 2 \cdot 3 - 5$

Where notably the terms with a factor of 4 (i.e. 3, 7, 11, 19, 43, 67, 163) have unique factorizations across the half-integers, whereas the ones without (i.e. 1, 2) only have a unique factorization across the integers.

It has qualitative similarities to the Janko group J_1.

Some interesting primality coincidences can be seen in the Ulam spiral for class 1 numbers that involve Lucas numbers:

$$4n^2 + p$$

$4n^2 + 3, n = 1...2$
$4n^2 + 19, n = 1...1$
$4n^2 + 1, n = 1...3$
$4n^2 + 43, n = 1...4$
$4n^2 + 67, n = 1...7$
Lucas(5) = 11 is absent
$4n^2 + 163, n = 1...18 + 1$

This leaves a question about the other class 1 numbers, which I conjecture relate to $3 \cdot A007814(k)$[41]:

$4n^2 + 1, n = 1...3$
$4n^2 + 2, n = 1...0$
$4n^2 + 7, n = 1...6$
$4n^2 + 11, n = 1...0$

Additionally, there are some other mathematical coincidences surrounding class 1 and Lucas numbers[42]:

$$Class1(k-1) + Class1(k+1) - 7 + \frac{-1 + (-1)^k}{2} \approx Lucas(n)$$

$$\begin{array}{lclclcll}
Class1(1) + Class1(3) - 7 + 0 & = & 1 + 3 - 7 + 0 & = & -3 & = & Lucas(-3) + 1 & \\
Class1(2) + Class1(4) - 7 - 1 & = & 2 + 7 - 7 - 1 & = & 1 & = & Lucas(1) & \\
Class1(3) + Class1(5) - 7 + 0 & = & 3 + 11 - 7 + 0 & = & 7 & = & Lucas(4) & \\
Class1(4) + Class1(6) - 7 - 1 & = & 7 + 19 - 7 - 1 & = & 18 & = & Lucas(6) & \\
Class1(5) + Class1(7) - 7 + 0 & = & 11 + 43 - 7 + 0 & = & 47 & = & Lucas(8) & \\
Class1(6) + Class1(8) - 7 - 1 & = & 19 + 67 - 7 - 1 & = & 78 & = & Lucas(9) + 2 & = 3^4 - 3 \\
Class1(7) + Class1(9) - 7 + 0 & = & 43 + 163 - 7 + 0 & = & 199 & = & Lucas(11) &
\end{array}$$

Where notably -3 and 78 are not Lucas numbers (but -4 and 76 are). These may relate to music theory (3.4.1).

There is an additional mathematical coincidence approximately relating class 1 numbers and Lucas numbers based on "$1.27 \cdot (1.557)^n$" [43]:

$$y = \lfloor \frac{7}{4} \cdot (1 + \frac{\pi}{2e})^{n-1} \rceil$$

1, 2, 3, 4, 7, 11, 17, 27, 43, 67, 106, 167...

Where the bolded six values are the sorted Lucas numbers (with progressively worse accuracy), and 1, 2, 3, 7, 11, 17, 43, 67, 167 are essentially class 1 numbers (with errors for 19 and 163).

One last observation about class 1 numbers worth mentioning:

$$2 \cdot 3^n + 1, n = 1...4$$

3, 7, 19, 55, 163

Which essentially maps out a fraction of the class 1 numbers (i.e. $7 \pm 4, 19, 55 \pm 12, 163$ or 3, 11, 19, 43, 67, 163).

4.2.2 Lucky Numbers of Euler

The lucky numbers of Euler are[44]:

2, 3, 5, 11, 17, 41

Some authors have conflicting views on whether 1 should be prepended to the list; I personally agree with the On-Line Encyclopedia of Integer Sequences's definition including 6 total, not 7. Each lucky number of Euler can be matched with a class 1 number via: $LuckyEuler(n) = Class1(n+3) \cdot 4 - 1$, so arguably $\frac{1}{2}$, $\frac{3}{4}$, and 1 meet some criteria for the definition.

The lucky numbers of Euler are typically defined by values such that the quadratic $y = n^2 + n + p$ produces primes for all n in $[0, p-2]$, which relates to prime-rich diagonals on the Ulam spiral. By shifting perspective, we can instead look at a relationship between the Fibonacci numbers and the lucky number of Euler.

$$n^2 + n + p$$

$n^2 + n + 1, n = 1...(3 \cdot 1)$
$n^2 + n + 5, n = 1...(3 \cdot 1)$
Fibonacci(3) = 2 is absent
$n^2 + n + 11, n = 1...(3 \cdot 3)$
$n^2 + n + 17, n = 1...(3 \cdot 5)$
Fibonacci(6) = 8 is absent
$n^2 + n + 41, n = 1...(3 \cdot 13)$

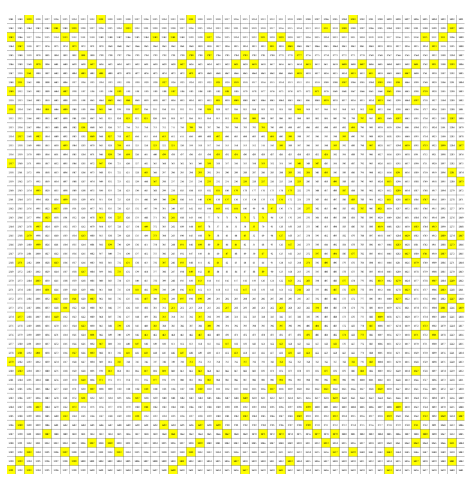

Figure 4.2: Ulam spiral centered on 41, emphasizing the prime diagonal $n^2 + n + 41$.

This leaves a question about the other lucky numbers of Euler, which I conjecture relate to $A001511(k)$ mod 3[45]:

$(n^2+n+\frac{1}{2})\cdot 2, n = 1...2$
$(n^2+n+\frac{3}{4})\cdot 4, n = 1...1$
$n^2+n+2, n = 1...0$
$n^2+n+3, n = 1...1$

Analogous to the class 1 numbers, there seems to be a mathematical coincidence relating to Fibonacci numbers:

$$LuckyEuler(k-1) + LuckyEuler(k+1) - 1 \approx Fibonacci(n)$$

$$\begin{aligned}
\tfrac{1}{2}+1-1 &= \tfrac{1}{2} = Fibonacci(0)+\tfrac{1}{2} = Fibonacci(1)-\tfrac{1}{2} = Fibonacci(2)-\tfrac{1}{2} \\
\tfrac{3}{4}+2-1 &= \tfrac{7}{4} = Fibonacci(3)-\tfrac{1}{4} \\
1+3-1 &= 3 = Fibonacci(4) \\
2+5-1 &= 6 = Fibonacci(5)+1 \\
3+11-1 &= 13 = Fibonacci(7) \\
5+17-1 &= 21 = Fibonacci(8) \\
11+41-1 &= 51 = Fibonacci(10)-4
\end{aligned}$$

Although it is more difficult to make sense of. The indices of the nearest non-negative Fibonacci indices are 0, 1, 2, 3, 4, 5, 7, 8, 10, and I find it curious that $0^2+1^2+2^2-3^2+4^2-5^2+7^2+8^2-10^2 = 0$ where $3^2+5^2+10^2 = 2\cdot 67$. The indices themselves feel like Fibonacci numbers unioned with $3k+1$. I think they may arise after four out of five layers of music theory to fill in prior holes, possibly using a square root and reversal of the time parameter.

Like the class 1 numbers, an equation can be constructed:

$$y = \lfloor(\frac{5}{4})\cdot(1+\frac{\pi}{2e})^{n-2}\rceil$$

1, 1, 2, 3, 5, 8, $3\cdot(5-1), 19, 3\cdot(11-1), 3\cdot(17-1), 76, 3\cdot(41-1)$

Which essentially relates the Fibonacci numbers and the lucky numbers of Euler.

Finally, I want to show a relationship to an inefficient and definitional primality test (from Wilson's theorem):

$$\frac{(n-2)!-1}{n}$$

Which will result in an integer when an input n is prime. A question can be asked about whether outputs can be prime integers as well.

$$\frac{(prime(n)-2)!-1}{prime(n)}$$

Searching this range one finds 5 examples below $prime(220)$:

4, 10, 18, 34, 76

Which may relate to the lucky numbers of Euler as follows:

$2\cdot\{\frac{1}{2}+1, \frac{3}{4}, 1+0, 2, 3-1, 5, 11-2, 17, 41-3\} = \{3, \frac{3}{2}, 2, 2^2, 2^2, 2\cdot 5, 2\cdot 3^2, 2\cdot 17, 2^2\cdot 19\}$

Where notably 4 is written twice. It is reminiscent of both the prime factorization of the order of the Higman-Janko-McKay group (i.e. $2^7\cdot 3^5\cdot 5\cdot 17\cdot 19$) and also the anomaly where $A061006(4) = 2$[13].

4.2.3 "Ibrishimova Numbers"

The "Ibrishimova numbers" are[46]:

5, 13, 17, 53, 61, 107, 251, 283, 1367

And are defined as odd primes p that exist in the hailstone sequence of $p+1$ in the Collatz conjecture. They seem to be related to the balanced primes $A006562(1) = 5$, $A006562(2) = 53$, $A006562(20) = 1367$[47].

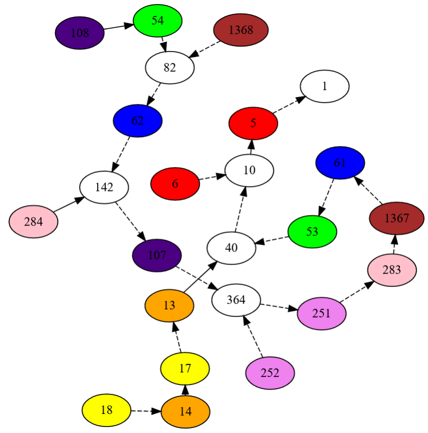

Figure 4.3: "Ibrishimova numbers" and the paths they take in the Collatz conjecture. Solid lines represent 1 step; dashed lines represent 2+ steps.

They have a relationship with the Fibonacci sequence via:

$$\lceil \sqrt{n} - 3 \rceil$$

0, 1, 2, 5, 5, 8, 13, 14, 34

An observation about "Ibrishimova numbers" is analogous with the class 1 numbers:

$$6^n + 61 + 10 \cdot (-1)^n, n = 1...4$$

57, 107, 267, 1367

Which essentially maps out a fraction of the "Ibrishimova numbers" (i.e. $57 \pm 2^{3-1}, 107, 267 \pm 2^{5-1}, 1367$ or 53, 61, 107, 251, 283, 1367).

4.2.4 "Compressor Numbers"

The "compressor numbers" are:

3, $\frac{7}{2}$, 4, 7, 11, 13, 20, 23, 36

Six of them are defined by:

$\lfloor \frac{1}{k} \sum_{i=1}^{k} prime(i) \rceil$

Where k is in $2 \cdot \{1, 2, 3, 5, 7, 11\}$, which defines 3, 4, 7, 13, 20, 36.
The remaining three were essentially found with intuition. The sequence has the Lucas number connection:

$3 - 3 = 0$
$\frac{\frac{7}{2}+4}{2} - 4 = \frac{-1}{4}$
$7 - 7 = 0$
$\frac{11+13}{2} - 11 = +1$
$\frac{20+23}{2} - 18 = \frac{+7}{2}$
$36 - 29 = +7$

They also have a property similar to class 1 numbers when adding adjacent terms $C_{n-1} + C_{n+1}$:

$3 + 4 = 7 = Fibonacci(6) - 1 = Lucas(4)$
$\frac{7}{2} + 7 = \frac{21}{2} = Fibonacci(6) + \frac{5}{2} = Lucas(5) - \frac{1}{2}$
$4 + 11 = 15 = Fibonacci(7) + 2 = Lucas(6) - 3$
$7 + 13 = 20 = Fibonacci(8) - 1 = Lucas(6) + 2$
$11 + 20 = 31 = Fibonacci(9) - 3 = Lucas(7) + 2$
$13 + 23 = 36 = Fibonacci(9) + 2$
$20 + 36 = 56 = Fibonacci(10) + 1$

The "compressor numbers" have qualitative properties similar to the Mathieu group M_{23}.

4.2.5 "Lucky Numbers of Martin"

The "lucky numbers of Martin" are[48]:

2, 3, 4, 5, 7, 8, 13

The "lucky numbers of Martin" are defined by values p such that the quadratic $y = |Numerator(\frac{n^2+n-p^2}{4})|$ produces primes or 1 for all n in $[1, 2p-2]$. Besides 8, the terms are the {divisors of 12} + 1.

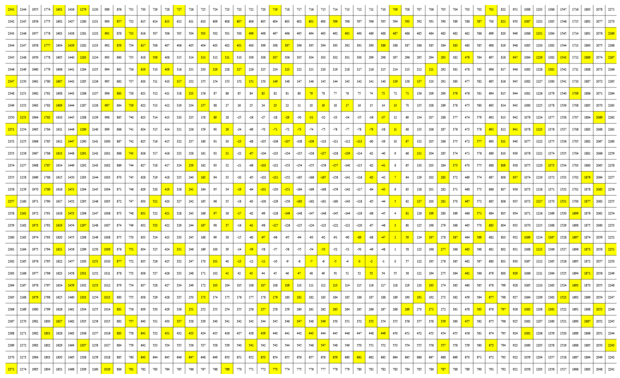

Figure 4.4: Ulam spiral centered on -169, emphasizing the prime diagonal $n^2 + n - 13^2$.

There is a relationship to the "compressor numbers" via $\lfloor \frac{Compressor(k)}{3} \rfloor + 1 = LuckyMartin(n)$:

$\lfloor \frac{36}{3} \rfloor + 1 = 13$
$\lfloor \frac{23}{3} \rfloor + 1 = 8$
$\lfloor \frac{20}{3} \rfloor + 1 = 7$
$\lfloor \frac{13}{3} \rfloor + 1 = 5$
$\lfloor \frac{11}{3} \rfloor + 1 = 4$
$\lfloor \frac{7}{3} \rfloor + 1 = 3$
$\lfloor \frac{4}{3} \rfloor + 1 = 2$
$\lfloor \frac{\frac{7}{2}}{3} \rfloor + 1 = 2$
$\lfloor \frac{3}{3} \rfloor + 1 = 2$

The "lucky numbers of Martin" have a soft relationship to the class 1 numbers:

$$\frac{12}{x} + 1, x = 1...12$$

$13, 7, 5, 4, \frac{17}{5}, 3, \frac{19}{7}, \frac{5}{2}, \frac{7}{3}, \frac{11}{5}, \frac{23}{11}, 2$

Which encodes 12 numbers, 6 of which are "lucky numbers of Martin" of the form {divisors of 12} + 1. If you take these numbers and apply the transform $n^2 - 6$ you get:

$$(\frac{12}{x} + 1)^2 - 6, x = 1...12$$

$163, 43, 19, 10, \frac{139}{25}, 3, \frac{67}{49}, \frac{1}{4}, \frac{-5}{9}, \frac{-29}{25}, \frac{-197}{121}, -2$

Which have prime factorizations that include every class 1 number as well as { 5, 29, 139, 197 }, where notably:

$$Numerator(2 \cdot 3^n + 1), n = -4...4$$

$83, 29, 11, 5, 3, 7, 19, 55, 163$

This approximates both the class 1 numbers and the other four terms if you consider 55 ± 12 and $2 \cdot 83 \pm \sim 29$.

Additionally, if you look at {divisors of 28} + 1, you get more peculiar behavior, particularly when looking at the prime factors of $y = n^2 + n - 29^2$, which seems to be related to the "Ibrishimova numbers" (particularly 5, 13, 17).

Also, I believe the "lucky numbers of Martin" have chaotic properties (because there are not nine terms in the sequence), and I believe it has qualitative properties similar to the Rudvalis group Ru.

4.2.6 Class 2 Numbers

The class 2 numbers are[49]:

15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403, 427

Which are related to class 1 numbers. Class 2 indicates there are two unique factorizations of the corresponding imaginary quadratic field.

Three particularly interesting values are 88, 148, 232, or $4 \cdot \{22, 37, 58\}$, which have properties related to almost integers [50].

The class 2 numbers mostly approximate primes when divided by 4 and rounded:

4, 5, 6, 9, 10, 13, 13, 22, 23, 29, 31, 37, 47, 58, 59, 67, 101, 107

Which have composite numbers {4, 6, 9, 10, 22, 58}. I like to interpret 22 as approximating 17 and 25 and 58 as approximating 53 and 61, which results in an interpretation where the "Ibrishimova numbers" {5, 13, 17, 53, 61, 107} and prime squares $(\{2, 3, 5\})^2$ are included. You can extend this further by making 37 approximate the "Ibrishimova number" $37^2 - 2$, resulting in seven representations out of nine (excluding 251 and 283).

4.2.7 "Serendipitous Numbers"

My favorite example of mathematical beauty involves the "serendipitous numbers"[51]:

1, 3, 5, 11, 17, 30, 41, 59, 71

Which (except for 71) essentially correspond to both the sum of the first n primes as well as $prime(prime(n))$, with a max error of 1 at indices 0 and 5. Interestingly, the indices 0 and 5 are the stable Fibonacci points (0, 1, 5) minus the stable Lucas point at 1.

Notably the sum of the first n primes has an asymptotic value of $n^2 \cdot log_e(n)$, whereas the formula $prime(prime(n))$ has an asymptotic value of $n \cdot log_e(n)^2$, and yet for the indices $[0, 7]$ there is almost no error between the two.

Showing this in a more straightforward manner:

$$
\begin{aligned}
&1 = -1 + prime(1) = 0 + \{\} + 1 \\
&3 = prime(prime(1)) = 1 + \{2\} \\
&5 = prime(prime(2)) = 0 + \{2 + 3\} \\
&11 = prime(prime(3)) = 1 + \{2 + 3 + 5\} \\
&17 = prime(prime(4)) = 0 + \{2 + 3 + 5 + 7\} \\
&30 = -1 + prime(prime(5)) = 1 + \{2 + 3 + 5 + 7 + 11\} + 1 \\
&41 = prime(prime(6)) = 0 + \{2 + 3 + 5 + 7 + 11 + 13\} \\
&59 = prime(prime(7)) = 1 + \{2 + 3 + 5 + 7 + 11 + 13 + 17\} \\
&71 = (-1 + 5) + prime(prime(8)) = 0 + \{2 + 3 + 5 + 7 + 11 + 13 + 17 + 19\} + (1 - 7)
\end{aligned}
$$

Where, notably, there are lesser of twin primes at 3, 5, 11, 17, 29, 41, 59, and 71. Additionally:

$$
\begin{aligned}
&71 - 1 = 70 \\
&59 - 2 = 57 \\
&41 - 4 = 37 \\
&30 - 8 = 22 \\
&17 - 16 = 1 \\
&11 - 32 = -21 \\
&5 - 64 = -59 \\
&3 - 128 = -125 \\
&1 - 256 = -255
\end{aligned}
$$

Where the terms essentially include 22 and 58 twice and 37 once, which relate to the class 2 numbers.

A likely question would be: what is with the 71?

The core answer to that lies in integer solutions of the form $\sqrt{n! + 1}$, where 5, 11, and 71 are the only known integer outputs. Ultimately, it boils down to intuition; I prefer the sequence with 71 more than without.

Relating to the logistic map (2.1.1), I believe there is a possibility that primes congruent to $-1 \bmod 6$ are in some sense more primitive (at least from our perspective of primes). If you observe these primes, an unusual inflection point can be seen for primes 59, 71, 83, 89 where $CumulativeSum(Primes_{6n\text{-}1}(n)) - (2n - 1)^2$ briefly turns negative:

$$CumulativeSum(\{5, 11, 17 ... 131, 137, 149\}) - (\{1, 3, 5 ... 31, 33, 35\})^2 = \{4, 7, 8, 7, 4, 5, 4, 1, -4, -5, -2, -1, 4, 7, 8, 19, 28, 41\}$$

Which could be unrelated or a coincidence, but I find it interesting.

The prime factorization of the order of the monster group also contains all of the "serendipitous numbers".

As a final note, I believe this sequence has imaginary properties.

4.3 Five Fibonaccis

This section focuses on a group of five Fibonacci-like sequences: two already-known standard Fibonaccis (Fibonacci and Lucas numbers); two chaotic Fibonaccis I define precursors to; and one imaginary Fibonacci I define a precursor to.

4.3.1 Fibonacci Numbers

The Fibonacci numbers are[52]:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...

They have stable points $Fibonacci(0) = 0$, $Fibonacci(1) = 1$, and $Fibonacci(5) = 5$.
Some difference equations of Fibonacci numbers (for divisors of 120 and 11):

$$\begin{aligned}
&Fibonacci(n) - Fibonacci(n-1) = 1 \cdot Fibonacci(n-2)\\
&Fibonacci(n) - Fibonacci(n-2) = 1 \cdot Fibonacci(n-1)\\
&Fibonacci(n) - Fibonacci(n-3) = 2 \cdot Fibonacci(n-2)\\
&Fibonacci(n) - Fibonacci(n-4) = 1 \cdot Lucas(n-2)\\
&Fibonacci(n) - Fibonacci(n-5) = 3 \cdot Fibonacci(n-2) - 1 \cdot Fibonacci(n-3)\\
&Fibonacci(n) - Fibonacci(n-6) = 4 \cdot Fibonacci(n-3)\\
&Fibonacci(n) - Fibonacci(n-8) = 3 \cdot Lucas(n-4)\\
&Fibonacci(n) - Fibonacci(n-10) = 11 \cdot Fibonacci(n-5)\\
&Fibonacci(n) - Fibonacci(n-11) = 3 \cdot Fibonacci(n-5) + 13 \cdot Fibonacci(n-6)\\
&Fibonacci(n) - Fibonacci(n-12) = 8 \cdot Lucas(n-6)\\
&Fibonacci(n) - Fibonacci(n-15) = 8 \cdot Fibonacci(n-6) + 26 \cdot Fibonacci(n-8)\\
&Fibonacci(n) - Fibonacci(n-20) = 55 \cdot Lucas(n-10)\\
&Fibonacci(n) - Fibonacci(n-24) = 144 \cdot Lucas(n-12)\\
&Fibonacci(n) - Fibonacci(n-30) = 1364 \cdot Fibonacci(n-15)\\
&Fibonacci(n) - Fibonacci(n-40) = 6765 \cdot Lucas(n-20)\\
&Fibonacci(n) - Fibonacci(n-60) = 832040 \cdot Lucas(n-30)\\
&Fibonacci(n) - Fibonacci(n-120) = 1548008755920 \cdot Lucas(n-60)
\end{aligned}$$

Where I believe the unusual forms for 5, 11, and 15 relate to Plimpton 322 (4.5.2). These exceptions could also relate to the integer solutions to $\sqrt{n!+1}$, which are 5, 11, 71.

4.3.2 Lucas Numbers

The Lucas numbers are $Lucas(n) = Fibonacci(n-1) + Fibonacci(n+1)$[53]:

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199...

They have stable points $Lucas(-1) = -1$ and $Lucas(1) = 1$.
Some difference equations of Lucas numbers (for divisors of 120 and 11):

$$\begin{aligned}
&Lucas(n) - Lucas(n-1) = 1 \cdot Fibonacci(n-0)\\
&Lucas(n) - Lucas(n-2) = 1 \cdot Lucas(n-1)\\
&Lucas(n) - Lucas(n-3) = 2 \cdot Lucas(n-2)\\
&Lucas(n) - Lucas(n-4) = 5 \cdot Fibonacci(n-2)\\
&Lucas(n) - Lucas(n-5) = 15 \cdot Fibonacci(n-2) - 7 \cdot Lucas(n-3)\\
&Lucas(n) - Lucas(n-6) = 4 \cdot Lucas(n-3)\\
&Lucas(n) - Lucas(n-8) = 15 \cdot Fibonacci(n-4)\\
&Lucas(n) - Lucas(n-10) = 11 \cdot Lucas(n-5)\\
&Lucas(n) - Lucas(n-11) = 15 \cdot Fibonacci(n-5) + 7 \cdot Lucas(n-6)\\
&Lucas(n) - Lucas(n-12) = 40 \cdot Fibonacci(n-6)\\
&Lucas(n) - Lucas(n-15) = 40 \cdot Fibonacci(n-7) + 18 \cdot Lucas(n-8)\\
&Lucas(n) - Lucas(n-20) = 275 \cdot Fibonacci(n-10)\\
&Lucas(n) - Lucas(n-24) = 720 \cdot Fibonacci(n-12)\\
&Lucas(n) - Lucas(n-30) = 1364 \cdot Lucas(n-15)\\
&Lucas(n) - Lucas(n-40) = 33825 \cdot Fibonacci(n-20)\\
&Lucas(n) - Lucas(n-60) = 4160200 \cdot Fibonacci(n-30)\\
&Lucas(n) - Lucas(n-120) = 7740043779600 \cdot Fibonacci(n-60)
\end{aligned}$$

Where if you check the prime factorizations, they typically include primes that are { divisors of 120 } + 1. A notable exception is for the last value, 7740043779600, which has a prime factorization $2^4 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41 \cdot 61 \cdot 2521$, where notably the last value is $\frac{7!}{2} + 1$. Also notable, the coefficients are often 5 times the analogs in the Fibonacci version.

4.3.3 "Midas Numbers"

The "Midas numbers" are $Midas(n) = Fibonacci(n+1) + Lucas(n)$[54]

3, 2, 5, 7, 12, 19, 31, 50, 81, 131...

They have a single stable point at $Midas(-1) = -1$. They also have the interesting properties $Midas(-4) = 9$, $Midas(0) = 3$, and $Midas(8) = 81$.

Perhaps the most peculiar aspect is when you divide each term by the index you get the following:

Index	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
Integer	-4	3	-2	3	-2	1	$\frac{3}{0}$	2	2	2	3	4	5	7	8
Fraction	$+\frac{1}{6}$	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$					$+\frac{1}{2}$	$+\frac{1}{3}$		$-\frac{1}{5}$	$+\frac{1}{6}$	$+\frac{1}{7}$	$+\frac{1}{8}$

Where the values of interest lie between indices $[-6, +8]$. Notably, the term $\frac{3}{0}$ is a term without an obvious interpretation, although I suggest the solution -1 for the following reasons:

1) Counting backwards from index 6, one finds the terms 5, 4, 3, 2... that if extrapolated result in -1 at index 0.
2) There is a negative term on every negative even index, and $\frac{Midas(0)}{0} = -1$ seems like a reasonable extrapolation.
3) -1 is the only stable point for $Midas(n)$, where $Midas(-1) = -1$.
4) if -1 is included, the integer terms become {-2, -1, 1, 2, 3}, which are the five possible cell values in all known 3×3 matrices of the trees of primitive Pythagorean triples.

Some difference equations of "Midas numbers" (for divisors of 120 and 11):

$$
\begin{aligned}
&Midas(n) - Midas(n-1) = 1 \cdot Midas(n-2)\\
&Midas(n) - Midas(n-2) = 1 \cdot Midas(n-1)\\
&Midas(n) - Midas(n-3) = 2 \cdot Midas(n-2)\\
&Midas(n) - Midas(n-4) = 1 \cdot (Midas(n-2) + 2 \cdot Midas(n-3))\\
&Midas(n) - Midas(n-5) = 3 \cdot Fibonacci(n-2) + 10 \cdot Fibonacci(n-3)\\
&Midas(n) - Midas(n-6) = 4 \cdot Midas(n-3)\\
&Midas(n) - Midas(n-8) = 3 \cdot (Midas(n-4) + 2 \cdot Midas(n-5))\\
&Midas(n) - Midas(n-10) = 11 \cdot Midas(n-5)\\
&Midas(n) - Midas(n-11) = 3 \cdot Midas(n-4) + 10 \cdot Midas(n-6)\\
&Midas(n) - Midas(n-12) = 8 \cdot (Midas(n-6) + 2 \cdot Midas(n-7))\\
&Midas(n) - Midas(n-15) = 98 \cdot Fibonacci(n-7) + 10 \cdot Lucas(n-8)\\
&Midas(n) - Midas(n-20) = 55 \cdot (Midas(n-10) + 2 \cdot Midas(n-11))\\
&Midas(n) - Midas(n-24) = 144 \cdot (Midas(n-12) + 2 \cdot Midas(n-13))\\
&Midas(n) - Midas(n-30) = 1364 \cdot Midas(n-15)\\
&Midas(n) - Midas(n-40) = 6765 \cdot (Midas(n-20) + 2 \cdot Midas(n-21))\\
&Midas(n) - Midas(n-60) = 832040 \cdot (Midas(n-30) + 2 \cdot Midas(n-31))\\
&Midas(n) - Midas(n-120) = 1548008755920 \cdot (Midas(n-60) + 2 \cdot Midas(n-61))
\end{aligned}
$$

Where notably there are 6 solutions of the form $a \cdot Midas(n-c)$ and 8 solutions of the form $a \cdot (Midas(n-c) + 2 \cdot Midas(n-c-1))$. The first class corresponds to divisors of 30 (except 5 and 15); the second class corresponds to 120÷{divisors of 30}.

Despite being close to the sequence I want, I think this is a precursor to the chaotic sequence I want.

4.3.4 "Pemdas Numbers"

The "Pemdas numbers" are $Pemdas(n) = Fibonacci(n) + 7 \cdot Fibonacci(n-1)$[55]:

7, 1, 8, 9, 17, 26, 43, 69, 112, 181...

They have a single stable point at $Pemdas(1) = 1$. They also have the interesting properties $Pemdas(-4) = 32$ and $Pemdas(2) = 8$.

Like the "Midas numbers" when you divide each term by the index you get the following:

Index	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
Integer	39	-27	19	-14	5	-8	6	-6	6	$\frac{7}{0}$	1	4	3	4	5	7	10	14	20
Fraction		$-\frac{1}{8}$	$+\frac{1}{7}$	$+\frac{1}{6}$	$+\frac{1}{5}$		$+\frac{1}{3}$	$-\frac{1}{2}$						$+\frac{1}{4}$	$+\frac{1}{5}$	$+\frac{1}{6}$	$-\frac{1}{7}$		$+\frac{1}{9}$

Where the values of interest lie between indices $[-9, +9]$. Notably, the term $\frac{7}{0}$ is a term without an obvious interpretation, and I speculate it may be -24 based on the Ulam spiral connection with Fibonacci numbers and lucky numbers of Euler (4.2.2).

Some difference equations of "Pemdas numbers" (for divisors of 120 and 11):

$Pemdas(n) - Pemdas(n-1) = 1 \cdot Pemdas(n-2)$
$Pemdas(n) - Pemdas(n-2) = 1 \cdot Pemdas(n-1)$
$Pemdas(n) - Pemdas(n-3) = 2 \cdot Pemdas(n-2)$
$Pemdas(n) - Pemdas(n-4) = 3 \cdot Midas(n-3)$
$Pemdas(n) - Pemdas(n-5) = 1 \cdot Pemdas(n-3) + 11 \cdot Lucas(n-4)$
$Pemdas(n) - Pemdas(n-6) = 4 \cdot Pemdas(n-3)$
$Pemdas(n) - Pemdas(n-8) = 15 \cdot Midas(n-5)$
$Pemdas(n) - Pemdas(n-10) = 11 \cdot Pemdas(n-5)$
$Pemdas(n) - Pemdas(n-11) = 15 \cdot Pemdas(n-6) + 11 \cdot Lucas(n-7)$
$Pemdas(n) - Pemdas(n-12) = 40 \cdot Midas(n-7)$
$Pemdas(n) - Pemdas(n-15) = 8 \cdot Pemdas(n-7) + 34 \cdot Pemdas(n-8)$
$Pemdas(n) - Pemdas(n-20) = 275 \cdot Midas(n-11)$
$Pemdas(n) - Pemdas(n-24) = 720 \cdot Midas(n-13)$
$Pemdas(n) - Pemdas(n-30) = 1364 \cdot Pemdas(n-15)$
$Pemdas(n) - Pemdas(n-40) = 33825 \cdot Midas(n-21)$
$Pemdas(n) - Pemdas(n-60) = 4160200 \cdot Midas(n-31)$
$Pemdas(n) - Pemdas(n-120) = 7740043779600 \cdot Midas(n-61)$

Where notably there are 6 solutions of the form $a \cdot Pemdas(n-c)$ and 8 solutions of the form $a \cdot Midas(n-c)$, where the difference compared to the previous section is due to the fact that the "Pemdas numbers" had not yet been defined.

And, like before, I think this is a precursor to the chaotic sequence I want.

4.3.5 "Codas Numbers"

The "Codas numbers" are $Codas(n) = 2 \cdot Lucas(n) + Fibonacci(n)$[56]:

4, 3, 7, 10, 17, 27, 44, 71, 115, 186...

They have a single stable point at $Codas(-1) = -1$.

When you divide each term by the index you get the following:

Index	-4	-3	-2	-1	0	1	2	3	4
Integer	-3	2	-3	1	$\frac{4}{0}$	3	3	3	4
Fraction	$+\frac{1}{4}$		$+\frac{1}{2}$				$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{1}{4}$

Where the values of interest lie between indices $[-4, +4]$.

Some difference equations of "Codas numbers" (for divisors of 120 and 11):

$Codas(n) - Codas(n-1) = 1 \cdot Codas(n-2)$
$Codas(n) - Codas(n-2) = 1 \cdot Codas(n-1)$
$Codas(n) - Codas(n-3) = 2 \cdot Codas(n-2)$
$Codas(n) - Codas(n-4) = 1 \cdot (Lucas(n-2) + 10 \cdot Fibonacci(n-2))$
$Codas(n) - Codas(n-5) = 5 \cdot Fibonacci(n-2) + 13 \cdot Fibonacci(n-3)$
$Codas(n) - Codas(n-6) = 4 \cdot Codas(n-3)$
$Codas(n) - Codas(n-8) = 3 \cdot (Lucas(n-4) + 10 \cdot Fibonacci(n-4))$
$Codas(n) - Codas(n-10) = 11 \cdot Codas(n-5)$
$Codas(n) - Codas(n-11) = 59 \cdot Fibonacci(n-5) + 1 \cdot Lucas(n-6)$
$Codas(n) - Codas(n-12) = 8 \cdot (Lucas(n-6) + 10 \cdot Fibonacci(n-6))$
$Codas(n) - Codas(n-15) = 156 \cdot Fibonacci(n-7) + 2 \cdot Lucas(n-8)$
$Codas(n) - Codas(n-20) = 55 \cdot (Lucas(n-10) + 10 \cdot Fibonacci(n-10))$
$Codas(n) - Codas(n-24) = 144 \cdot (Lucas(n-12) + 10 \cdot Fibonacci(n-12))$
$Codas(n) - Codas(n-30) = 1364 \cdot Codas(n-15)$
$Codas(n) - Codas(n-40) = 6765 \cdot (Lucas(n-20) + 10 \cdot Fibonacci(n-20))$
$Codas(n) - Codas(n-60) = 832040 \cdot (Lucas(n-30) + 10 \cdot Fibonacci(n-30))$
$Codas(n) - Codas(n-120) = 1548008755920 \cdot (Lucas(n-60) + 10 \cdot Fibonacci(n-60))$

Where notably there are 6 solutions of the form $a \cdot Codas(n-c)$ and 8 solutions of the form $a \cdot (Lucas(n-c) + 10 \cdot Fibonacci(n-c))$.

I also believe this sequence has imaginary properties and is a precursor to a better sequence.

4.4 Trees of Primitive Pythagorean Triples

One of the best summaries of trees of primitive Pythagorean triples is by Mathologer[57], and I suggest watching the video. It basically covers how the generation of the tree can be summarized by a group of rules based on the Fibonacci numbers.

The goal of the tree of primitive Pythagorean triples is to enumerate all primitive right angle triangles with integer sides. Thus the sides of the triangle must be of the form $a^2 + b^2 = c^2$ and a, b, and c must be integers. Primitive means that the three terms a, b, c share no common factors. For instance, $(3, 4, 5)$ is a primitive Pythagorean triple because $3^2 + 4^2 = 5^2$, but $(6, 8, 10)$ is not primitive because each term can be divided by 2 to get a simpler triangle: the $(3, 4, 5)$ triangle.

Here is an example of a tree of primitive Pythagorean triples, the Berggrens tree, discussed next:

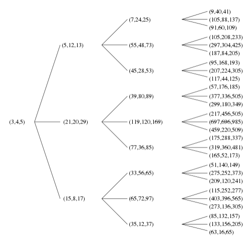

Figure 4.5: The Berggrens tree of primitive Pythagorean triples from Wikipedia[58].

4.4.1 Berggrens's Tree

The first tree of primitive Pythagorean triples found is attributed to B. Berggrens in 1934. The root of the tree is the $(3, 4, 5)$ triangle as a column vector. When you multiply one of the below matrices (found by F. J. M. Barning) by a primitive Pythagorean triple as a right-hand side column vector, you get a new primitive Pythagorean triple:

Type	Matrix
Plato Family	$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$
Fermat Family	$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$
Pythagoras Family	$\begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$

One interesting aspect I would like to highlight is that the absolute value of a cell does not change, just whether a cell is positive or negative, which can lead to the observation that applying the remaining set of negative values results in:

"Quasi-Parent" Matrix	$\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}$

The reason I call it a "Quasi-Parent" is because it gets the sign of the parent triple wrong, although that can easily be fixed by taking the absolute value.

4.4.2 Price's Tree

The second tree of primitive Pythagorean triples found is attributed to H. Lee Price in 2008[59]. The root of the tree is also a $(3, 4, 5)$ triangle as a column vector. When you multiply one of the below matrices by a primitive Pythagorean triple as a column vector, you get a new primitive Pythagorean triple:

Type	Matrix
Left Family	$\begin{pmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{pmatrix}$
Middle Family	$\begin{pmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}$
Right Family	$\begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

Like before, the absolute value of a cell does not change, just whether a cell is positive or negative, which leads to a similar observation by applying the remaining set of negative values:

$$\text{"Quasi-Cycle" Matrix} \quad \begin{pmatrix} -2 & 1 & 1 \\ 2 & 2 & -2 \\ 2 & 1 & -3 \end{pmatrix}$$

I call it a "Quasi-Cycle" matrix because iterating on any primitive Pythagorean triple seems to eventually lead to a cycle if you take the absolute value after each iteration.

The details are that when you use the Pythagoras family of triples (i.e. $((2k+1), \lfloor\frac{(2k+1)^2}{2}\rfloor, \lceil\frac{(2k+1)^2}{2}\rceil)$) or the Plato family of triples (i.e. $(4k^2-1, 4k, 4k^2+1)$) and apply the matrix k or fewer times, you always seem to cycle back to the original triple. Notably, the Pythagoras family has an odd quality to it (from the $(2k+1)$), whereas the Plato family has an even quality to it (from the $4k$).

The time it takes to cycle seems to be[60]:

1, 2, 3, 3, 5, 6, 4, 4, 9, 6, 11, 10, 9, 14, 5, 5, 12, 18, 12, 10, 7, 12, 23, 21, 8, 26, 20, 9, 29, 30, 6, 6, 33, 22, 35, 9, 20, 30...

Where the cycle seems to take the maximal number of iterations at indices equal to the Queneau numbers[61]:

1, 2, 3, 5, 6, 9, 11, 14, 18, 23, 26, 29, 30, 33, 35, 39, 41, 50, 51, 53, 65, 69, 74, 81, 83, 86, 89, 90, 95, 98, 99, 105, 113...

When you look at the subset of the above sequence that is prime, you seem to get the Sophie Germain primes[62]:

2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, 419, 431, 443, 491, 509, 593...

Which is a neat connection to a problem about cycling.

You can also consider what happens when the reverse cycle path is taken (by taking the inverse of the matrix):

$$\text{"Reverse Quasi-Cycle" Matrix} \quad \begin{pmatrix} \frac{-1}{2} & \frac{+1}{2} & \frac{-1}{2} \\ \frac{+1}{4} & \frac{+1}{2} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{+1}{2} & \frac{-3}{4} \end{pmatrix}$$

Which seems to work similarly although you need to apply the absolute value and remove any denominator of 2 should it exist.

4.4.3 The Search for The Third Tree

The search for the third tree of primitive Pythagorean triples is my favorite open problem in mathematics.

I did a brute force search of all 3×3 matrices using {-2, -1, 1, 2, 3}, and I only found the six 3×3 matrices defined by Berggrens and Price, which means the solution is somewhat resilient to naive searches.

One noteworthy thing I want to mention: both of the previous trees can be constructed with Fibonacci-like methods, which is explained well by both Price[59] and Mathologer[57].

Based on this, I posit the third tree has a Lucas-like connection that combines the other two methods. The relevant observation here is that the sum of the six known matrices is:

$$\text{"Lucas-like" Matrix} \quad \begin{pmatrix} 7 & 3 & 7 \\ 4 & 3 & 12 \\ 4 & 3 & 18 \end{pmatrix}$$

Where the trace of the matrix is $7+3+18=28$. When you compare these to the Lucas numbers (i.e. 2, 1, 3, 4, 7, 11, 18, 29...) you find only one mismatch in the matrix itself.

Originally, I thought that the third matrix would share two already-used matrices (one from Berggrens and one from Price, specifically those that do not use a -2 term), but after thinking about it more, I think only one matrix is reused (just not literally), specifically the only one with two negative signs by Price. The rationale is that the trees have an average of two negative terms per matrix.

If you add the matrix with two negatives by Price to the above summation again you get:

$$\text{"Fibonacci-like" Matrix} \quad \begin{pmatrix} 9 & 4 & 8 \\ 6 & 1 & 14 \\ 6 & 2 & 21 \end{pmatrix}$$

Which has a trace $9+1+21=31$. When you compare these to the Fibonacci numbers (i.e. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34...) you find five mismatches in the matrix itself where subtracting one results in a Fibonacci number.

That next realization was: what if I combine the two methods from the "Reverse Quasi-Cycle" and the "Quasi-Parent" matrices (derived in the two previous sections)?

My interpretation of this idea was to compute $(\{\{\frac{-1}{2},\frac{1}{2},\frac{-1}{2}\},\{\frac{1}{4},\frac{1}{2},\frac{-1}{4}\},\{\frac{-1}{4},\frac{1}{2},\frac{-3}{4}\}\}\cdot\{\{1,2,-2\},\{2,1,-2\},\{2,2,-3\}\})^{-1}$, which results in: {{-2, 3, 3}, {-6, 2, 6}, {-6, 3, 7}}.

Checking the behavior, the matrix seems to work with a caveat: it occasionally fails to generate a primitive triple, but it generates a Pythagorean triple.

The next step for me was to brute force the solution with this new knowledge. The amount of permutations gets much easier to compute once you know the matrix cell values are only $\pm\{2,3,6,7\}$.

```
import numpy as np
import math
import copy

matrix_options = [-6, -3, -2, 2, 3, 6, 7] #[-2, -1, 1, 2, 3]
test_cases = [np.array([3,4,5]), np.array([5, 12, 13])]
base_matrix = np.array([[0, 0, 0], [0, 0, 0], [0, 0, 0]])
output_matrix = np.array([0, 0, 0])

def primitive_pythagorean(vector):
  if not vector[0]**2 + vector[1]**2 == vector[2]**2:
    return False
  if vector[0] <= 0 or vector[1] <= 0 or vector[2] <= 0:
    return False
  if math.gcd(vector[0], vector[1], vector[2]) != 1:
    return False
  if vector[1] % 2 != 0:
    return False
  return True

for m00 in matrix_options:
  base_matrix[0,0]=m00
  for m01 in matrix_options:
    base_matrix[0,1]=m01
    for m02 in matrix_options:
      base_matrix[0,2]=m02
      for m10 in matrix_options:
        base_matrix[1,0]=m10
        for m11 in matrix_options:
          base_matrix[1,1]=m11
          for m12 in matrix_options:
            base_matrix[1,2]=m12
            for m20 in matrix_options:
              base_matrix[2,0]=m20
              for m21 in matrix_options:
                base_matrix[2,1]=m21
                for m22 in matrix_options:
                  base_matrix[2,2]=m22
                  works=True
                  for test_case in test_cases:
                    np.matmul(base_matrix, test_case, out=output_matrix)
                    if not primitive_pythagorean(output_matrix):
                        works=False
                        break
                  if works:
                    print(base_matrix)
```

Type	Matrix			
"Imaginary" Matrices	$\begin{pmatrix}2&-3&3\\6&-2&6\\6&-3&7\end{pmatrix}$	$\begin{pmatrix}2&3&3\\6&2&6\\6&3&7\end{pmatrix}$	$\begin{pmatrix}-2&3&3\\-6&2&6\\-6&3&7\end{pmatrix}$	$\begin{pmatrix}2&3&-3\\-6&-2&6\\-6&-3&7\end{pmatrix}$
"Negative Chaotic" Matrices	$\begin{pmatrix}3&-6&6\\2&3&-2\\2&-6&7\end{pmatrix}$	$\begin{pmatrix}3&2&-2\\-6&3&6\\-6&2&7\end{pmatrix}$	$\begin{pmatrix}6&-3&3\\2&6&-2\\2&-3&7\end{pmatrix}$	
"Positive Chaotic" Matrices	$\begin{pmatrix}3&6&6\\-2&3&2\\2&6&7\end{pmatrix}$	$\begin{pmatrix}3&-2&2\\6&3&6\\6&2&7\end{pmatrix}$	$\begin{pmatrix}6&3&3\\-2&6&2\\2&3&7\end{pmatrix}$	

I label the above matrices as precursors because they *almost* work (note that only two test cases are used).

Where I conjecture that one has imaginary qualities and the other two have the properties of "entangled chaotic twins". What is weird is that all ten vanish if you include more test cases, because they cannot preserve primitivity of the Pythagorean triples. This is as opposed to the case for { -2, -1, 1, 2, 3 } where only 1 out of 7 vanish as more test cases are included (which is where we get the Berggrens and Price matrices from).

The reason for the imaginary classification primarily deals with the overlapping negatives in the matrix definitions, which does not happen elsewhere. The "Imaginary" Matrices also have the same $\pm$ sign signatures as the Barning (Berggrens) matrices in three out of four cases.

The "entangled chaotic twins" on the other hand seem to use three different matrix definitions (by absolute value), which is the basis for the "chaotic" claim. Nonetheless, the absolute value matrices would hypothetically be shared between the two conjectured trees, hence the "entangled" claim. Similarly, the central chaotic matrices share $\pm$ sign signatures with two Price matrices.

It is worth mentioning that "V. E. Firstov showed generally that only three such trichotomy trees exist"[63] (hence the title of this section). Originally, I thought I might dispute the proof (without knowing Russian), but after thinking about it further I think there are five trees: three trichotomy trees (standard and imaginary) and two dichotomy trees (chaotic), which essentially sidesteps the proof altogether. I think there may be more trees after that, but I am largely curious about these five because they sort of unify the theories presented in this paper.

Until now I have overconstrained my brute force search. You can find interesting behavior by simplifying the code to:

```
def primitive_pythagorean(vector):
  if not vector[0]**2 + vector[1]**2 == vector[2]**2:
    return False
  #if vector[0] <= 0 or vector[1] <= 0 or vector[2] <= 0:
  #  return False
  if math.gcd(vector[0], vector[1], vector[2]) != 1:
    return False
 # if vector[1] % 2 != 0:
 #   return False
  return True
```

First, you can find eight signed variants of each of the following two matrices that seem to work for all test cases:

Type	Matrices
"First Chaotic" Matrices	$\begin{pmatrix} 6 & 3 & 6 \\ 3 & -2 & 2 \\ 6 & 2 & 7 \end{pmatrix}$
"Second Chaotic" Matrices	$\begin{pmatrix} 3 & -2 & 2 \\ 6 & 3 & 6 \\ 6 & 2 & 7 \end{pmatrix}$

Also, if you just search among the first two test cases, you can find two sets of 16 signed matrices that are variants of:

Type	Matrices
"Underconstrained Imaginary" Matrices	$\begin{pmatrix} 6 & 2 & 6 \\ 2 & 3 & 3 \\ 6 & 3 & 7 \end{pmatrix}$
"Overconstrained Imaginary" Matrices	$\begin{pmatrix} 2 & 3 & 3 \\ 6 & 2 & 6 \\ 6 & 3 & 7 \end{pmatrix}$

Moreover, the only two entirely positive matrices that meet these two test cases are the above matrices.

I believe the chaotic matrices probably use irrational coefficients, whereas the imaginary matrices might use rational coefficients based on work by Holly Krieger surrounding rational cycles in the Mandelbrot set[64]. For the imaginary matrices, this hypothetically means using the map $a(n+1) = a(n)^2 - \frac{29}{16}$ for $a(0) = \frac{-7}{4}, a(1) = \frac{5}{4}, a(2) = \frac{-1}{4}$, which cycles every three terms.

I am also curious if trees of primitive Pythagorean triples can be braided in some sense. For instance, Beggrens's tree could bifurcate from the Fermat family branch into the Plato and Pythagoras family branches (which notably have properties like evens and odds respectively).

The next steps are to try to infer the seven matrices corresponding to the conjectured three trees, although due to time constraints (and feasibility), I am setting this aside in the interest of publishing this paper. I consider trees of Pythagorean triples to be one of the most fascinating and promising pieces of mathematics in this paper, and I hope this cute caterpillar becomes a beautiful butterfly.

4.5 Babylonian Tablets

This section focuses on the base 60 numeral system, which may have originated between 5000 BC and 2000 BC[65]. Many authors (myself included) accidentally refer to this branch of mathematics as Babylonian mathematics, although this is a misnomer since the system predated the Babylonians. The tablets I talk about were likely created by Babylonians between 2000 BC and 1600 BC.

4.5.1 Base 60 Writing and Counting

Base 60 has a clear mismatch between writing and counting. While I could explain the details myself, I believe the best way to learn base 60 writing and counting is to view two Numberphile videos on 1) cuneiform writing (Alex Bellos)[66] and 2) base 60 counting (Thomas Woolley)[67].

The summary of the differences is that base 60 writing uses 6 groups of 10, whereas base 60 counting uses 5 groups of 12. Normally, if this sort of mismatch existed, you would expect mathematicians to get lazy and redefine the base to use 5 groups of 12, for notational convenience, but that seems not to be the case in the commonly taught history of base 60 throughout a very long history.

Some conjectured truthiness about base 60: 1) that cuneiform numbers do not use zero because zero was not discovered and 2) that cuneiform numbers are base 60 despite using only 59 digits.

For convenience, I refer to cuneiform numbers as base 60, but I conjecture they are actually base 59 with an embedded concept of epsilons. Related, I conjecture that both truthiness statements above are incorrect, and the mistake lies in our combined projection of our mathematics onto their mathematics.

4.5.2 Plimpton 322

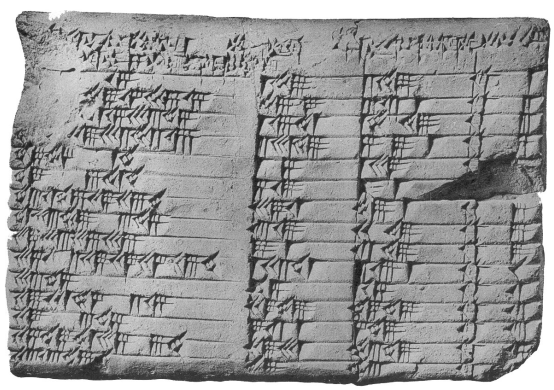

Figure 4.6: Photo of Plimpton 322 from Wikipedia[68].

The earliest known computation of primitive Pythagorean triples can be seen on Plimpton 322[69], a ∼1800 BC Babylonian tablet with a wealth of information on Wikipedia by smart authors of varied opinions.

The first thing I noticed about the tablet, albeit mundane, is that the 2nd column from the right contains a lot of 1's and 49's for the last base 60 digit:

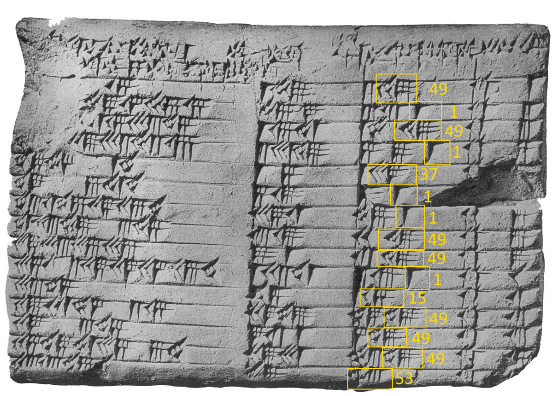

Figure 4.7: Modified photo of Plimpton 322 from Wikipedia[68] highlighting patterns in final digits.

Based on the exceptions for {37, 15, 53}, I relate the final digits to the class 2 numbers (4.2.6), specifically $4 \cdot \{22, 37, 58\}$.

My interest in Plimpton 322 concerns a term in the header of the second and third columns: ÍB.SI$_8$. I hoped I might contribute to its meaning, but I am concerned that speculating might be more harmful than helpful.

For more information, I would suggest reading the Wikipedia page, which will contain much less bias than here.

4.5.3 YBC 7289

Figure 4.8: Photo of YBC 7289 from Wikipedia[70].

Historians seem to focus on YBC 7289 because it is the best approximation of $\sqrt{2}$ in early mathematics. I would like to propose that this assertion may not be the full picture, since it is possible that it could be even closer than we are aware, or even exact by definition, if our understanding of base 60 is incomplete.

I believe that, unlike Plimpton 322, YBC 7289 was made by a single individual, which is why the handwriting is poorly planned and there is an apparent thumbprint while fixing a mistake.

I believe YBC 7289 was created by a student who had difficulty remembering the two numbers listed, so it was created for convenience. It is not surprising it could be convenient to remember $\sqrt{2}$, but the other number seems far less useful.

My current hypothesis is that ÍB.SI$_8$ uses the detail that $7^2 - 8 \approx 30\sqrt{2} - \sqrt{2}$, and any mathematician who needs to compute it must know these two constants.

Another aspect worth mentioning is that the distances between $\sqrt{2}$ and 37 and $30\sqrt{2}$ are exactly $36 - \frac{1}{1+\sqrt{2}}$ and roughly $4 + \sqrt{2}$, where $1 + \sqrt{2}$ is the silver ratio.

The 37 is taken from 4 particularly interesting residues mod 60: 1, 25, 37, 49 (2.3.3), which are congruent to 1 mod 12, but 13 is notably excluded. This could also relate to 22, 37, 58 from the class 2 numbers, using the mathematical coincidences: $22 + 3 = 5^2$, $58 + 3 = 1^2 + 60$, $22 + 3^3 = 7^2$, $58 - 3^2 = 7^2$.

You could instead compare the distances between $\sqrt{2}$ and 13 and $30\sqrt{2}$, which yields distances of exactly $12 - \frac{1}{1+\sqrt{2}}$ and roughly $28 + \sqrt{2}$. The accuracy on both terms could explain the behavior of divisors of 12 and 28 as they relate to "lucky numbers of Martin" (4.2.5).

4.5.4 IM 67118

Figure 4.9: Photo of IM 67118 from Wikipedia[71].

I believe the procedure for ÍB.SI$_8$ may actually be defined in IM 67118[72], at least partially. I believe their writing system using 6 groups of 10 relates to the tablet through "A = 0.75"[72] (for 6 groups) and "c = 1.25"[72] (for 10 subgroups), which are mentioned on the Wikipedia page, and could explain why their counting system does not match their writing system. There may be additional evidence that the numbers 6 and 10 are special. Of a separate tablet: "It should be pointed out that the problem on YBC 6967 actually solves the equation $x - \frac{100}{x} = x - \frac{60}{x} = c$"[69].

There is some interesting geometric intuition listed on Wikipedia:

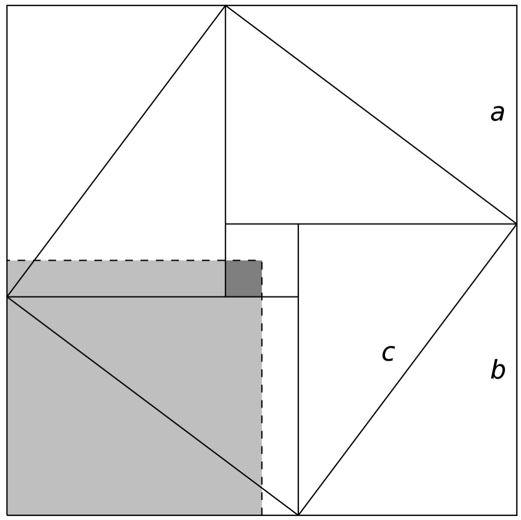

Figure 4.10: Geometry diagram relating IM 67118 to gnomons, from Wikipedia[73].

"Possible geometric basis for solution of IM 67118. Solid lines of the figure show stage 1; dashed lines and shading show stage 2. The central square has side $b - a$. The light gray region is the gnomon of area $A = ab$. The dark gray square (of side $(b-a)/2$) completes the gnomon to a square of side $(b+a)/2$. Adding $(b-a)/2$ to the horizontal dimension of the completed square and subtracting it from the vertical dimension produces the desired rectangle." - Wikipedia, IM 67118[72]

This is not my area of expertise (partially due to IM 67118 being strange math written in prose in an ancient language), so I will merely direct you to the Wikipedia page.

4.5.5 MS 3971

Unfortunately, I do not have permission to post a picture of the MS 3971 tablet (to my knowledge), although such photos can be found on The Internet.

The tablet derives "five diagonals", which notably correspond to "16/15(?)", "5/3", "3/2", "4/3", "6/5"[74]. My hypothesis from this is that historical base 60 has a lot in common with music theory, where the standard 12-tone octave[29] can be seen below, emphasis mine:

Semitone	Transition	Musical Interval
0	1	unison
1	$\mathbf{\frac{16}{15}}$	semitone
2	$\frac{9}{8}$	major second
3	$\mathbf{\frac{6}{5}}$	minor third
4	$\frac{5}{4}$	major third
5	$\mathbf{\frac{4}{3}}$	perfect fourth
6	$\frac{45}{32}$	diatonic tritone
7	$\mathbf{\frac{3}{2}}$	perfect fifth
8	$\frac{8}{5}$	minor sixth
9	$\mathbf{\frac{5}{3}}$	major sixth
10	$\frac{9}{5}$	minor seventh
11	$\frac{15}{8}$	major seventh
12	2	octave

Where Babylonian math seems to have an embedded concept of the semitone, major sixth, perfect fifth, perfect fourth, and minor third. It seems likely the author would not have overlooked the 11th semitone, but perhaps it is statistically less likely to encounter and they missed it. The leading value that is much smaller suggests a misalignment similar to the Lucas number connection for the 12-tone octave (3.4.1) using the shifted indices {1, 3, 5, 7, 11}.

This is circumstantial evidence that base 60 math relates to music theory, which could explain one of my favorite quotes on the subject: "[Sexagesimal] use has also always included (and continues to include) inconsistencies in where and how various bases are to represent numbers even within a single text"[75].

For these reasons, I believe base 60 should be given additional consideration as being an exception to standard algebra as we know it. During the arc of human history, we are but a small fraction, and I think it lacks humility to say our mathematical notations are fundamentally superior merely because of the belief that history builds upon itself, becoming greater with time.

4.6 "Epsilonic Bases"

I posit that Babylonian base 60 is best described as an "epsilonic base", where "epsilonic bases" can be split into at least two categories: "positive epsilonic" and "negative epsilonic".

For Babylonian base 60, I believe their base is best described as "base 59 adjoin positive epsilon", and the cool mathematics that followed is possibly a side effect of the "serendipitous numbers" (4.2.7).

I believe a side effect of using fewer digits is that numbers do not really have unique interpretations, which is consistent with what I know about base 60. This would explain the musical nature, where if computation is repeated enough it can reduce and converge in a statistical way onto a group of answers.

Thus, I believe that "epsilonic bases" are fundamentally linked to music theory; where they have statistical implications as much as they have concrete implications.

I also believe base 60 is a floating-point arithmetic, since it cannot represent trailing zeroes. The implication here is that the representation of zero would arguably be an infinitesimal with a 50:50 probability of being even or odd, and I think paradoxes like these led to arguments about whether zero is a number.

4.6.1 Plimpton 322, Revisited

My hypothesis that base 60 is an "epsilonic base" can be tested, and to remove bias I am writing the methodology section before performing the trial or looking at Plimpton 322 again. Unfortunately, since I have less experience with Babylonian, this will be the only one I do.

My methodology will take Babylonian digits after a computation is made and consider them for two categories: 1) agrees with modern math and 2) fails to conform to modern math.

I will consider two explanations that could justify this failure: 1) the null hypothesis, that the author made a lot of errors and 2) the "epsilonic hypothesis" which can only be true if the mistake rate occurs at around 1 in 59 or 60 digits. For the purposes of this trial, being strictly between 1 in 59 or 60 is considered stronger evidence than just being statistically significant.

Some data will be omitted from the trial, specifically: 1) the rightmost column for all fifteen rows, 2) the column headers, 3) leading 1's in the leftmost column if they existed (I claim that it is unlikely from the translation "from which 1 is torn out"[69] and the whitespace on the lowest rows), 4) damaged data, as defined by Wikipedia[69], and 5) [00] is not included in "ancient digit count".

Errors are subjective, so I will take the commonly agreed upon amount, which is six[69].

I will consider two metrics: 1) "digit aggregation", comparing "ancient digit count" to "modern digit count" and 2) "mistake frequency".

The goal of "digit aggregation" is to count and compare two metrics: "ancient digit count" and "modern digit count". "Ancient digit count" tallies post-computation sexagesimal digits that are visible, except [00] is not included in "ancient digit count". "Modern digit count" is the number of sexagesimal digits according to modern mathematics.

The hypothesis is that "ancient digit count", when raised to the power [sic] of $\frac{59}{60}$, will relate to "modern digit count". This simulates how we project our mathematics onto base 60 math. Recall that base 59 would be less compact than base 60, so $\frac{59}{60}$ should be the correct exponent [sic]. We then compare the adjusted "ancient digit count" to the "modern digit count" and check how significant it is. If it is off by a floor or ceiling operation I think that is good evidence (albeit with sample size one, so testing more than Plimpton 322 is important).

Notably, perceived errors do not matter for "digit aggregation" unless they add or subtract sexagesimal digits.

The goal of "mistake frequency" is to determine the probability a digit contains no mistakes, which should hover around 1 in 59 or 60, at least in cases where the Babylonian author(s) performed few intermediate calculations (because errors should accrue over time). The goal is to count the "conjectured digit mistakes" within "ancient digit count", where "conjectured digit mistakes" disagree with modern math. Then we compute the "mistake frequency" as a percentage. Afterwards, we construct a "normalized mistake frequency" with $\frac{|expected - actual|}{expected}$, where expected is 1.68076% (the arithmetic-geometric mean of $\frac{1}{59}$ and $\frac{1}{60}$ as a percentage).

Here is my interpretation of Plimpton 322, and I propose the original author(s) made 0 errors:

[area of] "takiltum of the diagonal from which 1 is torn out so that the width comes up"	"ÍB.SI$_8$ of the width"	"ÍB.SI$_8$ of the diagonal"	"its line"[69]	
59 **[00]** 15	01 59	02 49	11	1
56 56 58 14 **56** 15	56 07	**03 12 01**	11	2
55 07 41 15 33 45	01 16 41	01 50 49	11	3
53 10 29 32 52 16	03 31 49	05 09 01	11	4
48 54 01 40	01 05	01 37	11	5
47 06 41 40	05 19	08 01	11	6
43 11 56 28 26 40	38 11	59 01	11	7
41 33 **59** 03 45	13 19	20 49	11	8
38 33 36 36	**09** 01	12 49	11	9
35 10 02 28 27 24 26 40	01 22 41	02 16 01	11	10
33 45	45	01 15	11	11
29 21 54 02 15	27 59	48 49	11	12
27 **[00]** 03 45	**07 12 01**	04 49	11	13
25 48 51 35 06 40	29 31	53 49	11	14
23 13 46 40	**57**	53	11	15

Differences with modern base 60 are bolded (where I exclude the modern interpretation to avoid confusion).

The Wikipedia page for Plimpton 322 never mentions the hidden-in-plain-sight column of elevens (11's) that I propose is the denominator of a fraction. Notably the 11th row would then be $\frac{11}{11}$ and correspond to a $(3,4,5)$ triangle, which is extremely strong evidence of correctness, in my opinion. A source that at least acknowledges additional writing can be found transcribes it as "ki"[76], possibly indicating an aesthetic connection between language and mathematics (similar to words like unity or none). This fits the column header, where line could mean a geometric line, (x, y) coordinate, or fraction instead of a line of text.

Figure 4.11: Modified photo of Plimpton 322 from Wikipedia[68] highlighting a hidden column of elevens (11's) that might be denominators of rationals.

Another disagreement between my interpretation and others is taking [00] as an error compared to modern methodology. I believe Babylonian base 60 shows whitespace for zeroes as an embellishment and means of proofreading, but I suggest [00] is a no-operation in Babylonian mathematics. Notably, this is like Mayan base 20, which also has a very different symbol for zero compared to every other symbol.

Finally, the other major difference between my version and Wikipedia is that I propose the final row has a 57 (not 56). The rationale is complicated. Firstly, the ÍB.SI$_8$ columns all seem to have odd numbers for their final digit, but Wikipedia lists only two competing theories that ascribe an even number to at least one of the two columns. Moreover, this shows a point where the magnitude of the two columns has apparently flipped, similarly to column 13, and I suggest all later terms would have width terms that look larger than the diagonal (but are not). Another reason has to do with how the author writes 7's and how cracks tend to follow fault lines but diverge chaotically when a faultline ends (e.g. the top of YBC 7289). According to my theory the number must be six or greater, but cannot be even, which leaves the options of 7 or 9. 9 is more likely with no additional information, but because of how the edge broke, I think it provides strong evidence that the number itself is a 57.

I take the most inspiration from Friberg (2007)[77] and separately Høyrup (2010)[78]. I like every author for different reasons, but the subject matter is difficult, so I name those who I believe set aside preconceptions effectively. These authors are also among the most recent, and they have presumably learned a lot from earlier authors.

Early on, I noticed that the ratios in the leftmost column seemed to be roughly from 1 to $1 - \frac{1+\sqrt{5}}{2}$ with a particularly interesting value near $\frac{\sqrt{\pi}}{2}$ in the fourth row. It is worth noting that many authors agree there should be more than 15 terms, which would make the ratio observations mostly irrelevant. I dissent from that opinion, because I believe the author stops at $\frac{15}{11}$ because $\frac{16}{11}$ exceeds $\sqrt{2}$. I also believe there is a hidden threshold of $(\frac{7}{6})^2$ that explains why our interpretation of the numbers suddenly becomes half even for the final row.

If we interpret the columns from right-to-left as the natural progression of calculation, then it is not hard to see how many terms are calculated using modern math. A lot of authors seem to believe the calculations are left-to-right and that some values are broken off, so it is worthwhile to explore why the values seem to be written right-to-left. First, the values on the right are the simplest (the counting numbers one to fifteen) whereas the number of digits in the leftmost column is consistently the largest. Next, if you look at the second and third columns (regardless of direction), one of the column headings is partially omitted (the one on the left), which usually indicates redundant information.

Similarly, there is a piece of evidence that suggests there are no missing columns: the author tends to have a faint line that traces the separations between columns. In the case of the hidden 11's, there is a separating line that overlaps with the tens place, and since there is only a single column header, this might indicate the elevens and 1-15 are related (hence the theory about being a rational number). Similarly, sliding leftwards, a column line can be seen separating the other columns. That being said, no such line is visible on the leftmost side of tablet, which may indicate that there is nothing to separate (i.e. the table is complete, with the possible exception of rows 16+).

If we project our mathematics onto Plimpton 322, nine of the fifteen rows contain no errors, so computation usually works without conflict. Almost all the other columns have no substantial issues. An example of a logical stretch is row 13, where the sexagesimal numbers 07 12 01 and 04 49 are 161^2 and 289, and it is not *that* hard to imagine how you could turn that into a $(161, 240, 289)$ triangle. The one column that modern math does not adequately explain is row 2. Here the author goes from the sexagesimal numbers 56 07 (3367) and 03 12 01 (11521) when the triangle should be $(3367, 3456, 4825)$ in modern math. Unlike row 13, I cannot fathom a way the ancient author could have derived the fourth column using an incorrect value and no trivial method to remedy it. That is like going into a final exam, getting the first part of a three-part question wrong, but magically fixing your errors by the second and third parts even though they depend on the initial computation. Additionally, the implicit long sides of the triangle are: $\{$ 120, 3456, 4800, 13500, 72, 360, 2700, 960, 600, 6480, 60, 2400, 240, 2700, 45$\}$. These values are divisible by 60 except in the 2nd, 5th, and 15th columns, where the 2nd column is the one with the most warped interpretation.

Another relevant point is that some of the perceived errors look incredibly incorrect using modern math, to a point where the errors are so glaring that it is hard to imagine any academic missing them. It is hard to imagine a world where the first known computation of Pythagorean triples had a table where the short side of the triangle is longer than the hypotenuse in two out of fifteen cases, despite having tidy and predictive handwriting.

I believe this is strong circumstantial evidence that Babylonian base 60 sometimes overlaps with our modern mathematics, but the mathematics behind it is very different, resulting in differences that make it look non-rigorous when it actually has full rigor, as long as the author of a tablet knew how to compute (not everyone author would know everything).

This leads to the following summary:

metric	column 4	column 3	column 2	total
"ancient digit count"	65 [$\cancel{71}$]	32	33	130
"modern digit count"	68 [$\cancel{75}$]	31	33	132
"conjectured digit mistakes"	$\frac{2}{65}$	$\frac{5}{32}$	$\frac{3}{33}$	$\frac{10}{130}$
"mistake frequency"	3.08%	15.63%	9.09%	7.69%
"normalized mistake frequency"	1.83x	9.30x	5.41x	4.58x

Notably my hypothesis was wrong in both cases, although a modified hypothesis fits the data for "digit aggregation".

My first hypothesis was that the "ancient digit count" raised to the power $\frac{59}{60}$ would roughly equal "modern digit count", but there were actually more modern digits to represent the same thing, despite a true base 60 having an additional digit (i.e. zero) to work with (so it is wrong in the opposite direction of my hypothesis). For the second hypothesis, the "mistake frequency" is nowhere near 1.68076%, and it is even inconsistent between columns. So the reader can see I was wrong.

I am convinced my hypothesis around "mistake frequency" is useless, probably due to the unpredictability of the digits of intermediate calculations. That being said, the ceiled "normalized mistake frequency" for columns 2 and 3 is $6\times$ and $10\times$, which are the number of groups and subgroups in base 60.

If the reader will permit it, I would like to offer a modified hypothesis for "ancient digit count" and "modern digit count". When two unrelated terms are used for calculations, "ancient digit count" multiplied by $\frac{59}{60}$ will roughly equal "modern digit count" (I used exponentiation before). When two related terms are used for calculations, the nontrivial similarities result in a simpler representation than a modern base system (and thus fewer digits will be used).

For columns 2 and 3, the data uses an "ancient digit count" of $32 + 33$ to represent a "modern digit count" of $31 + 33$, which results in $(32+33) \cdot \frac{59}{60} \approx (31+33)$, which is within a twelfth of a digit for the modified hypothesis. For column 4, the two values are related (being two sides of a right triangle), so the composite metric requires fewer digits than common sense would dictate, which also meets the modified hypothesis. That is my best explanation of what is going on, although most readers would probably be more convinced that the ancient tablet just had errors on it.

One overarching point I want to make is that Babylonian base 60 might be confusing because normally when a number has more digits it is bigger, but I believe that is not always true in Babylonian base 60, unintuitively. This could explain why so many historians think historical base 60 is ill-defined.

As fairness would dictate, I falsified my own "epsilonic hypothesis", so take the following section with a grain of salt.

4.6.2 Conjectured "Epsilonic Bases"

If "epsilonic bases" exist, I believe Babylonian base 60 is among nine conjectured "nonalternating epsilonic bases":

Groups	Subgroups	"Abstract Base"	"Crude Base"	"Epsilon Class"
0	1	0	2	-2ϵ
2	3	6	5	$+\epsilon$
4	6	24	5^2	$-\epsilon$
6	10	60	59	$+\epsilon$
8	15	120	11^2	$-\epsilon$
10	7	70	71	$-\epsilon$
12	3	36	35	$+\epsilon$
14	2	28	3^3	$+\epsilon$
16	1	16	15	$+\epsilon$

For these conjectured bases, I want to note that $5, 11^2, 15$ are boundaries and maxima, if you ignore the "degenerate base" with zero groups. Also, if you sum the columns of epsilons, they all cancel. A further qualitative point is that the music theory bases proposed earlier are: 12, 29, 70, 169, 408, which has a vague similarity to the "abstract bases" 16, 28, 36, 70, 120.

Of the "non-degenerate bases", there are 3 "negative epsilonic" and 5 "positive epsilonic". I believe these "epsilonic bases" may relate to the positive real axis of the Riemann zeta function and "imaginary bifurcations".

Unfortunately, the above table does not explain Native American bases like the Maya numeral system and Kaktovik numerals (both base 20).

This led me to think of how different bases could be defined, and I concluded that the above class of "epsilonic bases" may be defined similarly to how we check for divisibility by 9 in base 10 (or more generally divisibility by $b-1$ in base b). An example in base 10 would be determining if 211104 is divisible by 9 by adding the digits: $2+1+1+1+0+4=9$ and since the result is divisible by 9, so is the original.

This leads to the notion of "alternating epsilonic bases", which I conjecture would help explain the Maya numeral system:

Groups	Subgroups	"Abstract Base"	"Crude Base"	"Epsilon Class"
1	1	1	2	$-\epsilon$
2	3	6	7	$-\epsilon$
4	5	20	19	$+\epsilon$
8	7	56	57	$-\epsilon$
7	9	63	62	$+\epsilon$
3	11	33	2^5	$+\epsilon$
2	13	26	5^2	$+\epsilon$
1	15	15	2^4	$-\epsilon$

This system would be based on the divisibility trick where you can determine a number's remainder mod $b+1$ in base b by alternating between adding and subtracting digits to find the residue. An example in base 10 would be determining if 25795 is divisible by 11 by adding and subtracting alternating digits: $2-5+7-9+5=0$ and since the result is divisible by 11, so is the original. A similar type of alternating addition and subtraction is pretty explicit when counting in another base 20 system, Yoruba numerals, which is circumstantial evidence in favor of this theory.

But I conjecture there is yet another related group defined from infinity downwards:

Lattice	"Crude Rank"	"Epsilon Class"
Unit	1	
G2	2	$-\frac{1}{\infty}$
A3	3	
D3	3	
F4	5	$-\epsilon$
D5	5	
E6	5	$+\epsilon$
E7	6	$+\epsilon$
E8	7	$+\epsilon$
Leech Lattice	24	$+\frac{1}{\infty}$

The above was essentially derived from the observation that three particularly prominent historical bases have been $\{10, 20, 60\} = 10 \cdot \{1!, 2!, 3!\}$ where $\{10, 20, 60\} + 5 = \{2^4 - 1, 5^2, 2^6 + 1\}$.

I then anticipated a similar version would hold for $\{4, 8, 24\} = 4 \cdot \{1!, 2!, 3!\}$ where $\{4, 8, 24\} + 1 = \{5, 3^2, 5^2\}$.

While I did not have a goal while creating this, the idea may conceptually relate to group theory's 27 sporadic groups and 18 countably infinite families. Differences in rank could be explained by things like Dynkin diagrams, but I know very little.

I believe that these systems can be related to five stages in the logistic map: the first stage being essentially normal bases (e.g. base 10) for the evens and odds before bifurcation begins, the second stage being normal bases as well (e.g. base 10) between the first bifurcation and the onset of chaos, the third stage being "alternating epsilonic bases" (e.g. base 20) defined for evens, the fourth stage being "nonalternating epsilonic bases" (e.g. base 60) for odds, with the fifth stage defined by the infinity group.

I suspect that if these "epsilonic bases" exist, then they relate back to chaos theory.

In order to combine four types into one superseding type, I suggest the possibility of using surreal numbers and a separate music-theoretic number system. The idea is to stitch together multiple number systems:

1) (0): Integers
2) ($\frac{1}{4}$): Eisenstein and Gaussian primes
3) ($\frac{1}{2}$): Rationals
4) (1): Reals
5) (2): Complex
6) (4): Quaternions
7) (8): Octonions
8) (16): Sedenions
9) (∞): Leech Lattice and E8 Lattice

This could relate to my first conjecture about the logistic map as it relates to primes and "hyperstructured numbers" on the horizontal axis and the E8 and Leech lattices on the vertical axis, since the boundaries of 0 (integers) and infinity (lattices) (2.1) are present.

I only recently learned about the Langlands program, but I believe the table above may relate to the higher dimensional integration. Notably, the Ramanujan tau function, which is related to the Langlands program, has degree 24, which arguably could relate to the Leech lattice.

Another explanation of my goal could be a derivation of a field with a single element[79]. It is worth noting that I often ask myself what my goal is, and I rarely have an answer I deem acceptable, but perhaps the concept of a field with a single element is my target when I use words like unify. This might mean there is a valid way to construct fields similar to finite fields (which use primes) with "hyperstructured numbers" instead, although it would require some creativity.

Notably, the below table may be related. It is heavily inspired by the Cayley–Dickson construction:[80]

	Dimension	2^d+1	Whole	P. Factors	Ordered	Π Commutative	Π Associative	Π Alternative	Π Power-assoc.	Nontrivial 0 divisors
Integer	0	2	True	True	True	True	True	True	True	False
Rational	$\frac{1}{2}$	$\sqrt{2}+1$	False	True	True	True	True	True	True	False
Real	1	3	False	False	True	True	True	True	True	False
Complex	2	5	False	False	False	True	True	True	True	False
Quaternion	4	17	False	False	False	False	True	True	True	False
Octonion	8	257	False	False	False	False	False	True	True	False
Sedenion	16	65537	False	False	False	False	False	False	True	True

I originally thought I would release this paper without being contrarian on any proof, but I realize there is some utility if I encourage skepticism. Because attempting to disprove century-old proofs can be problematic, I would like to encourage the reader to be even more skeptical of me here.

One proof that has felt lackluster to me is Cantor's diagonal argument, which proves the real numbers are uncountably infinite[81]. The core of the proof lies in the ability to construct a number that is guaranteed to differ with each enumerated element on at least one digit. In base 2 and higher, I do not believe this proof is controversial, but the problem with the proof becomes a little more apparent when you consider bases between 1 and 2 (e.g. phinary or separately base $\frac{3}{2}$). The problem with these intermediate bases (e.g. phinary) is they can have redundant representations of numbers that are not trivial like binary. While I think phinary is a good example of potential pitfalls, I think the actual mechanism by which rational numbers might be extended to the real numbers would involve base $\frac{3}{2}$. Notably, the argument that the reals might still be a countable infinity is not that unusual (despite the proof to the contrary), since the axiom of choice normally implies there is a well-ordering over the reals.

I strongly advise against taking this thought experiment too seriously, but acknowledging minority viewpoints is important in all fields, even math. My problem with proofs largely boils down to how we seem to take proofs and put them in a display case for everyone to see, but the irony is we never really take them out of the display case and reexamine them. It is a way of categorizing a problem as solved so we never really have to look at it again, which strikes me as less logical and more of a lazy tool that helps us focus on abstractions. I would be more sympathetic to the *proofs are logical* argument if proofs acknowledged circumstantial evidence against correctness (like the phinary and axiom of choice arguments above).

The reason I bring this up is because there is a standard argument that going past infinity is impossible, and I think this is only true if rigid interpretations are used. The outline of the idea is that going to 110% of infinity within the context of the real numbers is not meaningful for a naive interpretation, but if you consider the data within the context of a larger data type (e.g. complex numbers) it might derive new meaning. Similarly, going to 110% of infinity in the context of complex numbers is not meaningful naively, but if you consider the data within the context of quaternions it might be. Thus, arguably, the cause of many bifurcations may be reaching an infinity in some sense. This is far from axiomatic, but I believe lots of mathematics is aspirational and philosophical.

The key point is there is always more than one way to solve a problem, and using only one or two bases (e.g. base 10 and base 2) is not beautiful. If the world only had English (with 26 characters) and Mandarin (with thousands of characters) and the programming languages Python (object-oriented) and Lisp (functional), I would be very unhappy. Here we do not even seem to have distinctions of paradigms, we simply have base 10 and eccentric bases like phinary (rarely taught).

If you ask me why so many people hate math, I would argue math mirrors how we think, and human thought is often messy and chaotic, not a structured and logical system. By having a singular way to do arithmetic, we exclude everyone who thinks differently. Those who succeed were merely lucky their paradigm won in the war of ideas. I am not against rigor, it is merely not how I think, and I would still be empathetic if the roles had been reversed. It is worth noting that the system is not binary either, there are infinite systems, all equally valid, and presenting multiple perspectives is a moral obligation of any field that is seriously invested in growth, so long as those perspectives are harmless in nature.

If you are interested in studying ancient numeral systems, these four have stood out to me:

Base 60, ancient Sumerians, Akkadian, and Babylonians of modern Iraq

Base 10 without zero, modern Egyptians

Base 20, modern Eskaleut from Russia to Canada to Greenland; modern Mayans and ancient Olmecs of modern Mexico; modern Yoruba around Nigeria

Base 10 with zero, modern Indians

4.6.3 Base 120

I believe base 60 is good, but I also believe it has flaws that result in inconsistent treatment of zero. Because of this, I suggest the possibility of using base 120 with 8 groups of 15, where this formulation can partially be seen in the table of primes up to 120 listed two ways:

Primes in groups of $5^2 - 1 = 24$, by conjecture in standard counting order and including 2 as a prime:

1	5	7	11	13	17	19	23
25	29	31	35	37	41	43	47
49	53	55	59	61	65	67	71
73	77	79	83	85	89	91	95
97	101	103	107	109	113	115	119

Primes in groups of 15, by conjecture from infinity backwards and including 4:

1	5	7	11	13
17	19	23	25	29
31	35	37	41	43
47	49	53	55	59
61	65	67	71	73
77	79	83	85	89
91	95	97	101	103
107	109	113	115	119

Where notably, I think the numbers 1,2,4,5 are all half-prime in some sense (which is ultra controversial, and the goal is not to redefine the prime numbers).

The idea is that the first table would represent zero upwards (real numbers) and the second table would represent infinity downward (p-adic numbers), and the two might be combined into an abstract Fibonacci sequence. Thus, the identities for Fibonacci-like sequences using divisors of 120 and 11 (4.3) would be used. Particularly notable are the identities of the 120 offset's prime factorizations, which for the terms 1548008755920 and 7740043779600 include the divisor $\frac{7!}{2} + 1 = 2521$, a possible mechanism for allowing nested subspaces.

120 also corresponds to both a minimal and maximal point in the Riemann zeta function across integers for $zeta(-3) = \frac{1}{120}$. This occurs when using two metrics with values $\frac{2}{3}$ and 16 (2.3.3).

I also believe when the secrets of the above "epsilonic bases" are solved (if they exist), they may be combined into an equation that is base $\frac{3}{2}$ in some sense, with very fundamental properties that are tied to lots of important subjects (e.g. power fractional parts and Waring's problem).

4.7 Grimm's conjecture

Grimm's conjecture is probably my favorite modern conjecture. A good description from Wikipedia states: "[for] each element of a set of consecutive composite numbers one can assign a distinct prime that divides it"[82].

A missed opportunity with Grimm's conjecture is quantitatively assigning values to each group of composite numbers. Specifically the cross product of minimal and maximal sequences, and additive and multiplicative perspectives, seen here:

Underlying Composite Span	[4]	[6]	[8, 10]	[12]	[14, 16]	[18]	[20, 22]	[24, 28]	[30]	[32, 36]	[38, 40]	[42]	[44, 46]	[48, 52]
Minimal Multiplicative	2	2	30	2	42	2	30	2730	2	5610	30	2	66	2730
Maximal Multiplicative	2	3	30	3	70	3	385	2730	5	7854	1235	7	1265	23205
Minimal Additive	2	2	10	2	12	2	10	30	2	38	10	2	16	30
Maximal Additive	2	3	10	3	14	3	23	30	5	40	37	7	39	45

I have two pieces of this table I am interested in. First, the minimal and maximal additive values at indices 8 and 10, where the additive terms are 30-30 and 38-40, where the following column has the same span of composites. Weird, but not impossible to be pure coincidence.

The other part has to do with how 2730 appears twice. It is not the most unusual number, it is formed out of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2730$ and essentially occurs around composites 5^2 and 7^2, and part of my interest lies in the fact that all terms are {divisors of 12} plus 1. However, the main reason I am interested in it relates to known bounds of the Riemann hypothesis, particularly from Schoenfeld in 1976[14]:

$|\pi(x) - li(x)| < \frac{1}{8\pi}\sqrt{x} \cdot log_e(x)$, for all $x \geq 2657$
$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x} \cdot {log_e}^2(x)$, for all $x \geq 73.2$

With the prime counting function $\pi(x)$, the logarithmic integral $li(x)$, and Chebyshev's second function $\psi(x)$.

Where notably $2657 + 73.2 = 2730.2 \approx 2730$. It is not the first connection most people would make, but that was what sparked my interest in Grimm's conjecture.

I computed the values above manually, but it would be easier and less error-prone to automate the process. My first thought on computing these values was to use an algorithm called the stable marriage problem (which is intended to match one prime to each composite). Formalizing the four sequences may involve using this algorithm with integer or logarithmic costs (possibly negative) associated with either prime-optimal or composite-optimal matches.

4.8 Shell Sort

Shell sort is a sorting algorithm that uses number theory to sort data. This is a rare property, and the only other sorting algorithm I know with this property is comb sort. It is named after Donald Shell.

The primary motivation for Shell sort is sorting data in-place (without using additional memory) with an unusually simple implementation. Its simplicity, however, is deceptive, because its efficiency relies of the gap sequence used, which defines the groups that are sorted. This leads to a lot of misinformation about the algorithm, e.g. I thought the algorithm had quadratic complexity in college, and a popular and well-curated computer science resource, Big-O Cheat Sheet, has misinformation on the subject where it seems to assume Pratt's gap sequence (which is already somewhat problematic), but does not use Pratt's best case, which is $O(n \cdot log_e(n)^2)$ just like the expected and worst case[83][84].

My personal motivation for including Shell sort in this paper is that I was originally planning to make my first paper a satirical one called SHELLSORT$^{\text{and number theory}*}$. It is arguably the best explanation for why I got back into pure mathematics and part of why Shell sort is last in this paper.

In abstract terms, Shell sort uses insertion sort as a subroutine, where insertion sort is the fastest standard algorithm for sorting small arrays (e.g. one of the most practical sorting algorithms, timsort, uses it on arrays with < 64 elements[85]).

To explain how the gap sequences work, I will explain with the following gap sequence, Tokuda's $\lceil \frac{9 \cdot (\frac{9}{4})^n - 4}{5} \rceil$[86]:

1, 4, 9, 20, 46, 103, 233, 525, 1182, 2660, 5985, 13467, 30301...

If the algorithm is asked to sort an array of 1000 elements, it essentially breaks the problem down into subroutines of insertion sort of sizes 525 then 233 then 103 and so on until it reaches 1:

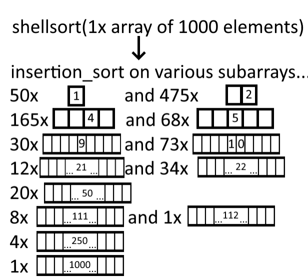

Figure 4.12: An abstraction of how Shell sort starts by sorting many small arrays and scales to sort larger arrays over time.

The interesting part lies in how these arrays overlap, since it is more useful to swap values earlier when they can be moved a distance of 525 or 233 rather than later when they can only be moved a distance of 1 or 4 at a time.

There is a truthiness that has evolved within the realm of research, specifically: "Gonnet and Baeza-Yates observed that Shell sort makes the fewest comparisons on average when the ratios of successive gaps are roughly equal to 2.2"[86]. I cannot debunk this fact, but there is circumstantial evidence that this is not true. For example, the gap sequence that is usually regarded as the best is Marcin Ciura's empirically determined:

1, 4, 10, 23, 57, 132, 301, 701, 1750

Which has the lowest ratio of $\frac{301}{132} = 2.2803...$ between consecutive terms but an average ratio of $\sqrt[8]{1750} = 2.5431....$ Additionally, there is a relatively obscure gap sequence that competes with Tokuda's despite being defined by $\lceil 2.48^n \rceil$[87]. Moreover, recent work has noticed a fractal-like structure in the gap sequences[88] that makes it difficult to determine good ratios. A possible explanation for why Gonnet and Baeza-Yates believe in ratios near 2.2 may lie in the two common ways of approaching the problem, fixed gap sequences and input size-dependent gap sequences (usually oblivious), where Gonnet and Baeza-Yates did the latter. I think my primary goal here is to provide a cautionary tale and indicate that making fewer assumptions can generally help with problems as complicated as Shell sort.

4.8.1 A New Empirical Sequence

I have largely researched Shell sort manually, rather than automating away the logic like other authors. During my time researching, I have largely converged on the following sequence that is conceptually similar to Ciura's methodology:

1, 4, 10, 23, 57, 145, 364, 920, 2990

My methodology sums the comparisons done across multiple trials using different input array sizes to compare two gap sequences. For example, 1,000,000 trials using random elements of input sizes $[750, 1740]$ with gap sequences A and B. While this methodology is vaguely similar to Marcin Ciura's I want to raise to the attention of the readers that Marcin Ciura's methodology uses an accept or reject system that might be better represented by the median than the average[89].

A typical comparison between Marcin Ciura's sequence and mine with 1,000,000 trials each can be found here:

Gap Sequence	Comparisons for random input size [55, 130]	Comparisons for [130, 325]	Comparisons for [325, 750]	Comparisons for [750, 1740]	Comparisons for [1740, 4000]
Marcin Ciura's	668128175	2129003608	6174930622	16978797347	45400828700
This paper	668128175	2125105785	6154715320	16936039838	45367825569
Reduction (%)	0%	-0.18308%	-0.32737%	-0.25182%	-0.07269%

So, as you can see, the improvement is *incredibly* marginal. The reason there is no difference for input sizes $[55, 130]$ is because both gap sequences are identical for the values 1, 4, 10, 23, 57, so the primary reason for inclusion is to show that the method of comparison meets at least one benchmark for correctness.

The first aspect I would like to highlight is that Marcin Ciura's version and mine seem to suggest the possibility that the central connection might be $2 \cdot Pell(n+2) - \frac{cos(\frac{\pi n}{2}) - sin(\frac{\pi n}{2}) + 1}{2}$, where $Pell(n)$ represents the Pell numbers. Notably, when you try this sequence, it does not perform efficiently, but I do not consider that disqualifying. For reference, here are the three versions side-by-side:

Marcin Ciura's	1	4	10	23	57	132	301	701	1750
Modified Pell	1	4	10	23	57	140	338	816	1970
This paper	1	4	10	23	57	145	364	920	2990

Where notably the ratios of the Pell numbers are roughly $1 + \sqrt{2} \approx 2.41421$. I am mostly curious about this version using the Pell numbers because it may be useful for two cases: 1) finding a Big O complexity of a reasonably efficient and modern Shell sort gap sequence and 2) the possibility of being applied to sorting networks, similar to Pratt's Shell sort network[84].

It is worth noting that the fastest known parallel sorting network (asymptotically) that I am aware of is zig-zag sort, which is based on Shell sort. Unlike most sorting networks, it runs in $O(log_e(n))$ instead of $O(log_e(n)^2)$ parallel time, although the known coefficient is impractical (19600 constructive, 2700 nonconstructive)[90].

That aside, discussing my gap sequence further, here is an important detail that is either a point of interest or a bias of mine:

$$\lfloor \frac{\{2 \cdot 6, 28, 2 \cdot 496, 8128\}}{e} \rfloor = \{4, 10, 364, 2990\}$$

Where e is 2.71828..., {6, 28, 496, 8128} are the first four perfect numbers, and { 4, 10, 364, 2990} are four of the elements of my gap sequence. If true, it would certainly be fascinating, but it is more likely due to bias on my part, especially since my methods are, again, manual.

Another interesting aspect of my sequence is that it uses a very large gap at the tail end, where $\frac{2990}{920} = \frac{13}{4} = 3.25$, which is notably much larger than the 2.2 suggested by Gonnet and Baeza-Yates. Notably, as the number of terms in a sequence increases it gets more difficult to test, so this data point could easily be invalidated.

Another point I want to make is that when you observe the successive ratios of the cumulative sum of my gap sequence, you get:

5, 3, 2.53333, 2.5, 2.52632, 2.51667, 2.52318, 2.96194

Where notably all the terms are $\frac{5}{2}$ or greater. This observation could also be a bias of mine, since I used similar checks while trying to find better gap sequences. Notably, the geometric mean $\sqrt{2.96194} \cdot \sqrt{\frac{5}{2}} \approx e$ could allude to alternating strategies.

Another interesting detail is related to Pratt's gap sequence, which uses the 3-smooth numbers:

1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81, 96, 108, 128, 144...

These values can be squared to make Shell sort more practical for small arrays, although there are no proofs surrounding the time complexity of this variant. This results in numbers of the form $4^a \cdot 9^b$, which can be seen here:

1, 4, 9, 16, 36, 64, 81, 144, 256, 324, 576, 729, 1024, 1296, 2304, 2916, 4096, 5184, 6561, 9216, 11664, 16384, 20736...

When you subtract the first nine values from my new gap sequence you get:

$$\{0, 0, 1, 7, 21, 81, 283, 776, 2734\} = \{0, 0, 1, 7, 3 \cdot 7, 3^4, prime(61), 2^3 \cdot prime(5^2), 2 \cdot 1367\}$$

Where notably the prime values highlighted are essentially class 1 numbers for half-integers (i.e. 3 and 7) and the minimum, middle, and maximum "Ibrishimova numbers" (i.e. 5, 61, 1367).

It seems somewhat arbitrary, but you can apply a similar logic starting from the companion Pell numbers minus powers of two, $CompanionPell(n) - 2^{n-1}$:

1, 4, 10, 26, 66, 166, 414, 1026, 2530

And when you subtract my gap sequence you get a similar set of values:

$$\{0, 0, 0, 3, 9, 21, 50, 106, -460\} = \{0, 0, 0, 3, 3^2, 3 \cdot 7, 2 \cdot 5^2, 2 \cdot 53, -2^2 \cdot 5 \cdot 23\}$$

Where the primary differences are 23 (a "compressor number") is used instead of 1367 (an "Ibrishimova number") and the final term is negative.

At the end of writing this paper, I may have improved my gap sequence marginally using:

$$2 \cdot \{2, 3 - 1, 5, 11 - 2, 17, 41 - 3\} = \{4, 4, 10, 18, 34, 76\}$$

From the "lucky numbers of Euler" (4.2.2), specifically by using:

$$\{1, 4, 10, 25, 76\} \approx \{1, agm(4, 4), 10, agm(18, 34), 76\}$$

With less testing, thinner efficiency margins, and fewer elements in the sequence. The effective ratio is $\sqrt[4]{76} \approx 2.952591...$.

4.8.2 Out-of-Place Shell Sort

Generally speaking, it goes against the spirit of Shell sort to sort data out-of-place (i.e. using additional memory). When the goal is practicality, it makes less sense to create a modified algorithm with extra memory and complexity, but the goal is research and improving the sorting speed for large arrays.

Because of this, I claim a new version of Shell sort can be made with much better cache utilization. The idea boils down to two abstract operations: unwind and rewind.

The unwind operation maps values from a single array with n terms into g subarrays of size $[\lfloor \frac{n}{g} \rfloor, \lceil \frac{n}{g} \rceil]$.
The rewind operation maps values from g subarrays of size $[\lfloor \frac{n}{g} \rfloor, \lceil \frac{n}{g} \rceil]$ back into a single array with n terms.

Because the effort is wasted going back and forth unnecessarily, the work can likely be optimized into a single step (using modular arithmetic) and alternating between the original array and an auxiliary array each time.

I believe that just by solving this alternate implementation, key insights about Shell sort may come to light, since there is a nontrivial amount of work that needs to be done to solve the out-of-place variant. Not only that, but once it is solved it allows for more efficient cache utilization that will speed up research of Shell sort for large input sizes.

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Afterword

5.1 Final Thoughts

About halfway through writing this paper, I discovered I use philosophy a lot. While it is simple to avoid applied math; it is much harder to avoid philosophy in research.

My general goal is to propose new ideas and have fun while writing my first math paper. This has amplified the scatterbrained style, and I expect to see hundreds of errors in this paper. With a tone between satire and reconstruction, this paper is almost mathematical fanfiction.

I tend to ignore certain proofs in the mathematics community, especially proofs of optimality and impossibility. My outlook has usually been: The appearance of a limitation, real or perceived, can be more debilitating than an actual handicap. My math prefers open-mindedness and empathy over logic.

There are many details of this paper I criticize. Foremost among them is superfluous content measured against Antoine de Saint-Exupéry's quote, "Perfection is achieved, not when there is nothing more to add, but when there is nothing left to take away." The problem is my creative process relies **heavily** on mathematical coincidences, so removing mathematical coincidences fundamentally alters the paper, and my hope is that the paper is better with these small mathematical artifacts and their personality.

It is worth noting I named sequences after individuals (to the best of my ability), and the goal here is to give the right of first refusal to the original author, where they may choose to personally name the sequence (or keep it named after themselves). These sequences are not mine to name, so the idea is to give credit where credit is due.

Other than that, I hope the paper speaks for itself. To leave you with a quote by Florence Nightingale: "The world is put back by the death of everyone who has to sacrifice the development of his or her peculiar gifts to conventionality."

5.2 About The Author

I have far fewer math credentials than most mathematicians. My highest degree in math is part of a triple major from a 2-year college. I have also failed a math course where 40% of the class got A's (Applied Numerical Methods). I even quit mathematics 10 years ago, shortly after thinking I solved the Riemann hypothesis. Nonetheless, I ended up doing mathematics while creating 2D virtual reality games that eventually brought me back to using math, and I started pure math again on 29 May 2019. My last job was in software, and I often use code to add some empiricism to my research.