The diagonal sections of bivariate Archimedean copulas and the estimation of parameters

Son Kum-Chol, Song Jin-A, Ro Song-Chol

Abstract

We introduce new concepts-a generator of degree α and a diagonal section of degree α for any real number $\alpha\,(\geq 1)$. A diagonal section of degree α is the one of the bivariate Archimedean copula with a generator of degree α. Generators of many well-known parametric families of bivariate Archimedean copulas, including those of Clayton, Frank and Gumbel-Hougaard, are of degree α.

In this article, we show that each bivariate Archimedean copula with a generator of degree $\alpha\,(\geq 1)$ is uniquely determined by its diagonal section. An asymptotic representation of these copulas in terms of corresponding diagonal sections is obtained. We also provide a sufficient condition to be a diagonal section of degree $\alpha(\geq 1)$. These results allow us to construct several statistical inference procedures for bivariate Archimedean copulas. Since diagonal sections of copulas are absolutely continuous, we suggest a parametric estimation procedure for bivariate Archimedean copulas based on the likelihood of a full sample from the diagonal section.

Keywords: Archimedean copula, generator of degree α, diagonal section of degree α, estimation of parameters

1. Introduction

We begin with some notations. Assume that a function h is defined on a subset of real numbers that includes a left-sided neighborhood of t_0, $(t_0-\varepsilon, t_0]$. Then, the left-sided limit of h as $t \to t_0$ is denoted by $\lim_{t\to t_0-0} h(t)$, and the left-sided derivative of h is denoted by $h'(t_0^-)$. Also, we write I for the interval $[0,1]$.

A bivariate copula C is a function from I^2 to I with the following properties:

· $\forall u,v \in [0,1]$, $C(u,0)=C(0,v)=0$, $C(u,1)=u$, $C(1,v)=v$,

· $\forall u_1,v_1,u_2,v_2 \in [0,1]$ $(u_1 \le u_2, v_1 \le v_2)$, $C(u_2,v_2)-C(u_1,v_2)-C(u_2,v_1)+C(u_1,v_1)\ge 0$.

As Sklar showed, for any bivariate cumulative distribution function $H(x,y)$ with margins $F(x)$ and $G(y)$, there exists a copula C such that $H(x,y)=C(F(x),G(y))$ for all $x,y\in R$ and C is uniquely determined on $RanF \times RanG$. This fact can be analyzed that copulas characterize the joint stochastic behavior between random variables independent of their marginal distributions.

The diagonal section of a copula C is defined by $\delta(u)=C(u,u), u\in I$.

A copula of the form

$$C(u,v)=\varphi^{-1}\{\min[\varphi(0),\varphi(u)+\varphi(u)]\}$$

for some convex, strictly decreasing function $\varphi: I\to[0,\infty]$ such that $\varphi(1)=0$ is called an Archimedean copula. If $\varphi(0)=\infty$, then φ is called a strict generator. For further discussion, see Genest and MacKay (1986 a, b), Marshall and Olkin (1988), among others.

Genest and Rivest(1993) showed that every bivariate Archimedean copula C is uniquely determined by its Kendall distribution function, the distribution function of $C(U,V)$, where

(U,V) is a bivariate random vector with the distribution function $C(u,v)$. Naturally, this led to construct a statistical inference procedure based on the moments of Kendall distribution functions in any class of Archimedean copulas. However, they concluded that estimating parameters in parametric families of Archimedean copulas was an open problem.

Since diagonal sections of copulas have the following properties, we thought that if every Archimedean copula of a parametric family has one-to-one relationship with its diagonal section, it would be advantageous to use diagonal sections to estimate parameters:

· δ_C is the distribution function of $\max(U,V)$,

· δ_C satisfies Lipschitz condition with constant 2 so that it is absolutely continuous.

Sungur and Yang(1996) claimed that every diagonal section of Archimedean copulas uniquely determined the corresponding copula. But this is not true for Archimedean copulas with φ such that $\varphi'(1^-)=0$, generally. (They denoted the generators by g, not φ). To the best of our knowledge, the largest class of Archimedean copulas determined uniquely by their corresponding diagonal sections is given by $\delta'(1^-)=2$, which is called Frank's condition (see Frank(1996)). However, unfortunately, there exist several parametric families of Archimedean copulas whose elements do not fulfill Frank's condition. Typically, the elements of both Gumbel-Hougaard and Joe families fulfill Frank's condition only when $\theta=1$, while they are uniquely determined by their diagonal sections, as we showed below.

This paper is organized as follows. In section 2, we suggest a specific class of bivariate Archimedean copulas whose elements are determined uniquely by their diagonal section. This is the class of bivariate Archimedean copulas with generators of degree $\alpha\,(\geq 1)$, introduced newly. We will see that many well-known parametric families of Archimedean copulas are included in that class. We also provide a sufficient condition to be the diagonal section of an Archimedean copula in the proposed class. In section 3, we suggest an estimation method of parameters of Archimedean copulas on the basis of the results of section 2. Our immediate problems briefly represented in section 4.

2. Generators and diagonal sections of degree $\alpha(\geq 1)$

Let C be a bivariate Archimedean copula generated by φ. Since $k\varphi$ is also a generator of the same copula C for any positive real number k, without loss of generality, throughout this article we restrict our consideration to the class of Archimedean generators satisfying

$$\varphi\left(\frac{1}{2}\right)=1. \tag{1}$$

We denote the class of these Archimedean generators by Ω.

Definition 1. Let $\varphi\in\Omega$. If

$$\lim_{u\to 1-0}\varphi(u)/(1-u)^{\alpha}\neq 0 \tag{2}$$

for some real number $\alpha(\geq 1)$, then we call φ a generator of degree α.

We denote the set of generators of degree α by Ω_α. We give some examples of the generators of degree $\alpha(\geq 1)$. Notice that the generators of all Archimedean copulas of the parametric families in Table 4.1, Chapter 4 of Nelson(2006) are of some degree $\alpha(\geq 1)$, except (4.2.18) showed in Example 6.

Example 1. The generator φ_θ of a copula C_θ of Clayton family((4.2.1) of Nelson(2006)) is $\varphi_\theta(u)=\dfrac{u^{-\theta}-1}{2^\theta-1}$, where $\theta\in[-1,\infty)\setminus\{0\}$. Since

$$\lim_{u\to 1-0}\frac{\varphi_\theta(u)}{(1-u)}=-\varphi_\theta'(1^-)=\frac{\theta}{2^\theta-1}\neq 0$$

for any $\theta \in [-1,\infty)\setminus\{0\}$, we have $\varphi_\theta \in \Omega_1$.

Example 2. The generator φ_θ of a copula C_θ of Frank family((4.2.5) of Nelson(2006)) is $\varphi_\theta(u) = \dfrac{\ln\left(e^{-\theta}-1\right)/\left(e^{-\theta u}-1\right)}{\ln\left(e^{-\theta}-1\right)/\left(e^{-\theta/2}-1\right)}$, where $\theta \in (-\infty,\infty)\setminus\{0\}$. Then,

$$\lim_{u\to 1-0}\frac{\varphi_\theta(u)}{(1-u)} = -\varphi_\theta'\left(1^-\right) = \frac{\theta e^{-\theta}}{\left(1-e^{-\theta u}\right)\cdot \ln\left(e^{-\theta}-1\right)/\left(e^{-\theta/2}-1\right)} \neq 0$$

for any $\theta \in (-\infty,\infty)\setminus\{0\}$, so that $\varphi_\theta \in \Omega_1$.

Example 3. The generator φ_θ of a copula C_θ of Gumbel-Hougaard family((4.2.4) of Nelson(2006)) is $\varphi_\theta(u) = \left(-\dfrac{\ln u}{\ln 2}\right)^\theta$, where $\theta \in [1,\infty)$. Since

$$\lim_{u\to 1-0}\frac{\varphi_\theta(u)}{(1-u)^\theta} = \lim_{u\to 1-0}\left(\frac{-\ln u}{1-u}\right)^\theta (\ln 2)^{-\theta} = (\ln 2)^{-\theta} \neq 0$$

for any $\theta \ge 1$, we have $\varphi_\theta \in \Omega_\theta$.

Example 4. The generators $\varphi_\theta(u) = (2-2u)^\theta$, $\theta \in [1,\infty)$ generate the copulas of the family (4.2.2) of Nelson(2006). Then,

$$\lim_{u\to 1-0}\frac{\varphi_\theta(u)}{(1-u)^\theta} = \lim_{u\to 1-0}\left(\frac{2-2u}{1-u}\right)^\theta = 2^\theta \neq 0$$

for all $\theta \ge 1$ so that $\varphi_\theta \in \Omega_\theta$.

Example 5. The generator of a copula C_θ of Joe family((4.2.6) of Nelson(2006)) is $\varphi_\theta(u) = \dfrac{\ln\left[1-(1-u)^\theta\right]}{\ln\left[1-2^{-\theta}\right]}$, where $\theta \in [1,\infty)$. By l'Hospital's rule, we have

$$\lim_{u\to 1-0}\frac{\dfrac{\ln\left[1-(1-u)^{\theta}\right]}{\ln\left[1-2^{-\theta}\right]}}{(1-u)^{\theta}}=\lim_{u\to 1-0}\frac{1}{\ln\left[1-2^{-\theta}\right]}\frac{\ln\left[1-(1-u)^{\theta}\right]}{(1-u)^{\theta}}=$$

$$=\lim_{u\to 1-0}\frac{1}{\ln\left[1-2^{-\theta}\right]}\frac{\dfrac{1}{1-(1-u)^{\theta}}\cdot\left[-(1-u)^{\theta}\right]'}{\left[(1-u)^{\theta}\right]'}=-\frac{1}{\ln\left[1-2^{-\theta}\right]}\neq 0$$

for all $\theta\geq 1$, so that $\varphi_{\theta}\in\Omega_{\theta}$.

Example 6. The generators $\varphi_{\theta}(u)=e^{(2u-1)\theta/(u-1)},\theta\in[2,\infty)$ generate the copulas of the family (4.2.18) of Nelson(2006). By setting $t=\frac{1}{1-u}$ in $\lim_{u\to 1-0}\frac{e^{\frac{2u-1}{u-1}\theta}}{(1-u)^{\alpha}}$ and using l'Hospital's rule, we get

$$\lim_{u\to 1-0}\frac{e^{\frac{2u-1}{u-1}\theta}}{(1-u)^{\alpha}}=\lim_{t\to\infty}t^{\alpha}e^{(2-t)\theta}=0$$

so that φ_{θ} is not the generator of degree α for any $\alpha\ (\geq 1)$.

Remark 1. If $\alpha_1\neq\alpha_2$, α_1, $\alpha_2\geq 1$, then $\Omega_{\alpha_1}\bigcap\Omega_{\alpha_2}=\varnothing$.

Remark 2. If $\varphi\in\Omega_1$, then $\varphi'(1^-)\neq 0$ and conversely. And If $\varphi\in\Omega_{\alpha}$ for some $\alpha>1$, then $\varphi'(1^-)=0$.

Remark 3. If φ_1 and φ_2 are generators of same degree, then

$$\lim_{u\to 1-0}\frac{\varphi_1(u)}{\varphi_2(u)}\neq 0.$$

Definition 2. The diagonal section δ of a bivariate Archimedean copula with $\varphi\in\Omega_{\alpha}$ is called the diagonal section of degree α.

The set of diagonal sections of degree α is denoted by Δ_α.

Now, we define the quasi-inverse $f^{\{-1\}}$ of a surjection $f: I \to I$ as follows:

$$f^{\{-1\}}(u) = \sup\{x \mid f(x) = u\}, \quad u \in [0,1].$$

We also set $f^{\{0\}}(u) = u,\ u \in [0,1]$ and define $f^{\{n\}}$ and $f^{\{-n\}}$ recursively as follows:

$$f^{\{n\}}(u) = f\left[f^{\{n-1\}}(u)\right],\ u \in [0,1],$$

$$f^{\{-n\}}(u) = f^{\{-1\}}\left[f^{\{-(n-1)\}}(u)\right],\ u \in [0,1],$$

where n is an arbitrary positive integer. Then, we have the following lemma.

Lemma 1. Suppose that $f: I \to I$ is a nondecreasing surjection. If

$$f(u) < u, \quad u \in (0,1), \tag{3}$$

then,

$$f^{\{m+1\}}(u) \le f^{\{m\}}(u), \quad u \in [0,1] \tag{4}$$

for every integer m and

$$\lim_{n\to\infty} f^{\{n\}}(u) = 0, \quad u \in [0,1), \tag{5}$$

$$\lim_{n\to\infty} f^{\{-n\}}(u) = 1, \quad u \in (0,1]. \tag{6}$$

Also, if f is convex, $f^{\{n\}}$ and $f^{\{-n\}}$ are convex and concave for any positive integer n, respectively.

Proof. Because f is a nondecreasing surjection, f is continuous, $f^{\{-1\}}$ is strictly increasing, $f(0) = 0$ and $f(1) = 1$. (4) is obtained by applying f and $f^{\{-1\}}$ repeatedly to both sides of (3) and considering that they are nondecreasing.

To prove (5), we choose any $u \in (0,1)$. Then, the sequence $\left\{f^{\{n\}}(u), n \ge 1\right\}$ is convergent, because it is nonincreasing. If $0 < a = \lim_{n\to\infty} f^{\{n\}}(u)$, we have

$$f(a) = f\left[\lim_{n\to\infty} f^{\{n\}}(u)\right] = \lim_{n\to\infty} f^{\{n+1\}}(u) = a$$

by continuity of f, which contradicts (3). The case $u = 0$ is trivial. (6) is treated similarly.

Now, we show that if f is convex, $f^{\{-1\}}$ is concave. For any $v_1, v_2 \in [0,1]$, we set $f^{\{-1\}}(v_i) = u_i,\ i = 1,2$. By continuity of f, $f^{\{1\}}(u_i) = f(u_i) = v_i,\ i = 1,2$. Since f is convex,

$$f\left[\lambda u_1 + (1-\lambda)u_2\right] \le \lambda f(u_1) + (1-\lambda) f(u_2)$$

for every λ ($0<\lambda < 1$). Applying $f^{\{-1\}}$ to both sides of the above inequality, we obtain

$$\lambda f^{\{-1\}}(v_1) + (1-\lambda) f^{\{-1\}}(v_2) \le f^{\{-1\}}\left[\lambda v_1 + (1-\lambda)v_2\right],$$

so that $f^{\{-1\}}$ is concave. Here we used the fact that $f^{\{-1\}}$ is increasing and $f^{\{-1\}} \circ f(u) \ge u$ for every $u \in [0,1]$. Since f and $f^{\{-1\}}$ are both nondecreasing and they are convex and concave, respectively, $f^{\{n\}}$ and $f^{\{-n\}}$ are convex and concave respectively for any positive integer n. This is because the composition of a convex function and a nondecreasing convex function is convex and, similarly, the composition of a concave function and a nondecreasing concave function is concave(see Roberts and Varberg(1973)).□

Note that the diagonal section δ of an Archimedean copula with φ can be represented as

$$\delta(t) = \varphi^{-1}\left\{\min\left[\varphi(0), 2\varphi(t)\right]\right\},\ t \in [0,1] \qquad (7)$$

from the definition of an Archimedean copula, so that diagonal sections of Archimedean copulas satisfy all conditions of Lemma 1, except convexity. Here it is understood that $\varphi^{-1}(\varphi(0)/2)=0$ in the case of $\varphi(0)=\infty$. From (7), we get

$$\varphi\left[\delta(t)\right]=\min\left[\varphi(0),2\varphi(t)\right],\quad t\in I \tag{8}$$

Substituting $t=\delta^{\{n\}}(u), n=1,2,...$ in (8), in turn, we obtain

$$\varphi\left[\delta^{\{n\}}(u)\right]=\min\left[\varphi(0),2^{n}\varphi(u)\right],\quad n=1,2,... \tag{9}$$

for any $u\in[0,1]$. Analogously, substituting $t=\delta^{\{-n\}}(u), n=1,2,...$ in (8), in turn, and considering $\delta\left[\delta^{\{-1\}}(u)\right]=u$, $u\in I$, we obtain

$$\varphi\left[\delta^{\{-n\}}(u)\right]=2^{-n}\cdot\varphi(u),\quad n=1,2,.... \tag{10}$$

In the following lemma, we show that every generator of $\Omega_{\alpha}\,(\alpha\geq 1)$ is in one-to-one relationship with its diagonal section.

Lemma 2. Let δ_1 and δ_2 be diagonal sections of Archimedean copulas with generators φ_1 and φ_2 of the same degree $\alpha\,(\geq 1)$. Then the following are equivalent:

a) $\varphi_1\equiv\varphi_2$ on I

b) $\delta_1\equiv\delta_2$ on I .

Proof. The implication a)$\Rightarrow$b) is trivial. Conversely, suppose that $\delta_1\equiv\delta_2=f$ and choose any $u\in(0,1)$. Then, by (6), (10), (1) and Remark 3, we obtain

$$\frac{\varphi_1(u)}{\varphi_2(u)}=\frac{\varphi_1\left[f^{\{-n\}}(u)\right]}{\varphi_2\left[f^{\{-n\}}(u)\right]}=\lim_{n\to\infty}\frac{\varphi_1\left[f^{\{-n\}}(u)\right]}{\varphi_2\left[f^{\{-n\}}(u)\right]}=\lim_{n\to\infty}\frac{\varphi_1\left[f^{\{-n\}}(1/2)\right]}{\varphi_2\left[f^{\{-n\}}(1/2)\right]}=1,$$

as desired.□

Lemma 3. Every diagonal section of degree $\alpha\,(\geq 1)$ has the left-sided derivative at 1, which is equal to $2^{1/\alpha}$.

Proof. Let δ be a diagonal section of degree $\alpha(\geq 1)$ and φ be the generator of degee α such that δ is the diagonal section of Archimedean copula with φ. Denoting the limiting value of (2) by k,

$$\lim_{u\to 1-0}\frac{\varphi(u)}{(1-u)^{\alpha}}=k\neq 0. \tag{11}$$

If we set $u=\delta(t)$ in (11), $u\to 1-0$ is equivalent to $t\to 1-0$ so that we obtain

$$\lim_{t\to 1-0}\frac{\varphi(t)}{(1-\delta(t))^{\alpha}}=\frac{k}{2}, \tag{12}$$

by (8). From (11) and (12), we have

$$\lim_{t\to 1-0}\left(\frac{1-\delta(t)}{1-t}\right)^{\alpha}=\lim_{t\to 1-0}\frac{[1-\delta(t)]^{\alpha}}{\varphi(t)}\frac{\varphi(t)}{(1-t)^{\alpha}}=2,$$

as desired. □

The following remarks are obtained directly from Lemma 3.

Remark 4. If $\alpha_1\neq\alpha_2$, α_1, $\alpha_2\geq 1$, then $\Delta_{\alpha_1}\bigcap\Delta_{\alpha_2}=\varnothing$.

Remark 5. Archimedean copulas with generators of degree $\alpha(\geq 1)$ fulfill Frank's condition or not according as $\alpha=1$ or not.

Lemma 4. Let $\delta\in\Delta_{\alpha}\,(\alpha\geq 1)$. Then, $\lim_{n\to\infty}2^{n}\left[1-\delta^{\{-n\}}(u)\right]^{\alpha}\neq 0$ for any $u\in(0,1)$.

Proof. Let φ be the generator of degree α corresponding to δ. Fix any $u\in(0,1)$. Then, $\delta^{\{-n\}}(u)\to 1$ as $n\to\infty$ by Lemma 1 so that

$$\lim_{n\to\infty}\frac{\varphi\left[\delta^{\{-n\}}(u)\right]}{\left[1-\delta^{\{-n\}}(u)\right]^{\alpha}}=k\neq 0$$

from (11). Therefore, considering (10), we obtain

$$\lim_{n\to\infty} 2^n \left[1-\delta^{\{-n\}}(u)\right]^{\alpha} = \frac{\varphi(u)}{k} \neq 0, \tag{13}$$

as desired.□

Theorem 1. Let $\delta \in \Delta_{\alpha}(\alpha \geq 1)$ and φ be the generator of α corresponding to δ. Then,

$$\varphi(u) = \lim_{n\to\infty} \frac{\left[1-\delta^{\{-n\}}(u)\right]^{\alpha}}{\left[1-\delta^{\{-n\}}(1/2)\right]^{\alpha}}, \quad u \in (0,1]. \tag{14}$$

Proof. Fix any $u \in (0,1)$. From Lemma 4, it follows that the right side of (14) is well-defined. Then,

$$\lim_{n\to\infty} \frac{\left[1-\delta^{\{-n\}}(u)\right]^{\alpha}}{\left[1-\delta^{\{-n\}}(1/2)\right]^{\alpha}} =$$

$$= \lim_{n\to\infty} \frac{2^n \cdot \left[1-\delta^{\{-n\}}(u)\right]^{\alpha}}{2^n \cdot \left[1-\delta^{\{-n\}}(1/2)\right]^{\alpha}} = \frac{\varphi(u)/k}{1/k} = \varphi(u).$$

by (13) and (1). Because $\delta^{\{-n\}}(1) = 1$ for any positive integer n, the case of $u = 1$ is obvious.□

Now, we give several examples that we have seen earlier to illustrate (14).

Example 1′. Every generator of the copulas of Clayton family is of Ω_1 so that $\alpha = 1$ in (14). φ is strict for $\theta > 0$ and not strict for $-1 \leq \theta < 0$. Then, $\delta_{\theta}(u) = \left(2u^{-\theta}-1\right)^{-1/\theta}$ for $\theta > 0$ and $\delta_{\theta}(u) = \max\left[\left(2u^{-\theta}-1\right)^{-1/\theta}, 0\right]$ for $-1 \leq \theta < 0$. Hence, $\delta_{\theta}^{-n}(u) = \left[\frac{2^{-n}-1+u^{-\theta}}{2^{-n}}\right]^{-1/\theta}$

for all $\theta \in [-1,\infty) \setminus \{0\}$ and every positive integer n. Thus, by l'Hospital's rule, we obtain

$$\lim_{n\to\infty} \frac{\left[1-\delta_\theta^{\{-n\}}(u)\right]^\alpha}{\left[1-\delta_\theta^{\{-n\}}(1/2)\right]^\alpha} = \lim_{n\to\infty} \frac{1-\left[1+2^{-n}\left(u^{-\theta}-1\right)\right]^{-1/\theta}}{1-\left[1+2^{-n}\left(2^{\theta}-1\right)\right]^{-1/\theta}} = \lim_{x\to\infty} \frac{1-\left[1+2^{-x}\left(u^{-\theta}-1\right)\right]^{-1/\theta}}{1-\left[1+2^{-x}\left(2^{\theta}-1\right)\right]^{-1/\theta}} =$$

$$= \lim_{x\to\infty} \frac{(1/\theta)\cdot\left[1+2^{-x}\left(u^{-\theta}-1\right)\right]^{-1/\theta-1}\left(u^{-\theta}-1\right)\cdot\left(2^{-x}\right)'}{(1/\theta)\cdot\left[1+2^{-x}\left(2^{\theta}-1\right)\right]^{-1/\theta-1}\left(2^{\theta}-1\right)\cdot\left(2^{-x}\right)'} = \frac{\left(u^{-\theta}-1\right)}{\left(2^{\theta}-1\right)} = \varphi_\theta(u). \ \square$$

Example 2′. Every generator of the copulas of Frank family is of Ω_1 so that $\alpha = 1$ in (14).

The diagonal section δ_θ corresponding to φ_θ is $-\frac{1}{\theta}\ln\left[1+\frac{\left(e^{-\theta u}-1\right)^2}{e^{-\theta}-1}\right]$. Then, we have

$$\delta_\theta^{\{-n\}}(u) = -\frac{1}{\theta}\ln\left[1-\left(1-e^{-\theta}\right)\cdot\left(\frac{1-e^{-\theta u}}{1-e^{-\theta}}\right)^{2^{-n}}\right]$$

for all $\theta \in (-\infty,\infty) \setminus \{0\}$ and every positive integer n. By l'Hospital's rule, we obtain

$$\lim_{n\to\infty} \frac{\left[1-\delta_\theta^{\{-n\}}(u)\right]^\alpha}{\left[1-\delta_\theta^{\{-n\}}(1/2)\right]^\alpha} = \lim_{n\to\infty} \frac{1+\frac{1}{\theta}\ln\left[1-\left(1-e^{-\theta}\right)\left(\frac{1-e^{-\theta u}}{1-e^{-\theta}}\right)^{2^{-n}}\right]}{1+\frac{1}{\theta}\ln\left[1-\left(1-e^{-\theta}\right)\left(\frac{1-e^{-\theta/2}}{1-e^{-\theta}}\right)^{2^{-n}}\right]} =$$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{\theta}\ln\left[1-\left(1-e^{-\theta}\right)\left(\frac{1-e^{-\theta u}}{1-e^{-\theta}}\right)^{2^{-x}}\right]}{1+\frac{1}{\theta}\ln\left[1-\left(1-e^{-\theta}\right)\left(\frac{1-e^{-\theta/2}}{1-e^{-\theta}}\right)^{2^{-x}}\right]} =$$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{\theta}\dfrac{1}{1-\left(1-e^{-\theta}\right)\left(\dfrac{1-e^{-\theta u}}{1-e^{-\theta}}\right)^{2^{-x}}}\left(1-e^{-\theta}\right)\left(\dfrac{1-e^{-\theta u}}{1-e^{-\theta}}\right)^{2^{-x}}\ln\dfrac{1-e^{-\theta u}}{1-e^{-\theta}}\cdot\left(2^{-x}\right)'}{\dfrac{1}{\theta}\dfrac{1}{1-\left(1-e^{-\theta}\right)\left(\dfrac{1-e^{-\theta u}}{1-e^{-\theta}}\right)^{2^{-x}}}\left(1-e^{-\theta}\right)\left(\dfrac{1-e^{-\theta/2}}{1-e^{-\theta}}\right)^{2^{-x}}\ln\dfrac{1-e^{-\theta/2}}{1-e^{-\theta}}\cdot\left(2^{-x}\right)'} =$$

$$= \frac{\ln\left[\left(e^{-\theta}-1\right)/\left(e^{-\theta u}-1\right)\right]}{\ln\left[\left(e^{-\theta}-1\right)/\left(e^{-\theta/2}-1\right)\right]} = \varphi_\theta(u).\square$$

Example 3′. The generator φ_θ of a copula C_θ of Gumbell-Hougaard family is of Ω_θ so that $\alpha=\theta$ in (14). The diagonal section δ_θ corresponding to φ_θ is $\exp\left(2^{1/\theta}\cdot\ln u\right)$. Then, $\delta_\theta^{\{-n\}}(u)=\exp\left(2^{-n/\theta}\cdot\ln u\right)$ for all $\theta\in[1,\infty)$ and every positive integer n. Hence, we obtain

$$\lim_{n\to\infty}\frac{\left[1-\delta_\theta^{\{-n\}}(u)\right]^\theta}{\left[1-\delta_\theta^{\{-n\}}(1/2)\right]^\theta} = \lim_{n\to\infty}\frac{\left\{1-\exp\left[2^{-n/\theta}\cdot\ln(u)\right]\right\}^\theta}{\left\{1-\exp\left[2^{-n/\theta}\cdot\ln(1/2)\right]\right\}^\theta} = \lim_{x\to\infty}\frac{\left\{1-\exp\left[2^{-x/\theta}\cdot\ln(u)\right]\right\}^\theta}{\left\{1-\exp\left[2^{-x/\theta}\cdot\ln(1/2)\right]\right\}^\theta} =$$

$$=\left[\lim_{x\to\infty}\frac{\exp\left(2^{-x/\theta}\ln u\right)\cdot\ln u\cdot\left(2^{-x/\theta}\right)'}{\exp\left(2^{-x/\theta}\ln(1/2)\right)\cdot\ln(1/2)\cdot\left(2^{-x/\theta}\right)'}\right]^\theta = \left(-\frac{\ln u}{\ln 2}\right)^\theta = \varphi_\theta(u).\ \square$$

Example 4′. The generator φ_θ of a copula C_θ of the family is of Ω_θ so that $\alpha=\theta$ in (14). The diagonal section δ_θ corresponding to φ_θ is $\max\left(2^{1/\theta}\cdot u+1-2^{1/\theta},0\right)$. Then, $\delta_\theta^{-n}(u)=2^{-n/\theta}\cdot u+1-2^{-n/\theta}$ for all $\theta\in[1,\infty)$ and any positive integer n. Therefore, we obtain

$$\lim_{n\to\infty}\frac{\left[1-\delta_\theta^{\{-n\}}(u)\right]^\theta}{\left[1-\delta_\theta^{\{-n\}}(1/2)\right]^\theta} = \left[\lim_{n\to\infty}\frac{2^{-n/\theta}(1-u)}{2^{-n/\theta}\cdot(1/2)}\right]^\theta = [2-2u]^\theta = \varphi_\theta(u).\square$$

Example 5′. The generator φ_θ of a copula C_θ of Joe family is of Ω_θ so that $\alpha=\theta$ in

(14). The diagonal section δ_θ corresponding to φ_θ is $\delta_\theta(u)=1-(1-u)\left[2-(1-u)^\theta\right]^{\frac{1}{\theta}}$. Then, $\delta_\theta^{\{-n\}}(u)=1-\left\{1-\left[1-(1-u)^\theta\right]^{2^{-n}}\right\}^{\frac{1}{\theta}}$ for all $\theta\in[1,\infty)$ and any positive integer n so that, by l'Hospital's rule, we have

$$\lim_{n\to\infty}\frac{\left[1-\delta_\theta^{\{-n\}}(u)\right]^\theta}{\left[1-\delta_\theta^{\{-n\}}(1/2)\right]^\theta}=\lim_{n\to\infty}\frac{1-\left[1-(1-u)^\theta\right]^{2^{-n}}}{1-\left[1-2^{-\theta}\right]^{2^{-n}}}=\lim_{x\to\infty}\frac{1-\left[1-(1-u)^\theta\right]^{2^{-x}}}{1-\left[1-2^{-\theta}\right]^{2^{-x}}}=$$

$$=\lim_{x\to\infty}\frac{\left[1-(1-u)^\theta\right]^{2^{-x}}\cdot\ln\left[1-(1-u)^\theta\right]\cdot\left(2^{-x}\right)'}{\left[1-2^{-\theta}\right]^{2^{-x}}\cdot\ln\left(1-2^{-\theta}\right)\cdot\left(2^{-x}\right)'}=$$

$$=\frac{\ln\left[1-(1-u)^\theta\right]}{\ln\left(1-2^{-\theta}\right)}=\varphi_\theta(u).$$

Theorem 1 suggests that diagonal sections can play a role of function parameters in the class of all Archimedean copulas generated by $\varphi\in\Omega_\alpha\,(\alpha\geq 1)$. The next theorem gives an alternative representation of Archimedean copulas of this class through the diagonal section-a new function parameter.

Theorem 2. Let C be the Archimedean copula generated by $\varphi\in\Omega_\alpha\left(\alpha\geq 1\right)$ and δ be the diagonal section of C. Then C is expressed in terms of δ as follows:

$$C(u,v)=\lim_{n\to\infty}\delta^{\{n\}}\left\{1-\left\{\left[1-\delta^{\{-n\}}(u)\right]^\alpha+\left[1-\delta^{\{-n\}}(v)\right]^\alpha\right\}^{1/\alpha}\right\},(u,v)\in I^2. \qquad (15)$$

Proof. It is sufficient to show that $\varphi\left(C(u,v)\right)=\min\left\{\varphi(0),\varphi(u)+\varphi(v)\right\}$. To show this equality, let us apply the generator φ to the right side of (15). Then, by (9) and continuity of φ, we obtain

$$\varphi\left\{\lim_{n\to\infty}\delta^{\{n\}}\left[1-\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]^{1/\alpha}\right]\right\}=$$

$$=\lim_{n\to\infty}\varphi\left\{\delta^{\{n\}}\left[1-\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]^{1/\alpha}\right]\right\}=$$

$$=\lim_{n\to\infty}\min\left\{\varphi(0),2^{n}\cdot\varphi\left[1-\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]^{1/\alpha}\right]\right\}=$$

$$=\min\left\{\varphi(0),\lim_{n\to\infty}2^{n}\cdot\varphi\left[1-\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]^{1/\alpha}\right]\right\}.$$

In the last equality, the second term can be rewritten as

$$\lim_{n\to\infty}2^{n}\cdot\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]\frac{\varphi\left[1-\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]^{1/\alpha}\right]}{\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}}.$$

By (13), we have

$$\lim_{n\to\infty}2^{n}\cdot\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]=\frac{1}{k}\cdot\left(\varphi(u)+\varphi(v)\right).$$

And, by (11),

$$\lim_{n\to\infty}\frac{\varphi\left[1-\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]^{1/\alpha}\right]}{\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}}=k.$$

Therefore,

$$\lim_{n\to\infty}2^{n}\cdot\varphi\left[1-\left[\left(1-\delta^{\{-n\}}(u)\right)^{\alpha}+\left(1-\delta^{\{-n\}}(v)\right)^{\alpha}\right]^{1/\alpha}\right]=\varphi(u)+\varphi(v),$$

so that $\varphi\left[C(u,v)\right]=\min\left[\varphi(0),\varphi(u)+\varphi(v)\right]$, as desired.□

Now, let $\alpha(\geq 1)$ be a real number and define a new class $\Im_{\alpha}$ of all surjections $f:I\to I$ satisfying the following conditions:

(a) f is nondecreasing, but strictly increasing on it's support,

(b) $\forall u \in (0,1)$, $f(u) < u$,

(c) $\forall u \in (0,1)$, $\lim_{n\to\infty} 2^n \left[1 - f^{\{-n\}}(u)\right]^\alpha \neq 0$,

(d) f is convex.

Note that every diagonal section of Archimedean copulas satisfies (a) and (b). Also, from Lemma 4, it follows that any diagonal section of degree α satisfies (c).

The next theorem shows that $\Im_\alpha \subset \Delta_\alpha$.

Theorem 3. Let $f \in \Im_\alpha$. Setting $\delta = f$ in (14), $\varphi \in \Omega_\alpha$ and f is the diagonal section of the Archimedean copula with φ.

Proof. From (c), it follows that φ in (14) is well-defined on $(0,1]$. Also it is obvious that $\varphi(1/2) = 1$ and $\varphi(1) = 0$. As showed in Lemma 1, $f^{\{-n\}}(u)$ is concave and strictly increasing and x^α is convex and strictly increasing on $[0,\infty)$ so that $\left[1 - f^{\{-n\}}(u)\right]^\alpha$ is convex and strictly decreasing for any positive integer n. Therefore, φ is convex, because it is the limit of sequence of convex functions. (see Roberts and Varberg (1973)) Similarly, φ is decreasing, because it is the limit of sequence of decreasing functions, so that it is convex and strictly decreasing, i.e., φ is an Archimedean generator.

To show that f is the diagonal section of the Archimedean copula generated by φ, it is sufficient to show that $\varphi\left[f(u)\right] = 2 \cdot \varphi(u)$ on the support of f. For any u such that $f(u) > 0$, $f^{\{-1\}}(f(u)) = u$ so that we get

$$\varphi\left[f\left(u\right)\right]=\lim_{n\to\infty}\frac{\left[1-f^{\{-n\}}\left(f\left(u\right)\right)\right]^{\alpha}}{\left[1-f^{\{-n\}}\left(1/2\right)\right]^{\alpha}}=$$

$$=\lim_{n\to\infty}\frac{\left[1-f^{\{-(n-1)\}}\left(u\right)\right]^{\alpha}}{\left[1-f^{\{-(n-1)\}}\left(1/2\right)\right]^{\alpha}}\frac{2^{n-1}\cdot\left[1-f^{\{-(n-1)\}}\left(1/2\right)\right]^{\alpha}}{2^{n}\cdot\left[1-f^{\{-n\}}\left(1/2\right)\right]^{\alpha}}\cdot 2=$$

$$=2\cdot\varphi\left(u\right),$$

as desired.

To complete the proof, it remains to show that φ is a generator of degree α. Since $f^{\{-n\}}\left(u\right)\to 1$ for any $u\in\left(0,1\right]$ by Lemma 1, $\varphi\in\Omega_{\alpha}$ if and only if $\lim_{n\to\infty}\frac{\varphi\left(f^{\{-n\}}\left(u\right)\right)}{\left(1-f^{\{-n\}}\left(u\right)\right)^{\alpha}}\neq 0$. This follows from (10) and (c). □

Wlodzimierz(2012) showed that every strict and continuously differentiable generator φ such that $\varphi'\left(1^{-}\right)=-1$ is in one-to-one relationship with its diagonal section. Note that generators, satisfying (1), of Archimedean copulas with above generators are of Ω_{1}. Also, he introduced a family $\Im$, consisting of all bijections $f:I\to I$ such that

- f is strictly increasing,
- $\forall u\in\left(0,1\right)$, $f\left(u\right)<u$,
- f is convex,
- f is C^{2} on I and $\lim_{u\to 1-0}f'\left(u\right)=2$,

and showed that every $f\in\Im$ is the diagonal section of an Archimedean copula with strict and twice continuously differentiable generator such that $\varphi'\left(1^{-}\right)=-1$. It is obvious that

$\Im \subset \Im_1$.

3. Estimation of parameters based on the diagonal seciton.

In section 2, we showed that an Archimedean generator of degree α is in one-to-one correspondence with its diagonal section. Thus, every generator of $\bigcup_{\alpha \ge 1} \Omega_\alpha$ is in one-to-one relationship with its diagonal section of $\bigcup_{\alpha \ge 1} \Delta_\alpha$. Based on this result, a natural strategy for inference in a parametric Archimedean copula family is to construct an estimation procedure using the likelihood of a full sample from a diagonal section, if all generators, satisfying (1), of the parametric family belong to $\bigcup_{\alpha \ge 1} \Omega_\alpha$.

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample of n observations from a vector (X, Y) of continuous random variables with the joint distribution function $H(x, y)$ and known marginal distribution functions $F(x)$ and $G(y)$. Suppose that the copula C of (X, Y) is Archimedean and depends on a parameter θ to be estimate. Also, assume that the generator φ_θ of C_θ satisfying (1) is of degree $\alpha(\ge 1)$. Thus, $\{\varphi_\theta, \theta \in \Theta\} \subset \bigcup_{\alpha \ge 1} \Omega_\alpha$ so that $\{\delta_\theta, \theta \in \Theta\} \subset \bigcup_{\alpha \ge 1} \Delta_\alpha$, where δ_θ is the diagonal section of C_θ generated by φ_θ.

Setting $U_i = F(X_i)$, $V_i = G(Y_i)$, $i = 1, 2, \ldots, n$, then

$$(U_1, V_1), \ldots, (U_n, V_n) \tag{16}$$

is a sample from C_θ. If every C_θ is absolutely continuous, one can seek to construct estimation procedure based on the likelihood of the sample (16) directly, but it does not hold generally. For example, every copula of the families in Example 4 and 5 has both absolute and singular components, while any copula of families in the example 1, 2 and 3 is absolutely continuous.

Now, we set

$$W' = \left(W_1, \dots, W_n\right); \;\; W_i = \max\left\{U_i, V_i\right\}, \qquad i = 1, 2, \dots, n. \tag{17}$$

Then (17) is a sample from the diagonal section δ_θ. Denoting the density function of δ_θ by d_θ, the likelihood function of the sample (17) is as follows;

$$L\left(\theta; W\right) = \prod_{i=1}^{n} d_\theta\left(W_i\right) \tag{18}$$

Here, we choose an estimator $\hat{\theta}\left(W\right)$ that maximizes the likelihood function (18), that is,

$$L\left(\theta; W\right) \le L\left(\hat{\theta}\left(W\right); W\right).$$

4. Conclusion

It does not hold that each diagonal section uniquely determines its corresponding copula in Archimedean class. In this paper we have introduced new concept of generator of degree $\alpha\left(\ge 1\right)$ of an Archimedean copula and showed that every Archimedean copula with a generator of degree $\alpha\left(\ge 1\right)$ is determined uniquely by its diagonal section. Note that the Archimedean copulas generated by $\varphi \in \bigcup_{\alpha > 1} \Omega_\alpha$ do not fulfill Frank's condition.

Now, we have some problems for further research in statistical inference for bivariate Archimedean copulas:

· To assess the performance of the estimation method proposed in section 3,

· To construct a nonparametric estimator based on diagonal section and examine its performance.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Disclosure statement: No potential conflict of interest was reported by the authors.

References

Genest C, MacKay J (1986) The joy of copulas: Bivariate distributions with uniform marginals, Amer Statist 40: 280-285

Genest, C, Rivest, L-P (1993) Statistical inference procedures for bivariate archimedean copulas. J. Amer. Statist. Assoc. 88, 1034–1043

Marshall AW, Olkin I (1988) Families of multivariate distributions, J Amer Statist Assoc 83: 834-841

Nelsen RB(2006) An introduction to Copulas, Second Edition, Springer

Sungur EA, Yang Y (1996) Diagonal copulas of Archimedean class, Comm. Ststist. 25: 1659-1676

Frank MJ (1996) Diagonals of copulas and Schröeder's equation, Aequ. Math. 51 150

Wlodzimierz W(2012) Constructing Archimedean copulas from diagonal sections, Statist Probab Lett 82: 818-826

Roberts AW, Varberg DE(1973) Convex functions, Academic Press, New York and London