Numerical Calculation of Roots of Real Polynomial Functions, Convergent Method

Juan Jorge Isaac López[1]

ABSTRACT

The Newton-Raphson method is the most widely used numerical calculation method to determine the roots of Real polynomial functions, but it has the drawback that it does not always converge. The method proposed in this work establishes the convergence condition and the development of its application, and therefore will always converge towards the roots of the function. This will mean a conclusive advance for the determination of roots of Real polynomial functions. According to interpretation of the Abel-Ruffini theorem, the roots of polynomial functions of degree greater than 4 can only be determined by numerical calculation.

KEYWORDS: Newton-Raphson method, roots of Real polynomial functions, numerical calculation.

RESUMEN

El método Newton-Raphson es el método de cálculo numérico más utilizado para determinar raíces de funciones polinómicas Reales, pero tiene el inconveniente de que no siempre converge. El método que se propone en este artículo establece la condición de convergencia y el desarrollo de su aplicación, y en consecuencia siempre convergerá hacia las raíces de la función. Ello significaria un avance conclusivo para la determinación de raices de funciones polinomicas Reales. De acuerdo a interpretación del teorema de Abel-Ruffini, la determinación de raíces de funciones polinómicas de orden superior a 4 solo es posible mediante cálculo numérico.

PALABRAS CLAVE: Método Newton-Raphson, raíces de funciones polinomicas Reales, calculo numérico.

Introduction

Here we propose the calculation of the roots of a Real polynomial function of order H, $f(x) = \sum_{i=0}^{H} a_i x^i$; $f : \mathbb{R} \to \mathbb{R}$, in two phases:

First Phase: Numerical calculation of $H - 2$ roots of the function with a successive approximation method that starts for the calculation of each root with the value of each of the roots of the second derivative of the function.

[1]E-mail: pandealocha1@hotmail.com, pandealocha1@gmail.com. WhatsApp: +58 412-5558066

This start condition causes safe convergence towards the value of the root of the function.

Second Phase: Direct calculation of the two remaining roots through a second degree equation obtained once the $H-2$ roots of the function are known.

1. Especification of the Method

In this method, the first approximation to the value of a root of the function will be the value of a root of the second derivative of the function; the second approximation to the value of the root of the function will be the value of x of the intersection point of the abscissa axis with the line tangent to the function at the point whose abscissa corresponds to the value of the root of the second derivative of the function. For this new value of x, the tangent line to the function is specified, and from the point of intersection of that tangent line with the abscissa axis, another value of x is determined that will be even closer to the value of the root of the function. Each time this procedure is repeated, a value of x closer to the root of the function will be achieved until a value as close as desired to the value of the root of the function is obtained. *With this method there will always be convergence towards the value of the root of the function.*

To obtain the values of the roots of the second derivative of the Real polynomial function to be solved, the successive derivatives of such function are previously determined until the last derivative is a linear function. From this group of successive derivatives, the roots of the intermediate derivatives are calculated, starting from the last derivative if the order of the function to solve is odd and from the penultimate derivative if the order of the function to solve is even, until the values of the roots of the second derivative of the function to solve are calculated. The last derivative equal to zero is a linear equation and the penultimate derivative equal to zero is an equation of the second degree, both are equations of direct resolution. To calculate the roots of each intermediate derivative, we proceed with the approximation method described above.

The successive derivatives of a Real polynomial function are Real polynomial functions. The number of roots of the second derivative of a Real polynomial function is equal to $H-2$, where H is the order of the function. Thus, knowing the $H-2$ roots of the function, such function can be reduced to quadratic function that allows direct calculation of the remaining two roots. These two roots will be the smallest and the largest of the roots of the function.

A variant of this method will be to test the function until the value of the root is reached or to get a value as close as one wants to the root, from any approximation x obtained in the way described above that is different from the value of the root of the second derivative. The testing is done with values greater than x if x is greater than the value of the root of the second derivative of the function and with values less than x if x is less than the value of the root of the second derivative of the function. It is possible that the value of the root of the second derivative of the function coincides with the value of the root of the function.

2. Definitions

Let $P_0(x)$ be a Real polynomial function; domain $\in \mathbb{R}$, co-domain $\in \mathbb{R}$.
Let $x(k,n)$ be the ordinal root k of the function $P_n(x)$
Let $x(l,k,n)$ be the ordinal approximation l to the root $x(k,n)$
Starting from $n=1$, $P_n(x)$ is the nth derivative function of the Real polynomial function $P_0(x)$
Ec_n: $P_n(x)=0$

3. Application of the Method

Calculation of the Roots of the function $P_0(x)$

Let
$P_0(x) = x^5 - 19x^4 + 133x^3 - 421x^2 + 586x – 280$

$P_0\text{'}(x) = P_1(x) = 5x^4 - 76x^3 + 399x^2 - 842x + 586$
$P_0\text{''}(x) = P_2(x) = 20x^3 - 228x^2 + 798x - 842$
$P_0\text{'''}(x) = P_3(x) = 60x^2 - 456x + 798$
$P_0\text{''''}(x) = P_4(x) = 120x - 456$
$P_1(x); P_2(x); P_3(x); P_4(x)$ are the successive derivatives of the function $P_0(x)$

$H=5$: the order of $P_0(x)$ is odd $\rightarrow Ec_4$: $P_4(x)=0$; Ec_4: $120x – 456 = 0 \rightarrow x(1,4) = 456/120 = 3.80$
$x(1,4)$ corresponds to the value of the second derivative of the function $P_2(x)$, so $x(1,4)$ will be the first approximation to a root of the function $P_2(x)$. Then:
$x(1,1,2) = x(1,4) \rightarrow x(1,1,2) = 3.80$

The approximations to the roots of the function are defined by the following formula:

$$x(l+1,k,n) = \frac{-P_n(x(l,k,n)) + P_{n+1}(x(l,k,n)) * x(l,k,n)}{P_{n+1}(x(l,k,n))}$$

Thus,
$x(1,1,2) = 3.80; P_2(x(1,1,2)) = -4.48$
$x(2,1,2) = 3.734502924; P_2(x(2,1,2)) = -0.0005619475$
$x(3,1,2) = 3.73442045758701; P_2(x(3,1,2)) = -2.67369E-08$
$x(4,1,2) = 3.73442045719464; P_2(x(4,1,2)) = 0$
Then $x(1,2) = x(4,1,2) \rightarrow x(1,2) = 3.73442045719464$

$$\frac{P_2(x)}{(x - x(1,2))} = \frac{P_2(x)}{(x - 3.73442045719464)} = 20x^2 - 153.3115909x + 225.4700588$$

$20x^2 - 153.3115909x + 225.4700588 = 0 \rightarrow x(2,2) = 1.984337851; x(3,2) = 5.681241692$

$x(1,2); x(2,2); x(3,2)$ correspond to the values of each root of the second derivative of the function $P_0(x)$, so $x(1,2); x(2,2); x(3,2)$ will each be the first approximation to one of the roots of the function $P_0(x)$. Then:
$x(1,1,0) = x(1,2); x(1,2,0) = x(2,2); x(1,3,0) = x(3,2)$
$x(1,1,0) = 3.73442045719464; x(1,2,0) = 1.984337851; x(1,3,0) = 5.681241692$

Thus,
$x(1,1,0) = 3.73442045719464; P_0(x(1,1,0)) = -5.205518732$
$x(2,1,0) = 3.989557854; P_0(x(2,1,0)) = -0.188927438$
$x(3,1,0) = 3.999947413; P_0(x(3,1,0)) = -0.000946592$
$x(4,1,0) = 3.99999999861755; P_0(x(4,1,0)) = -2.48833E - 08$
$x(5,1,0) = 3.99999999999995; P_0(x(5,1,0)) = -9.09495E - 13$
$x(6,1,0) = 4; P_0(x(6,1,0)) = 0$
Then $x(1,0) = x(6,1,0) \rightarrow x(1,0) = 4$

$x(1,2,0) = 1.984337851; P_0(x(1,2,0)) = 0.470028549$
$x(2,2,0) = 1.999997258; P_0(x(2,2,0)) = 8.226E - 05$
$x(3,2,0) = 2; P_0(x(3,2,0)) = 0$
Then $x(2,0) = x(3,2,0) \rightarrow x(2,0) = 2$

$x(1,3,0) = 5.681241692$; $P_0(x(1,3,0)) = -26.02866987$
$x(2,3,0) = 5.122483594$; $P_0(x(2,3,0)) = -3.322773707$
$x(3,3,0) = 5.012417523$; $P_0(x(3,3,0)) = -0.302023732$
$x(4,3,0) = 5.000162194$; $P_0(x(4,3,0)) = -0.00389334$
$x(5,3,0) = 5.000000028$; $P_0(x(5,3,0)) = -6.72E - 07$
$x(6,3,0) = 5$; $P_0(x(6,3,0)) = 0$
Then $x(3,0) = x(6,3,0) \rightarrow x(3,0) = 5$

$$\frac{P_0(x)}{(x - x(1,0)) * (x - x(2,0)) * (x - x(3,0))} = \frac{P_0(x)}{(x-4) * (x-2) * (x-5)} = x^2 - 8x + 7$$

$x^2 - 8x + 7 = 0 \rightarrow x(4,0) = 1; x(5,0) = 7$

Then the roots of the function $P_0(x) = x^5 - 19x^4 + 133x^3 - 421x^2 + 586x - 280$ are $x(1,0) = 4$; $x(2,0) = 2$; $x(3,0) = 5$; $x(4,0) = 1$; $x(5,0) = 7$

4. Convergence in Real functions:

For Real functions $f(x)$; $f : \mathbb{R} \to \mathbb{R}$, if the method of the tangents (Newton-Raphson) is started at an inflexion point, it will always converge towards a $(x, f(x)))$=$(x, 0)$ point.

Conclusions:

The subject of this article is more than three centuries late. It is a conclusive advance on the Newton-Raphson method. This article proposes the novelty of the convergence condition and the development of its application for Real polynomial functions, with which everything related to the numerical calculation of roots of Real polynomial functions is settled. It also establishes the convergence condition for Real functions, which would make it possible to try developments of its application for the numerical calculation of roots of Real functions.

References:

Newton's method. URL: https://en.wikipedia.org/wiki/Newton's_method

Abel–Ruffini theorem. URL: https://en.wikipedia.org/wiki/Abel–Ruffini_theorem

Isaac Newton. URL: https://en.wikipedia.org/wiki/Isaac_Newton