Knot in weak-field geometrical optics

Miftachul Hadi[1, 2]

[1)] *Badan Riset dan Inovasi Nasional (BRIN), KST Habibie (Puspiptek) Gd 442, Serpong, Tangerang Selatan 15314, Banten, Indonesia.*

[2)] *Institute of Mathematical Sciences, Kb Kopi, Jalan Nuri I, No.68, Pengasinan, Gn Sindur 16340, Bogor, Indonesia. E-mail: instmathsci.id@gmail.com*

We construct the geometric optical knot in 3-dimensional Euclidean (vacuum or weak-field) space using the Abelian Chern-Simons integral and the variables (the Clebsch variables) of the complex scalar field, i.e. the function of amplitude and the phase related to the refractive index. The result of numerical simulation shows that in vacuum or weak-field space, there exists such a knot.

Keywords: *knot, geometrical optics, Abelian Chern-Simons integral, complex scalar field, phase, 3-dimensional Euclidean space, weak-field.*

I. INTRODUCTION

It is commonly believed there exists no topological object in the linear theory, such as Maxwell's theory. It is because a topological theory must be a non-linear theory[1]. The existence of a topological object, a knot, in Maxwell's linear theory so far has not been well known[2]. How could a knot exist in Maxwell's linear theory?

In Maxwell's theory, the electromagnetic fields (the set of the solutions of Maxwell equations) in a vacuum space has a subset field with a topological structure[1]. Any electromagnetic field is locally equal to a subset field i.e. any electromagnetic field can be obtained by patching together subset fields (except in a zero measure set) but globally different[1]. This means that the difference between the set of the subset fields and all the electromagnetic fields in Maxwell's theory in a vacuum is global instead of local since the subset fields obey the topological quantum condition that the electromagnetic helicity (consists of electric and magnetic helicities) is equal to an integer number[1].

The electromagnetic field satisfies a linear field equation, but a subset field satisfies a non-linear field equation. Both fields, the electromagnetic field, and a subset field, satisfy the linear field equation in the case of the weak field[3]. It means that a non-linear subset field theory reduces to Maxwell's linear theory in the case of the weak field. The space where the weak field lives approximately represents the vacuum space. The electromagnetic helicity or the electromagnetic knot could exist in the vacuum Maxwell's theory because of the vacuum Maxwell's theory is the weak field limit[3] of a non-linear subset field theory.

In this article, we propose there exists a knot in the geometrical optics, as a solution of the eikonal equation. The reason is indeed there exists a knot in Maxwell's theory[1–3] and the geometrical optics (the eikonal equation) can be derived from Maxwell's theory (Maxwell equations)[4–6]. We treat the geometrical optics as an Abelian $U(1)$ local gauge theory[7,8] the same as Maxwell's gauge theory. To the best of our knowledge, the formulation of a knot in geometrical optics (a geometric optical knot) has not been done yet[1,2,9,10].

II. SUBSET FIELDS PROPERTY AND MAPS $S^3 \to S^2$

Let us consider maps of subset fields (consisting of complex scalar fields) from a finite radius r to an infinite r implies from the stronger field to the weak field. A scalar field has properties that, by definition, its value for a finite r depends on the magnitude and the direction of the position vector, $\vec{r}$, but for an infinite r it is well-defined[3] (it depends on the magnitude only). In other words, for an infinite r, a scalar field is isotropic. Throughout this article, we will work with the classical scalar field.

The property of such scalar fields can be interpreted as maps $S^3 \to S^2$[1] where S^3 and S^2 are 3-dimensional and 2-dimensional spheres, respectively i.e. after identifying via stereographic projection, 3-dimensional physical space, $R^3 \cup \{\infty\}$, with the sphere S^3 and the complete complex plane, $C \cup \{\infty\}$, with the sphere S^2.

These maps $S^3 \to S^2$ can be classified in homotopy classes, labelled by the value of the corresponding Hopf indexes, integer numbers, the topological invariants[1,3]. The other names of the topological invariants are the topological charge, the winding number (the degree of a continuous mapping)[11]. The topological charge which is independent of the metric tensor could be interpreted as energy[12].

We see there exists (one) dimensional reduction in such maps. We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a scalar field for an infinite r. The property of a scalar field as a function of space seems likely in harmony with the property of space-time. Space-time could be locally anisotropic but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

III. HOPF INVARIANT AND ABELIAN CHERN-SIMONS

Let us discuss this more formally. As we mentioned above we have a scalar field as a function of the position vector, $\phi(\vec{r})$, with a property that can be interpreted using the non-trivial Hopf map written below[1,3]

$$\phi(\vec{r}) : S^3 \rightarrow S^2 \tag{1}$$

This non-trivial Hopf map is related to the Hopf invariant[13], $\mathcal{H}$, expressed as an integral[13–15]

$$\mathcal{H} = \int_{S^3} \omega \wedge d\omega \tag{2}$$

where ω is a 1-form on S^3[13].

The relation between the Hopf invariant and the Hopf index, h, can be written explicitly as[1]

$$\mathcal{H} = h\ \gamma^2 \tag{3}$$

where γ is the total strength of the field, that is the sum of the strengths of all the tubes formed by the integral lines of electric and magnetic fields[1].

It can be interpreted naturally that the Hopf invariant has a deep relationship with the Abelian Chern-Simons action[13] (the Abelian Chern-Simons integral) in gauge field theory and the self-helicity in magnetohydrodynamics[13]. The Hopf invariant is just the winding number of Gauss mapping[13]. Hopf invariant or the Chern-Simons integral is an important topological invariant to describe the topological characteristics of the knot family[13,16]. In a more precise expression, the Hopf invariant or the Chern-Simons integral is the total sum of all the self-linking and all the linking numbers of the knot family[13,16].

In 3-dimensional Euclidean space, the Chern-Simons integral could be written as[2,16]

$$h = \int_{E^3} \varepsilon^{\alpha\mu\nu}\ \vec{A}_\alpha\ \vec{F}_{\mu\nu}\ d^3r \tag{4}$$

where h is the electromagnetic helicity, a non-zero integer number (if h is zero it implies zero energy), $\varepsilon^{\alpha\mu\nu}$ is the Levi-Civita symbol, $\alpha, \mu, \nu = 1, 2, 3$ denote the 3-dimensional space where the geometric optical knot lives, $\vec{A}_\alpha$ is the $U(1)$ gauge potential, $\vec{F}_{\mu\nu}$ is the $U(1)$ gauge field tensor[16] (the field strength tensor), $\int_{E^3} d^3r$ shows that we work in 3-dimensional Euclidean space. In Maxwell's theory, this integer h determines the $\pi_3(S^2)$ topology of the electromagnetic knots[2].

IV. GEOMETRIC OPTICAL KNOT

Let us derive how to obtain the geometric optical knot. We will follow the steps of Ranada's work. Using the scalar field, $\phi(\vec{r}, t)$, the field strength can be written as[1]

$$\vec{F}_{\mu\nu} = \vec{f}_{\mu\nu} = \theta\ \frac{\partial_\mu\phi^*\ \partial_\nu\phi - \partial_\nu\phi^*\ \partial_\mu\phi}{(1+\phi^*\phi)^2} \tag{5}$$

where $\theta = 1/(2\pi i)$ and ϕ^* is the complex conjugate of the scalar field. We call eq.(5) as the non-linear field equation where the nonlinearity is shown by the $\phi^*\phi$ term. We see that a scalar field in a non-trivial Hopf map is written as $\phi(\vec{r})$. It differs from a scalar field $\phi(\vec{r}, t)$ in Maxwell's theory. This problem is solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[17].

In the case of the weak field, i.e. $\phi << 1$ so $\phi^*\phi << 1$ then the denominator in eq.(5) can be taken as being equal to one and $\vec{f}_{\mu\nu}(\phi)$ (5) is equivalent to the Maxwell linear theory[1] as written below

$$\vec{F}_{\mu\nu} = \vec{f}_{\mu\nu} = \theta\ (\partial_\mu\phi^*\ \partial_\nu\phi - \partial_\nu\phi^*\ \partial_\mu\phi) \tag{6}$$

We interpret Maxwell's linear theory in a vacuum as the same as the non-linear field theory in the case of the weak-field limit due to the field being taken far away from the source (electric charge or current).

Assume that in general case the scalar field could be written as[17]

$$\phi(\vec{r}, t) = \rho(\vec{r}, t)\ e^{iq(\vec{r}, t)} \tag{7}$$

and

$$f(\vec{r}, t) = -1/[2\pi(1+\rho^2)] \tag{8}$$

where $\rho(\vec{r}, t)$ is the amplitude, $q(\vec{r}, t)$ is the phase, $f(\vec{r}, t)$ is the function of amplitude. This assumption (7) is based on the wave point of view of the field. We could interpret the scalar field, ϕ, as the disturbance where the physical disturbance is the real part of ϕ[18].

In the case of the weak field (a vacuum space) the scalar field (7) and the function of amplitude (8) becomes

$$\phi(\vec{r}, t) = \rho_c\ e^{iq(\vec{r}, t)} \tag{9}$$

and

$$f = -1/[2\pi(1+\rho_c^2)] \tag{10}$$

respectively, where ρ_c is a constant amplitude so f is also a constant. We consider ρ_c as an analogy with a constant amplitude of electromagnetic wave in a vacuum space[31].

By using the components of the scalar field, f and q, eq.(6) can be written as[17]

$$\vec{F}_{\mu\nu} = \vec{f}_{\mu\nu} = \partial_\mu(f\ \partial_\nu q) - \partial_\nu(f\ \partial_\mu q) \tag{11}$$

where f and q are known as the Clebsch variables[17]. We assume that eq.(11) is equal to

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu \tag{12}$$

We call eqs.(11), (12) as the linear field equations.

By observing the equality of eq.(11) and (12), we see that[17]

$$\vec{A}_\nu = f\ \partial_\nu q \tag{13}$$

Eq.(13) shows that the gauge (vector) potential can be written using the Clebsch (scalar) variables.

By substituting eqs.(11), (13) into eq.(4), we obtain the geometric optical knot as written below

$$h_{\rm go} = \int_{E^3} \varepsilon^{\alpha\mu\nu} f\ \partial_\alpha q\ \{\partial_\mu(f\ \partial_\nu q) - \partial_\nu(f\ \partial_\mu q)\}\ d^3r \tag{14}$$

where the phase is[19,20]

$$q(\vec{r},t) = X(\psi_1 - ct) = X\left(\int_{r_1}^{r_2} n(\vec{r})\ d^3r - ct\right) \quad (15)$$

$X = f_\theta/c$, f_θ is the angular frequency, c is the speed of light in a vacuum space, $\psi_1(\vec{r})$ is also called the phase, t is time and $n(\vec{r})$ is the refractive index. The refractive index is the real scalar function of coordinates (vector position) with positive values, the slowness at a point[7]. The refractive index is typically supplied as known input, given, and we seek the solution, the phase[7]. The integral $\int_{r_1}^{r_2}\ d^3r$ shows the propagation of ray from the initial position, r_1, to the final position, r_2, in 3-dimensional space.

By substituting eq.(15) into eq.(14), we obtain

$$\begin{aligned} h_{\mathrm{go}} = \int_{r_1}^{r_2} & \varepsilon^{\alpha\mu\nu}\ f\ \partial_\alpha \left[X\left(\int_{r_1}^{r_2} n(\vec{r})\ d^3r - ct\right)\right] \\ & \left\{\partial_\mu \left\{f\ \partial_\nu \left[X\left(\int_{r_1}^{r_2} n(\vec{r})\ d^3r - ct\right)\right]\right\}\right. \\ & \left. -\ \partial_\nu \left\{f\ \partial_\mu \left[X\left(\int_{r_1}^{r_2} n(\vec{r})\ d^3r - ct\right)\right]\right\}\right\} d^3r \end{aligned} \quad (16)$$

We see from eq.(16) the geometric optical knot could be formulated in relation to the refractive index. It means that the knot could exist in the geometrical optics in the case of the weak-field limit. Due to h_{go} is non-zero integer number we consider eq.(16) as a topological quantum condition[1].

V. NUMERICAL SIMULATION

Let us calculate eq.(16) numerically to show the existence of the geometric optical knot in our computer. Numerical simulation is very important as a preliminary work in order to reduce wasting much time, energy, and money. We hope that this numerical simulation makes sure the experimentalists and could be used as a guidance for searching the geometric optical knot in a laboratory.

To simplify the complicated calculation of eq.(16), we will calculate the simpler one i.e. the magnetic knot[17] (the total geometric optical knot consists of the magnetic knot plus the electric knot) and we assume that the physical system has a rotational symmetry. Now, we have the magnetic knot equation in the form of rotational symmetry as written below

$$\int_{r_1}^{r_2} \vec{A}\cdot\vec{B}\ r^2 dr = h\ a \quad (17)$$

where $\vec{A}$ is magnetic (vector) potential, $\vec{B} = \vec{\nabla}\times\vec{A}$ is magnetic field, h is an integer number excluding zero, $a = \hbar c\mu_0 = 3.97\times 10^{-32}$J . s/C.

Using the value of parameters as follow: $f_\theta = 299792458$, $c = 299792458$, $n_0 = 1.6$, $A = 0.7499$, $r = 1$, $t = 1$, $\rho = \rho_c =$ linspace$(-3, 3, 1000)$ and by the help of AI and Octave, we obtain the graph of $\int_0^1 \vec{A}\cdot\vec{B}\ r^2\ dr$ versus ρ as **Fig. 1** below

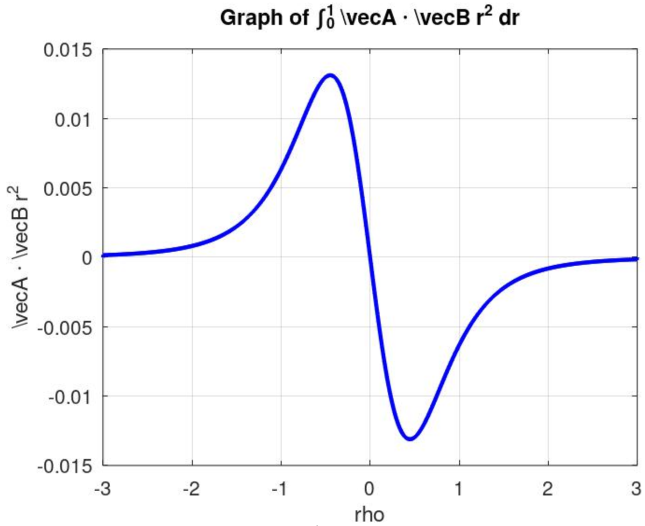

Fig. 1 The graph of $\int_0^1 \vec{A}\cdot\vec{B}\ r^2\ dr$ versus $\rho(\vec{r},t)$.

We see from **Fig. 1** above, roughly speaking, the interval $-3 < \rho < 0$ or $0 < \rho < 3$ gives the non-zero value of $\int_0^1 \vec{A}\cdot\vec{B}\ r^2\ dr$. More precisely, $\rho = -0.49493$ gives $\int_0^1 \vec{A}\cdot\vec{B}\ r^2 dr = 0.013093$. By substituting this value of integral to eq.(17) and ignore the units, we obtain $0.013093 = h\cdot 3.97\times 10^{-32}$. It gives $h = 3297984886649874\times 10^{14}$.

VI. DISCUSSION AND CONCLUSION

The mathematical study of knots began in physics in the nineteenth century. The names associated with the first work on knots are those of Carl Friedrich Gauss and Listing where Gauss gave an integral formula for the linking number of two knots in 3-dimensional space[22]. Notice that the electric and magnetic field lines could form closed loops and thus be linked, and this linking could provide the topological structure. Let two closed field lines be $\vec{c}_1(s)$ and $\vec{c}_2(s)$[2]. They are linked if they have a non-vanishing Gauss' integral. The Gauss' integral can be written as[2,23]

$$L(\vec{c}_1,\vec{c}_2) = \frac{1}{4\pi}\int\left(\frac{\vec{c}_1-\vec{c}_2}{|\vec{c}_1-\vec{c}_2|^3}\times\frac{\vec{c}_1}{ds_1}\right)\cdot\frac{\vec{c}_2}{ds_2}\ ds_1\ ds_2 \quad (18)$$

whereas for a single closed field line $c(s)$, the self-linking number $L(\vec{c},\vec{c})$ describes the knottedness[2,23].

Later, Maxwell interpreted Gauss' integral, i.e. the linking number of two knots as the energy required to move an electric charge in a knotted charged wire complement along the other knot. The other pioneers were Tait and Kelvin where the aim of Tait was to construct a table of knots[22] and Kelvin proposed the idea of a knot, topologically stable matter, that the atoms could be knots or links of vorticity lines of aether[2]. In other words, Kelvin constructed his theory of vortex atoms, according to which the matter is constituted by small knots formed by something like vortex lines of smoke[22].

A knot can be defined as a smooth-embedding of a circle in E^3[10], 3-dimensional Euclidean space[24]. In algebraic topology, a knot is defined by the Hopf index[2]. The Hopf index is related to the Hopf invariant[1]. In turn, the Hopf invariant is related to a non-trivial Hopf map[13]. Two knots are equivalent if one knot can be deformed continuously into the other without crossing itself[10].

In the case of the electromagnetic field, a knot could be formed by bending the electric and magnetic field lines (the geometric concept of magnetic lines of force - those lines of force are today designated by the symbol $\vec{H}$, the magnetic field - is due to Faraday[25]) so that they could form closed loops[2]. A set of closed loops in space forms a link[26]. These closed loops can be linked[2] (although links do not actually need to be linked[27]). The average of the linking integral over all field lines pairs together with the self-linking number of overall field lines giving rise to the electromagnetic helicity given by the Chern-Simons integral[2].

We see from **Fig. 1** above, roughly speaking, the interval $-3 < \rho < 0$ or $0 < \rho < 3$ gives the non-zero value of $\int_0^1 \vec{A} \cdot \vec{B}\ r^2\ dr$. It means that the non-zero value of integral shows the magnetic knot is a non-trivial knot. The magnetic knot could really exist. The integer number $h = 3297984886649874 \times 10^{14}$ shows the total sum of the linking number over all magnetic field lines pairs plus the self-linking number of overall magnetic field lines.

So far, we formulate the theoretical and numerical existence of the geometric optical knot. Does the electromagnetic (geometric optical) knot really exist in the universe or a laboratory? Ball lightning[29] probably is an electromagnetic knot in the universe[30] where tokamaks and devices constructed to produce fireball are two possible laboratory settings to observe ball lightning[30]. A (geometric optical) knot of light may be generated using tightly focused circularly polarized laser beams[23].

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