A New Identity For Prime Counting Function

Dedicated to my father , my first teacher

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“ No Hardy , 1729 it is a very interesting number , it is the smallest number expressible as a sum of two cubes in two different ways . ”
Srinivasa Ramanujan

Abstract

In this article, the author proves on a new identity (or equation) which asserts that for every natural number n the sum of the prime-counting function $\pi(2n)$ and the con-counting function $\bar{\pi}(2n)$ equals n , explicitly and simply $\forall\ n \in \mathbb{N}^*$ we have $\pi(2n) + \bar{\pi}(2n) = n$. The new identity (or equation) may have many applications in Number Theory and its related to one of the famous problems in Mathematics .

Notation and reminder

$\mathbb{N}^* := \{1,2,3,4,5,6,7,\dots\}$ *The natural numbers.*

$\mathbb{N}_{en} := \{2,4,6,8,10,12,14,\dots\}$ *The even numbers.*

$\mathbb{N}_{con} := \{9,15,21,25,27,33,35,\dots\}$ *The composite odd numbers.*

$\mathbb{P} := \{2,3,5,7,11,13,17,\dots\}$ *The prime numbers.*

$\mathbb{P}^* := \{3,5,7,11,13,17,19,\dots\}$ *The odd prime numbers.*

$\forall$: *The universal quantifier.*

Card A : *The number of elements in A* .

A∩B : *All elements that are members of both A and B*.

A∪B : *All elements that are members of both A or B*.

∅ : *The empty set is the unique set having no elements*.

Introduction

Definition 1(The prime-counting function $\pi(x)$) .
$\forall\ x > 0$ *we have* $\pi(x) = \text{Card}[0\ ,x] \cap \mathbb{P} = \text{Card}\{\ p \leq x : p \in \mathbb{P}\ \}$.
In other words, $\pi(x)$ *is the number of primes less than or equal to* x*.*

In 1838, Dirichlet observed that $\pi(x)$ can be well approximated by the logarithmic integral function $\text{li}(x) = \int_2^x \frac{dt}{\log t}$ or $\pi(x) \sim \text{li}(x)\ (x \to +\infty)$.
The celebrated prime number theorem, proved independently by de la Vallée Poussin and Hadamard in 1896, states that $\pi(x) \sim \frac{x}{\log x}\ (x \to +\infty)$.
Definition 2(The prime-counting function $\pi(2n)$) .
$\forall\ n \in \mathbb{N}^*$ *we have* $\pi(2n) = \text{Card}[1\ , 2n] \cap \mathbb{P} = \text{Card}\{p \leq 2n : p \in \mathbb{P}\}$.
In other words, $\pi(2n)$ *is the number of primes less than or equal to* $2n$*.*

Definition 3(The con-counting function $\bar{\pi}(2n)$) .
$\forall\ n \in \mathbb{N}^*$ *we have* $\bar{\pi}(2n) = \text{Card}[1\ , 2n] \cap \mathbb{N}_{con} = \text{Card}\{p \leq 2n : p \in \mathbb{N}_{con}\}$.
In other words, $\bar{\pi}(2n)$ *is the number of composite odd numbers less than* $2n$*.*

Definition 4(The en-counting function $\bar{\bar{\pi}}(2n)$) .
$\forall\ n \in \mathbb{N}^*$ *we have* $\bar{\bar{\pi}}(2n) = \text{Card}[1\ , 2n] \cap \mathbb{N}_{en} = \text{Card}\{p \leq 2n : p \in \mathbb{N}_{en}\}$.
In other words, $\bar{\bar{\pi}}(2n)$ *is the number of even numbers less than or equal to* $2n$*.*

For instance :

For $n = 1$ we have $\pi(2) = 1$ and $\bar{\pi}(2) = 0$ and $\bar{\bar{\pi}}(2) = 1$

For $n = 2$ we have $\pi(4) = 2$ and $\bar{\pi}(4) = 0$ and $\bar{\bar{\pi}}(4) = 2$

For $n = 3$ we have $\pi(6) = 3$ and $\bar{\pi}(6) = 0$ and $\bar{\bar{\pi}}(6) = 3$

For $n = 4$ we have $\pi(8) = 4$ and $\bar{\pi}(8) = 0$ and $\bar{\bar{\pi}}(8) = 4$

For $n = 5$ we have $\pi(10) = 4$ and $\bar{\pi}(10) = 1$ and $\bar{\bar{\pi}}(10) = 5$

For $n = 6$ we have $\pi(12) = 5$ and $\bar{\pi}(12) = 1$ and $\bar{\bar{\pi}}(12) = 6$

For $n = 7$ we have $\pi(14) = 6$ and $\bar{\pi}(14) = 1$ and $\bar{\bar{\pi}}(14) = 7$

For $n = 8$ we have $\pi(16) = 6$ and $\bar{\pi}(16) = 2$ and $\bar{\bar{\pi}}(16) = 8$

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Lemma. $\forall\, n \in \mathbb{N}^*$ *we have* $\bar{\bar{\pi}}(2n) = n$.

Proof. Indeed, $\forall\, n \in \mathbb{N}^*$ we have $\text{Card}[1\,,2n] \cap \mathbb{N}^* = 2n$, this means that the number of integers odd or even in the interval $[1\,,2n]$ is equal to $2n$, and $\text{Card}\{1\,,3\,,\ldots,2n-1\} = \text{Card}\{2\,,4\,,\ldots,2n\} = \bar{\bar{\pi}}(2n)$, this means that the number of odd numbers equal to the number of even numbers in $[1\,,2n]$, and $\text{Card}[1\,,2n] \cap \mathbb{N}^* = \text{Card}\{1\,,3\,,\ldots,2n-1\} + \bar{\bar{\pi}}(2n) = 2n$, thus $\bar{\bar{\pi}}(2n) = n$.

Theorem. $\forall\, n \in \mathbb{N}^*$ *we have* $\pi(2n) + \bar{\pi}(2n) + \bar{\bar{\pi}}(2n) = 2n$ *.*

Proof. Indeed, $\forall\, n \in \mathbb{N}^*$we have

$$[1\,,2n] \cap \mathbb{N}^* := \{1\} \cup \{[1\,,2n] \cap \mathbb{N}_{en}\} \cup \{[1\,,2n] \cap \mathbb{P}^*\} \cup \{[1\,,2n] \cap \mathbb{N}_{con}\}$$

where $\{1\} \cap \{[1\,,2n] \cap \mathbb{N}_{en}\} \cap \{[1\,,2n] \cap \mathbb{P}^*\} \cap \{[1\,,2n] \cap \mathbb{N}_{con}\} = \emptyset$

then, $\text{Card}[1\,,2n] \cap \mathbb{N}^* = \text{Card}\{1\} + \text{Card}[1\,,2n] \cap \mathbb{N}_{en} + \text{Card}[1\,,2n] \cap \mathbb{P}^*$

$$+ \text{Card}[1\,,2n] \cap \mathbb{N}_{con} = 2n$$

, then $1 + \bar{\bar{\pi}}(2n) + \pi(2n) - 1 + \bar{\pi}(2n) = 2n$

, thus $\pi(2n) + \bar{\pi}(2n) + \bar{\bar{\pi}}(2n) = 2n$.

For instance :

For $n = 1$ we have $\pi(2) + \bar{\pi}(2) + \bar{\bar{\pi}}(2) = 1 + 0 + 1 = 2 = 2.1$

For $n = 2$ we have $\pi(4) + \bar{\pi}(4) + \bar{\bar{\pi}}(4) = 2 + 0 + 2 = 4 = 2.2$

For $n = 3$ we have $\pi(6) + \bar{\pi}(6) + \bar{\bar{\pi}}(6) = 3 + 0 + 3 = 6 = 2.3$

For $n = 4$ we have $\pi(8) + \bar{\pi}(8) + \bar{\bar{\pi}}(8) = 4 + 0 + 4 = 8 = 2.4$

For $n = 5$ we have $\pi(10) + \bar{\pi}(10) + \bar{\bar{\pi}}(10) = 4 + 1 + 5 = 10 = 2.5$

For $n = 6$ we have $\pi(12) + \bar{\pi}(12) + \bar{\bar{\pi}}(12) = 5 + 1 + 6 = 12 = 2.6$

For $n = 7$ we have $\pi(14) + \bar{\pi}(14) + \bar{\bar{\pi}}(14) = 6 + 1 + 7 = 14 = 2.7$

For $n = 8$ we have $\pi(16) + \bar{\pi}(16) + \bar{\bar{\pi}}(16) = 6 + 2 + 8 = 16 = 2.8$

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Corollary(New identity). $\forall\, n \in \mathbb{N}^*$ *we have* $\pi(2n) + \bar{\pi}(2n) = n$.

Proof. $\forall\, n \in \mathbb{N}^*$ we have $\pi(2n) + \bar{\pi}(2n) + \bar{\bar{\pi}}(2n) = 2n$ and $\bar{\bar{\pi}}(2n) = n$, then $\pi(2n) + \bar{\pi}(2n) + n = 2n$, thus $\pi(2n) + \bar{\pi}(2n) = n$.

Remark. $\begin{cases} \bar{\pi}(2n) = 0 \text{ when } n \leq 4 \\ \bar{\pi}(2n) \geq 1 \text{ when } n > 4 \end{cases}$.

For instance :

For $n = 1$ we have $\pi(2) + \bar{\pi}(2) = 1 + 0 = 1$

For $n = 2$ we have $\pi(4) + \bar{\pi}(4) = 2 + 0 = 2$

For $n = 3$ we have $\pi(6) + \bar{\pi}(6) = 3 + 0 = 3$

For $n = 4$ we have $\pi(8) + \bar{\pi}(8) = 4 + 0 = 4$

For $n = 5$ we have $\pi(10) + \bar{\pi}(10) = 4 + 1 = 5$

For $n = 6$ we have $\pi(12) + \bar{\pi}(12) = 5 + 1 = 6$

For $n = 7$ we have $\pi(14) + \bar{\pi}(14) = 6 + 1 = 7$

For $n = 8$ we have $\pi(16) + \bar{\pi}(16) = 6 + 2 = 8$

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References

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