An equation that relates the energy density to the curvature of space-time in the ΛCDM model

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Abstracs

Starting from the Birkhoff-Jebsen theorem, some mathematical results on the spacetime curvature of the relativistic Schwarzschild cosmological model have been related to the space-time curvature and energy density in the ΛCDM model. With this result, an equation has been obtained that relates the Gaussian curvature of space-time with the energy density in the ΛCDM model. The equation found can facilitate the resolution of the Friedmann equation in some cases.

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Keywords

ΛCDM model, matter, energy density, curvature of space-time.

1.- Study of the curvature of space-time in the Schwarzschild model

1.1 Introduction

The first physical problem that concerns us is the calculation of the curvature of space-time caused by a point mass at a point located at a distance "r" from its center. This point will always be at a greater distance from the event horizon or Schwarzschild radius, “Rs”. Schwarzschild [1] solves the equations of the generalized relativity theory for an assumption of point mass and a surrounding empty space, establishing a metric and a space-time equation that turns out to be Flamm's paraboloid. This approach leads to a time-stationary space-time solution. The spherical symmetry of the problem geometrically simplifies its solution, resulting in a 2D surface, as represented in figure 1.

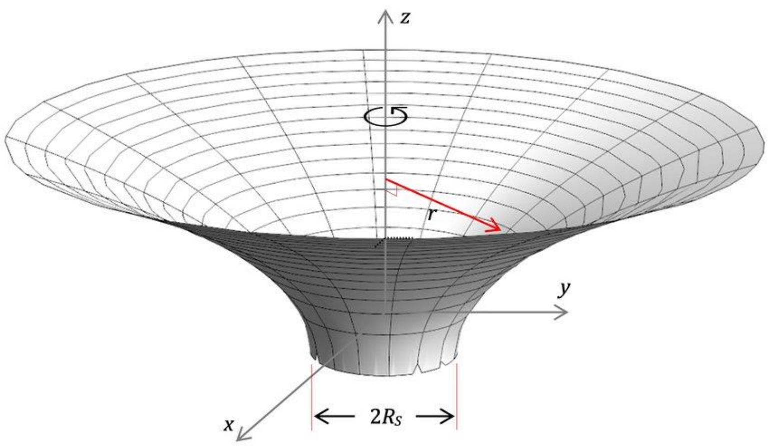


Fig 1. Space-time in the Schwarzschild model

1.2 Resolution of the mathematical problem

Flamm's paraboloid, mathematical solution to the proposed physical problem, is a surface inserted in a space R^3. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the center of the black hole "r" and the azimuth angle "φ". The problem admits a mathematical treatment of differential geometry of surfaces [2], and with it we are going to calculate values of the Gauss Curvature.

Surface parameters (r, φ)

$0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi$

parametric vector equation:

$x = r\cos\varphi$

$y = r\,\mathrm{sen}\varphi$

$z = 2(Rs(r - Rs))^{1/2}$

vector equation;

$f(x,y,z) = (r\cos\varphi,\ r\,\mathrm{sen}\varphi,\ 2(Rs(r - Rs))^{1/2})$

Determination of velocity, acceleration and normal vectors to the surface

$\eth f/\eth\varphi = (-r\,\mathrm{sen}\varphi, r\cos\varphi, 0)$

$\eth^2 f/\eth\varphi^2 = (-r\cos\varphi,\ -r\,\mathrm{sen}\varphi,\ 0)$

$\eth f/\eth r = (\cos\varphi, \mathrm{sen}\varphi, (r/Rs - 1)^{-1/2})$

$\eth^2 f/\eth r^2 = (0,\ 0,\ (-1/(2R_s)).\ (r/Rs - 1)^{-3/2})$

$$\eth f/\eth\varphi\eth r = (-\mathrm{sen}\varphi,\ \cos\varphi,\ 0)$$

$$n = (\eth f/\eth\varphi \times \eth f/\eth r) = (r\cos\varphi/(r/Rs - 1)^{1/2},\ r\,\mathrm{sen}\varphi/(r/Rs - 1)^{1/2},\ -r)$$

$$[n] = r\,((1/(r/Rs - 1)) + 1)^{1/2}$$

$$\mathbf{n} = n/[n]$$

Curvature and curvature parameters.

Gaussian curvature: $\mathbf{K = LN\text{-}M^2/EG\text{-}F^2}$

$L = \eth^2 f/\eth\varphi^2.\ \mathbf{n}$

$E = \eth f/\eth\varphi.\ \eth f/\eth\varphi$

$M = (\eth f/\eth\varphi\ \eth r).\ \mathbf{n}$

$F = \eth f/\eth\varphi.\ \eth f/\eth r$

$N = \eth^2 f/\eth r^2.\ \mathbf{n}$

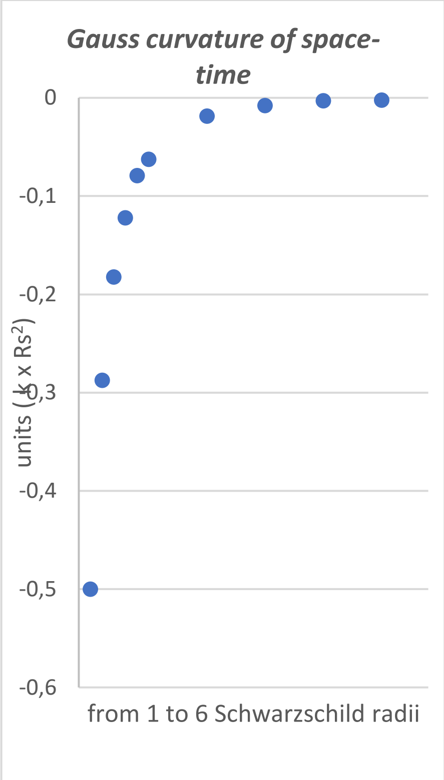
$G = \eth f/\eth r.\ \eth f/\eth r$

1.3 Results of curvature values

According to the above, we have calculated curvature values at 20 points located at a distance between 1 and 1400 Schwarzschild radii (Rs). [3].

Table 1. Curvature values according to the Schwarzschild model

Distance to the center of gravitational mass	Gaussian curvature value k
1Rs	**-0,5000 x Rs^{-2}**
1,2Rs	**-0,2873 x Rs^{-2}**
1,4Rs	**-0,1821 x Rs^{-2}**
1,6Rs	**-0,1220 x Rs^{-2}**
1,8 Rs	**-0,0790 x Rs^{-2}**
2Rs	**-0,0625 x Rs^{-2}**
3Rs	**-0,0186 x Rs^{-2}**
4Rs	**-0,0078 x Rs^{-2}**
5Rs	**-0,0030 x Rs^{-2}**
6Rs	**-0,0023 x Rs^{-2}**
60Rs	**-2,325.10^{-6} x Rs^{-2}**
80Rs	**-9,596.10^{-7} x Rs^{-2}**
100Rs	**-4,925.10^{-7} x Rs^{-2}**
200Rs	**-5,963.10^{-8} x Rs^{-2}**
400Rs	**-7.855.10^{-9} x Rs^{-2}**
600Rs	**-2,801.10^{-9} x Rs^{-2}**
800Rs	**-9,710.10^{-10} x Rs^{-2}**
1000Rs	**-5,059.10^{-10} x Rs^{-2}**
1200Rs	**-2,883.10^{-10}x Rs^{-2}**
1400Rs	**-1,810.10^{-10} x Rs^{-2}**



1.4 An equation to calculate the curvature of space-time according to the Schwarzschild model.

We are going to study the data of the Gaussian curvature by means of an equation obtained by fitting.

We have used an Excel program to determine a regression equation that turns out to be a potential function and we have used the 20 data obtained in the fit. The results is as follows:

FIT EQUATION

Set the equation between 1 and 1400 Schwarzschild radii.

Gaussian curvature $k = -0{,}5268\ (r/Rs)^{-3{,}054} \times Rs^{-2}$

Fit quality $R^2 = 0{,}9999$

Rounding decimals and considering that:

$$Rs = 2GM/c^2$$

we can express the fit equation for the Gaussian curvatures of space-time according to the Schwarzschild model as:

$$\mathbf{k = -GM/c^2r^3}$$

where k is the Gaussian curvature of space-time and r is the distance to mass M.

2.- An equation that relates the curvature of space-time and the energy density of space in the ΛCDM model

2.1 Introduction

The ΛCDM cosmological model [4] assumes of a continuous universe, with a constant energy density ρ that gives rise to a constant k space-time curvature. It is in this case where we are going to move to quantitatively relate that energy density and that curvature to be calculated.

We start from the formula found by us from the Schwarzschild model, [1], there we studied the curvature of space-time produced by a mass M at a distance "r" from it. According to the solution found by Schwarzschild to the Einstein equations and to the calculations made by us, we obtained a fitting equation that allowed us to calculate values of curvatures in the region between 1 Schwarzschild radius and 1400 Schwarzschild radii. The equation obtained was the following:

$$k = -GM/c^2r^3$$

where k is the Gaussian curvature of the Schwarzschild space-time.

Obviously according to this formula, the curvature will depend on the distance to the mass, however, we will see below that from this formula an equation is obtained that relates the energy density with the Gaussian curvature of space-time when the mass is uniformly distributed throughout the space, as is the case with the ΛCDM model. To relate both models the Schwarzschild. and the ΛCDM we are going to use the Birkhoff-Jebsen theorem.

2.2 Birkhoff–Jebsen theorem

In general relativity, Birkhoff's theorem [5] states that any spherically symmetric solution of the vacuum field equations must be statically and asymptotically flat. This means that the outer solution (that is, the spacetime outside a gravitational, non-rotating, spherical body) must be given by the Schwarzschild metric.

Following [6] we state this theorem as follows:

“If we have a spherical universe of mass-energy density ρ and radius r and within it a concentric sphere of radius r_s smaller than r, it is true that the acceleration due to gravity at any point on the surface of the sphere of relative radius r_s to an observer at its origin, depends solely on the mass-energy relation contained within this sphere”.

Thus, according to this, to calculate the curvature of the gravitational field of a point located at a distance "r_s" from the geometric center that we are considering in our continuous universe, it is only necessary to consider its interaction with the points that are at a radius smaller than "r_s", therefore, the mass "m" to be considered will only be that contained in the sphere of radius "r_s".

Obviously, the equation we are looking for must be consistent with this theorem and we will see that it is so.

2.3 An Equation relating Curvature and Energy Density

Let us consider applying our space-time curvature formula to a sphere of radius r of our supposed continuous universe of the ΛCDM model, the Birkhoff-Jebsen theorem assures us that if we want to calculate the space-time curvature on the surface of the sphere, you only have to take into account the interaction with the gravitational mass that is inside it. In addition, as the Birkhoff-Jebsen theorem assures us that the solution is given by the Schwarzschild metric, the curvature formula that we have obtained may be applicable in this case, taking into account that the interaction with the interior points of the sphere is, that is, the gravitational field on the surface of the sphere, by Gauss Theorem, is reduced to an interaction with a point mass of equal magnitude in the center of the sphere and in this case the equation for calculating curvatures is applicable of the Schwarzschild model that we have found.

Since the energy density “ρ” in the ΛCDM model is constant, it will be constant in every sphere that we are considering and thus we can write:

$$M = \rho.(4\pi r^3/3) \quad (1)$$

According to our curvature adjustment equation, we have:

$$K = -GM/c^2r^3 \qquad (2)$$

substituting (1) in (2) we get:

$$\mathbf{K = -4\pi G\rho/3c^2}$$

Equation found,

$$\mathbf{K/\rho = -4\pi G/3c^2 = -0{,}3104.10^{-26}\ m/Kg}$$

k is the Gaussian curvature (m^{-2}) and ρ_m is the energy density (Kg/m^{-3})

This formula relates the energy density of space to the curvature of space-time that generates it in the ΛCDM model.

Conclusions

We have come a long way to find our equation that relates the energy density to the curvature of spacetime in the ΛCDM model. Fist, we have studied the mathematical solution of space-time in the Schwarzschild relativistic model and with it we have calculated 20 space-time curvature values at different distances from the point gravitational mass that studies this assumption. These distances cover a diameter between 1 Schwarzschild radius and 1400 Schwarzschild radii. By studying the curvature values at these 20 points, we have fitted an equation that reproduces with a high degree of precision the spatiotemporal curvature values in the Schwarzschild model at these points and at intermediate points.

Then, using the Birkhoff-Jebsen theorem, we have related the Schwarzschild model to the ΛCDM cosmological model. Through this relationship we have been able to study in the ΛCDM model the energy density of space and the curvature of space-time, relating them by means of an equation. This equation is the object of our study, and we hope it will be very useful when it comes to finding solutions to the Friedmann equation in some cases, since it relates two of its variables, the curvature of space-time and the energy density that generates it.

References

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