Space-time as spinors

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Abstract

We uncover an implicit volume-preserving mapping from the $\mathbb{C}^4$ space of a bispinor onto the past cone belonging to an arbitrary space-time point in $\mathbb{R}^{3,1}$. The quotient group $SO(3,1)$ is shown to be given by $SO(8)/U(1) \times U(1) \times SO(3)$ and a simple geometrical interpretation is presented. We conclude by showing that the novel mapping allows the reformulation of many equations of motion of boson and fermion fields as integral equations over null cones that are devoid of field derivatives.

1 Introduction

A radical simplification of certain QFT calculations was reported in a number of groundbreaking papers [2, 3, 4, 5] published in the early years of this century. This led us to explore whether points, field derivatives, displacements and metrics could be eliminated from quantum field calculations entirely. Whilst the current introductory paper does not establish a *direct* link with the twistor programme, the family resemblence is undeniable in that it too is founded on the notion of spinorial spaces that become real physical quantities through bilinear forms.

2 Integration over the causal past cone

Integration over unbounded spacetime corresponds to the relativistically invariant delta in momentum space:

$$\iiiint_{-\infty}^{\infty} e^{ik_\nu x^\nu} d^4x = \delta^4(k_v)$$

..but what becomes of the RHS if the integration is restricted to the causal past relative to some arbitrary origin ? Since we have been unable to find the answer in the literature, we will perform an explicit calculation.

With no loss of generality, we choose a point in momentum space $k_\mu =$

$\{\omega, 0, 0, k\}$. Then

$$\iiiint_{x_\nu x^\nu \geq 0} e^{ik_\nu x^\nu} d^4x = \int_0^\infty dt \int_0^t e^{i(\omega t + ukr)} r^2 \; dr \; d\Omega$$

$$= \frac{2\pi}{ik} \int_0^\infty dt \int_0^t \left[e^{i(\omega t + kr)} - e^{i(\omega t - kr)} \right] r \; dr$$

$$= \frac{2\pi}{ik} \int_0^\infty \left[\frac{e^{i(\omega+k)t}}{k(\omega+k)} + \frac{e^{i(\omega-k)t}}{k(\omega-k)} - \frac{1}{k^2} \left[e^{i(\omega t + kr)} - e^{i(\omega t - kr)} \right]_0^t \right] dt$$

$$= \frac{2\pi}{k} \left[\frac{1}{k(\omega+k)^2} + \frac{1}{k(\omega-k)^2} - \frac{1}{k^2(\omega+k)} + \frac{1}{k^2(\omega-k)} \right] = \frac{4\pi}{(k_\nu k^\nu)^2} \quad (1)$$

This result is unsurprising - it had after all to be a Lorentz invariant function of homogeneous degree minus 4 - but interesting all the same, because it suggests that the past cone can be decomposed into a simple product of two independent null surfaces, each of which contributes one $\frac{1}{k^2}$ factor.

3 One cone

Spinors transform under one or other of the restricted Lorentz $SU(2)$ subgroups. As discussed in Section III of [6], null displacement vectors can be constructed from a spinor Λ according to the following prescription:

$$x_\nu = \Lambda^\dagger \sigma_\nu \Lambda \qquad \sigma_0 = \mathbf{I}_2$$

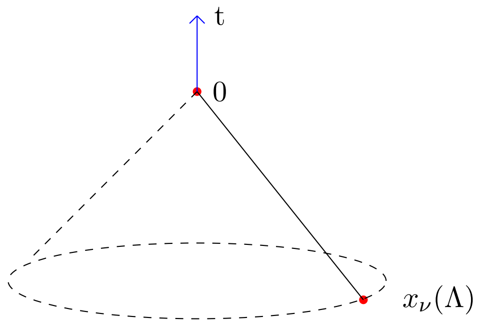
Parametrizing the spinor using radial coordinates

$$\Lambda = \begin{pmatrix} \lambda_\uparrow e^{i\phi_\uparrow} \\ \lambda_\downarrow e^{i\phi_\downarrow} \end{pmatrix} \qquad \Lambda^\dagger = \begin{pmatrix} \lambda_\uparrow e^{-i\phi_\uparrow} & \lambda_\downarrow e^{-i\phi_\downarrow} \end{pmatrix}$$

we obtain explicitly, in an arbitrary Lorentz frame:

$$\begin{cases} x_0 = \lambda_\uparrow^2 + \lambda_\downarrow^2 \\ x_1 = 2\lambda_\uparrow \lambda_\downarrow \cos\phi \\ x_2 = 2\lambda_\uparrow \lambda_\downarrow \sin\phi \\ x_3 = \lambda_\uparrow^2 - \lambda_\downarrow^2 \end{cases} \quad (2)$$

where we have used the clarifying substitution $\phi = \phi_\uparrow - \phi_\downarrow$. The overall phase $\Phi = \phi_\uparrow + \phi_\downarrow$ corresponds to a space-time degeneracy and is reflected in the nullity condition: $x_\nu \; x^\nu = \det \Lambda\Lambda^\dagger = 0$



The correspondence of an infinitesimal volume element in the linear $\mathbb{R}^4$ ($\mathbb{C}^2$) space of the spinor:

$$d\Lambda = \lambda_\uparrow \lambda_\downarrow \; d\lambda_\uparrow \; d\lambda_\downarrow \; d\phi \; d\Phi$$

to an infinitesimal surface element on the null cone: $dx_1 dx_2 dx_3|_{x_0=r}$ is given by the Jacobian:

$$\frac{\partial(x_1, x_2, x_3)}{\partial(\lambda_\uparrow, \lambda_\downarrow, \phi)} = 8 \begin{vmatrix} \lambda_\downarrow \cos\phi & \lambda_\uparrow \cos\phi & -\lambda_\uparrow \lambda_\downarrow \sin\phi \\ \lambda_\downarrow \sin\phi & \lambda_\uparrow \sin\phi & \lambda_\uparrow \lambda_\downarrow \cos\phi \\ 2\lambda_\uparrow & -2\lambda_\downarrow & 0 \end{vmatrix} = 8\lambda_\uparrow \lambda_\downarrow (\lambda_\uparrow^2 + \lambda_\downarrow^2)$$

So

$$d\Lambda = \lambda_\uparrow \lambda_\downarrow \, d\lambda_\uparrow \, d\lambda_\downarrow \, d\phi \, d\Phi = \lambda_\uparrow \lambda_\downarrow \frac{\partial(\lambda_\uparrow, \lambda_\downarrow, \phi)}{\partial(x_1, x_2, x_3)} dx_1 \, dx_2 \, dx_3 \, d\Phi = \frac{1}{8r} \, dx_1 \, dx_2 \, dx_3 \, d\Phi \tag{3}$$

Following trivial integration over the complex phase, Φ we obtain the well-known result in momentum representation space

$$\int e^{ik_\nu x^\nu} d\Lambda = 4\pi \int e^{ik_\nu x^\nu} r dr = \frac{1}{k^2} \tag{4}$$

which suggests that the past cone that yielded $\frac{4\pi}{k^4}$ in momentum space corresponds to the product of two spinorial spaces. We will now confirm this by an explicit real-space calculation.

4 Two cones generated by a bispinor

Consider a bispinor consisting of two null cone spinors transforming respectively as (1/2,0) and (0,1/2) representations of the SO$(3,1)$ Lorentz group.

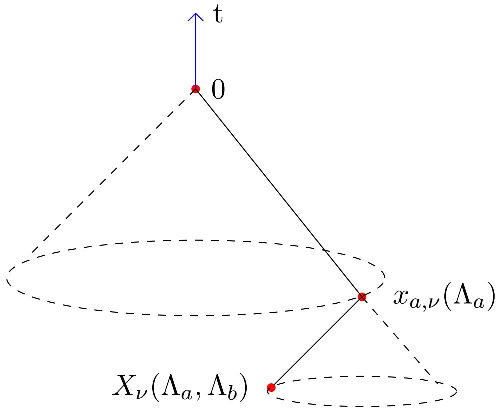
$$\Lambda := \begin{pmatrix} \Lambda_a \\ \Lambda_b \end{pmatrix} \qquad \overline{\Lambda} := \begin{pmatrix} \Lambda_b^\dagger & \Lambda_a^\dagger \end{pmatrix}$$

The total displacement vector is given by the sum of the two null displacements, or in gamma matrix notation:

$$X_\nu = \overline{\Lambda} \gamma_\nu \Lambda = \{t, \mathbf{r}\} \tag{5}$$

where

$$\begin{cases} t = r_a + r_b \\ \mathbf{r} = \mathbf{r}_a - \mathbf{r}_b \end{cases} \tag{6}$$



To investigate the mapping into space-time from the $\mathbb{R}^8$ ($\mathbb{C}^4$) space spanned by the bispinor, we note first that

$$X_\nu X^\nu = 2x_{a,\nu}x_b^\nu = 2(r_a r_b - \mathbf{r}_a \cdot \mathbf{r}_b) \geq 0$$

This means that every point in the $\mathbb{R}^8$ ($\mathbb{C}^4$) space of $\Lambda_a \times \Lambda_b$ maps onto a space-time point X_ν within the causal past relative to the origin.
According to (3), the 8-dimensional integral F over the $\mathbb{R}^8$ spinor space of an arbitrary real space function $f(x_\nu)$ can be expressed in terms of hybrid real and phase coordinates by

$$F = \int f(x_\nu) d\Lambda_a \, d\Lambda_b = \int \frac{f(x_\nu)}{64 r_a r_b} dx_{a,1}\, dx_{a,2}\, dx_{a,3}\, d\Phi_a\, dx_{b,1}\, dx_{b,2}\, dx_{b,3}\, d\Phi_b$$
$$= \int \frac{\pi^2 R f(x)}{32 r_a r_b} \frac{\partial R^2}{\partial t}_{|x,y,z} dx\, dy\, dz\, dt\, du\, d\phi_R = \int \frac{\pi^3 R f(x)}{16 r_a r_b} \frac{\partial R^2}{\partial t}_{|x,y,z} dx\, dy\, dz\, dt\, du \tag{7}$$

where $\mathbf{R} := \mathbf{r}_a + \mathbf{r}_b$ and we have chosen the polar axis in $\mathbf{R}$ to be along $\mathbf{r}$, with cosine $u := \mathbf{r} \cdot \mathbf{R}/rR$
ϕ_R is the azimuthal angle in the plane perpendicular to $\mathbf{r}$. It can be simply integrated out and replaced by a 2π factor, just like the spinor phases Φ_a, Φ_b.
We now set about obtaining expressions for R, r_a, r_b in terms of r, t, u.

$$2t = |\mathbf{R} + \mathbf{r}| + |\mathbf{R} - \mathbf{r}| \implies R^2 = \frac{t^4 - r^2 t^2}{t^2 - u^2 r^2} \tag{8}$$

$$\implies \frac{\partial R^2}{\partial t} = \frac{2t(t^4 - 2u^2 r^2 t^2 + u^2 r^4)}{(t^2 - u^2 r^2)^2} \tag{9}$$

$$4r_a r_b = \sqrt{(R^2+r^2)^2 - 4r^2R^2u^2} = \frac{\sqrt{t^8 - 2u^2r^4t^4 + u^4r^8 - 4u^2r^2(t^4 - r^2t^2)(t^2 - u^2r^2)}}{t^2 - u^2r^2}$$
$$= \frac{t^4 - 2u^2r^2t^2 + u^2r^4}{t^2 - u^2r^2} \tag{10}$$

So combining (8, 9, 10) we obtain:

$$\frac{R}{4r_a r_b}\frac{\partial R^2}{\partial t} = \frac{2t^2\sqrt{t^2 - r^2}}{(t^2 - u^2r^2)^{\frac{3}{2}}} = \frac{2\tau^2\sqrt{\tau^2 - 1}}{(\tau^2 - u^2)^{\frac{3}{2}}}$$

Integrating over u:

$$\int_{-1}^{1} \frac{R}{4r_a r_b}\frac{\partial R^2}{\partial t} du = 2\tau^2\sqrt{\tau^2 - 1^2}\int_{-1}^{1}\frac{du}{(\tau^2 - u^2)^{\frac{3}{2}}} = 4 \qquad (\tau = t/r)$$

where we have used indefinite integral No. 198 in the table published in [7] So finally (7) becomes:

$$F = \int f(x_\nu) d\Lambda_a d\Lambda_b = \pi^3 \int_{t\geq r} f(x_\nu) dx\, dy\, dz\, dt \tag{11}$$

This confirms that the mapping from the $\mathbb{R}^8$ space of the bispinor into Lorentz invariant volume elements of Minkowski spacetime relative to an arbitrary origin is indeed completely homogeneous. The 8 real dimensions of $\mathbb{C}^4$ are seen to map into the $\mathbb{R}^{3,1}$ spacetime, one phase angle for each spinor and an SO(3) describing the ellipsoid traced out by all possible junctions of the two null displacement vectors. In other words,

$$SO(8) \to SO(3,1) \times SO(3) \times U(1) \times U(1)$$

5 Applications

So far we have established how to get k^{-2n} integrals, however odd-power integrals k^{-2n-1} are also possible by virtue of the easily obtained identity

$$i\int x_\mu e^{ik_\nu x^\nu} d\Lambda = \frac{4\pi k_\mu}{k^4} \tag{12}$$

(4) and (12) taken together allow a great number of differential field equations to be recast as cone integrals in which field derivatives make no explicit appearance. For example, the free Dirac equation can be transformed into

$$i\int \not{x}\,\psi\, d\Lambda + m\iint \psi d\Lambda_a d\Lambda_b = 0 \tag{13}$$

This however turns out to be a vast subject that we will expound upon in subsequent papers.

Finally the complex Minkowski space $\mathcal{M}_C$ that is central to with twistor theory is also spanned by two cones according to the following modified form of (14)

$$X_\nu^C = \Lambda^\dagger \gamma_\nu \Lambda \tag{14}$$

where

References

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