Contraction of Ramanujan Formulas in the Letter to Hardy.

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0- Abstract:

In this paper we show an approach to the Ramanujan summation of series formulas, proving that it is possible a contracted version of them.

1- Introduction.

Srinivasa Ramanujan (1887-1920), the Hindu genius send to G. H. Hardy a letter in 1903 [1]. In this letter were a few discoveries and advanced (for that time) questions that he made by himself. In this paper we will focus on part "V: Theorems of summation of series". We will do a more modern contraction of the equations with a calculus approach. We will distinguish between "Ramanujan notation" and "Contracted notation".
I will use to contract negative parts of sequences my own operator (Subtractory), if you want to know more about negative-summation operator you can see [2].
As warning I will say that I do not test the veracity of any Ramanujan's equality so it can be wrong as we understand the mathematics in a numeric form today.

2- Section V contractions:

(1.1) Ramanujan notation

$$\frac{1}{1^3}\cdot\frac{1}{2^1}+\frac{1}{2^3}\cdot\frac{1}{2^2}+\frac{1}{3^3}\cdot\frac{1}{2^3}+\frac{1}{4^3}\cdot\frac{1}{2^4}+...=\frac{1}{6}(\log 2)^3-\frac{\pi^2}{12}\log 2+(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+...)$$

(1.2) Contracted notation

$$\sum_{\substack{n=1\\m=1}}^{\infty}\frac{1}{n^3\cdot 2^m}=\frac{1}{6}\log(2)^3-\frac{\pi}{12}\log(2)+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}$$

(2.1) Ramanujan notation

$$1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1\cdot 5}{4\cdot 8}\right)^4+25\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^4+...=\frac{2\sqrt{2}}{\sqrt{\pi}\{\Gamma(\frac{3}{4})\}}$$

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(2.2) Contracted notation

$$\sum_{n=1}^{\infty} 1+8n\left(\frac{\prod_{m=0}^{\infty} 1+4m}{\prod_{m=1}^{\infty} 4m}\right)^{4} = \frac{2\sqrt{2}}{\sqrt{\pi}\,\{\Gamma(\frac{3}{4})\}}$$

(3.1) Ramanujan notation

$$1-5\cdot\left(\frac{1}{2}\right)^{3}+9\cdot\left(\frac{1\cdot 3}{2\cdot 4}\right)^{3}-\ldots=\frac{2}{\pi}$$

(3.2) Contracted notation

$$1+2\left(\sum_{n=0}^{\infty} 8n\left(\frac{\prod_{m=1}^{\infty} 2m-1}{\prod_{m=1}^{\infty} 2m}\right)^{3}\right)+\left(\overset{\infty}{\underset{n=1}{\mathrm{P}}} 4n\left(\frac{\prod_{m=1}^{\infty} 2m-1}{\prod_{m=1}^{\infty} 2m}\right)^{3}\right)=\frac{2}{\pi}$$

(4.1) Ramanujan notation

$$\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\ldots=\frac{1}{24}$$

(4.2) Contracted notation

$$\sum_{n=1}^{\infty}\frac{n^{13}}{e^{(2n)\pi}-1}=\frac{1}{24}$$

(5.1) Ramanujan notation

$$\frac{\coth\pi}{1^{7}}+\frac{\coth 2\pi}{2^{7}}+\frac{\coth 3\pi}{3^{7}}+\ldots=\frac{19\pi^{7}}{56700}$$

(5.2) Contracted notation

$$\sum_{n=1}^{\infty}\frac{\coth n\pi}{n^{7}}=\frac{19\pi^{7}}{56700}$$

(6.1) Ramanujan notation

$$\frac{1}{1^{5}\cosh\frac{\pi}{2}}-\frac{1}{3^{5}\cosh\frac{3\pi}{2}}+\frac{1}{5^{5}\cosh\frac{5\pi}{2}}-\ldots=\frac{\pi^{5}}{768}$$

(6.2) Contracted notation

$$\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5\cosh\frac{(2n-1)\pi}{2}}\right)+2\left(\underset{n=1}{\overset{\infty}{\mathrm{P}}}\frac{1}{(4n-1)^5\cosh\frac{(4n-1)\pi}{2}}\right)=\frac{\pi^5}{768}$$

(7.1) Ramanujan notation

$$\left(\frac{1}{(1^2+2^2)(\sinh 3\pi-\sinh\pi)}\right)+\left(\frac{1}{(2^2+3^2)(\sinh 5\pi-\sinh\pi)}\right)+\left(\frac{1}{(3^2+4^2)(\sinh 7\pi-\sinh\pi)}\right)+\ldots=$$
$$=\frac{1}{2\sinh\pi}\left(\frac{1}{\pi}+\coth\pi-\frac{\pi}{2}\tanh^2\frac{\pi}{2}\right)$$

(7.2) Contracted notation

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+(n+1)^2)\cdot(\sinh(2n+1)\pi-\sinh\pi)}=\frac{1}{2\sinh\pi}\left(\frac{1}{\pi}+\coth\pi-\frac{\pi}{2}\tanh^2\frac{\pi}{2}\right)$$

(8.1) Ramanujan notation

$$\frac{1}{(25+\frac{1^4}{100})(e^{\pi}+1)}+\frac{3}{(25+\frac{3^4}{100})(e^{3\pi}+1)}+\frac{5}{(25+\frac{5^4}{100})(e^{5\pi}+1)}+\ldots=\frac{\pi}{8}\coth^2\frac{5\pi}{2}-\frac{4689}{11890}$$

(8.2) Contracted notation

$$\sum_{n=1}^{\infty}\frac{(2n-1)}{(25+\frac{(2n-1)^4}{100})(e^{(2n-1)\pi}+1)}=\frac{\pi}{8}\coth^2\frac{5\pi}{2}-\frac{4689}{11890}$$

(9.1) Ramanujan notation

$$\frac{1}{1^7\cosh\frac{1}{2}\pi\sqrt{3}}-\frac{1}{3^7\cosh\frac{3\pi}{2}\sqrt{3}}+\ldots=\frac{\pi^7}{23040}$$

(9.2) Contracted notation

$$\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^7\cosh\frac{(2n-1)\pi}{2}\sqrt{3}}\right)+2\left(\underset{n=1}{\overset{\infty}{\mathrm{P}}}\frac{1}{(4n-1)^7\cosh\frac{(4n-1)\pi}{2}\sqrt{3}}\right)=\frac{\pi^7}{23040}$$

(10.1) Ramanujan notation

$$\left\{1+\left(\frac{n}{1}\right)^3\right\}\left\{1+\left(\frac{n}{2}\right)^3\right\}\left\{1+\left(\frac{n}{3}\right)^3\right\}\ldots$$

Can always be exactly found if n is any integer positive or negative.

(10.2) Contracted notation

$$\prod_{m=1}^{\infty} \{ \ 1+\left(\frac{n}{m}\right)^{3} \ \}$$

Can always be exactly found if n is any integer positive or negative.

(11.1) Ramanujan notation

$$\frac{2}{3}\int_{0}^{1} \frac{\tan^{-1} x}{x} dx - \int_{0}^{2-\sqrt{3}} \frac{\tan^{-1} x}{x} dx = \frac{\pi}{12} \log 2 + \sqrt{3}$$

3- Conclusions.

As you can see almost every summation series from Ramanujan (except integral one) can be expressed as calculus contracted notation. I think, and this is just a comment, that nowadays we can do a more technical mathematics, with more precision in our calculus expressions.

4- References.

[1] Ramanujan, Srinivasa. S. Ramanujan to G. H. Hardy.
https://www.qedcat.com/misc/ramanujans_letter.jpg

[2] Millas Vera, Juan Elías. Resume of the serial operators theory.
https://vixra.org/pdf/2109.0029v1.pdf