Categorifying connections

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Abstract

The notion of a connection from differential geometry is employed in a category-theoretic context. We discuss the properties of holonomy from a tangent ∞-category perspective.

§0 Preamble

Let x and y be two distinct invariant objects. Then, there is a shortest path $\mathcal{P}$:x → y, which is a distance-minimizing curve (proper geodesic) between them. Cartan, Ehressman, Levi-Civita, et al. produced a particularly compelling industry for describing this sort of a path. The common terminology involves the notion of a "connection," i.e., a fibration between two elements or objects. However, they did not discuss these connections categorically. To give a proper picture of this story, we employ the notion of a *tangent ∞-category* of K. Bauer et al.

Let $\varphi(\mathbb{X},\mathbb{X})$:$\mathbb{X} \to \mathbb{X}$ be a fibration between quasicoherent schemes. We work in the category-enriched category **SchCat**. There is an intrinsic bundle (which is normal),

$$\boldsymbol{t}\mathbb{X} = \mathrm{H} \oplus \mathrm{V},$$

which is essentially affords an *Ehresmann connection* $x_i \to y_i$ to every object in $\mathbb{X}$. This gives us a smooth, projective morphism which allows us to do some bookkeeping apropos to any "twisting" that may occur over $\mathbb{X}$.

We will also borrow the notion of a "display tangent category" [Disp1] to license talk of our previously constructed notion of a hyperbolic block of display maps, $\mathbf{B}_\lambda$, which foliates open neighborhoods of some sort of geometrically or topologically closed structure. In a sense, the distinction between a tangent infinity category, and a projective restriction from a vector space to an ambient space of codimension 1 is not so clear. For instance, if we introduce a twist:

$$\boldsymbol{tw}: {}^{\mathrm{L}}\varepsilon \to \varepsilon^{\mathrm{L}\otimes \mathrm{R}}$$

and *lift* out of a desuspended point, then we can simply choose to model the lift as a morphism in a vect-enriched category.

§1 Cross Sections

Definition 1.1.0 A *connection* is a path $\mathcal{P}$:$\boldsymbol{\alpha} \to \boldsymbol{\beta}$ which admits a module A in the category **SchCat**.

Definition 1.1.1 A *cross-section* is an open topological neighborhood $\mathscr{U}(-)$ which covers a section of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Definition 1.1.2 A $\mathscr{C}$-module is an object $\mathscr{C}_{\mathrm{Mod}}$ which is "movable," i.e., exhibits holonomic transport along connections

One of the quintessential features of connections (categorically) is that they allow us to describe physical and dynamic motion along spaces. This is very geometrically interesting, but it should, in principle, be very categorically appetizing as well. We will use Grothendieck's notion of a homotopy type [Hot] to ease the discussion.

Let $\mathfrak{G}$ be a field. A connection (in our context) is essentially a map:

$$\mathfrak{G}_{\mathrm{HOT}} \to \mathbf{A},$$

where $\mathrm{A} \simeq \mathbf{TM}(\alpha)$ is a manifold tangent to some α. Quantum mechanically, this gives us a path interval $[\alpha, \mathbf{A}_{\mathrm{CENT}}]$

$$\begin{tikzcd} \mathfrak{G}_x \arrow[r, "\simeq"] \arrow[d, phantom, "\perp"] \arrow[dr, phantom, "\Downarrow An(x)"] & A \arrow[d, "Res"] \\ \mathfrak{G} \arrow[r, "\gamma"'] & A_{cent} \end{tikzcd}$$

Diagram I

An(x) takes the top of the diagram to the bottom by animating a small subcategory of $\mathfrak{G}^{\infty}$.

Let $\mathbf{C}$ be a chain of functors $\gamma^{n} \circ \gamma^{n-1} \circ \ldots \circ \gamma^{\inf(D)}$. Then, there is a *choice* of restriction $\mathcal{R}\boldsymbol{es}_{n}(\mathfrak{G})$ such that a countable cardinal is obtained. $\mathcal{R}\boldsymbol{es}_{n}(\mathfrak{G})$ is also called the codensity monad. The codensity monad obscures the twist by providing an ind-object for n to pull back to.[1]

As we see here, An(x) provides a pipe from isomorphism to a normal functor.

Definition 1.2.0 A *pipe* is an n-cell which maps to an (n-k)-cell, with (n-k)>0

Examples Say we have a 3-cell in some category, large or small. Then, if there is a map to a natural transformation, that is a pipe. A further map to an arrow would be a pipe. However, the evident map

$$(\mathbf{f}{:}a \to^{n} b) \to c$$

fails to be a pipe, because it reduces to an object, or in other words vanishes.

Our choice of n for this operation is syntactically crucial. If n is sufficiently large, it approximates a non-localized ∞-category Hot_{∞}. This makes the simplicial complex representing Hot_{∞} "locally smooth," but the smoothness is perhaps not global.

1 See [Cod]

Definition 1.3.0 A *locally smooth* map is a continuous ($\mathscr{C}^\infty$) map.

Lemma 1.4 A map $\mathscr{C}^\infty(A) \to^{n} \mathscr{C}^\infty(A)$ is locally smooth for $n > \aleph_0$.

Proof Let $\aleph_0$ be a supercompact cardinal. Then, there is a projection

$$\aleph_0 \rightrightarrows \lambda(n),$$

where $\lambda(n)$ is some proper function.

Definition 1.5.1 A *proper function* is a function of the form

$$f(x) = *_{*}*_{*}* \vee {}_{*}*_{*}*_{*}$$

where every asterisk is a "wildcard."

Definition 1.5.2 A *wildcard* is a formal power series, object, etc., which is chosen deterministically, but is initially non-determined.

§2 Determination Conditions

Definition 2.0.0 A *determination* is an action which connects a point to a choice.

Proposition 2.1.0 Let $X \to^{n} Y$ be a connection. Let n be comparable with m. Then, we have the classical relationship:

$$n\mathbf{R}m,$$

and we have

$$(X \to^{n} Y) \to (X \to^{m}) \simeq \mathrm{Fun}(\mathrm{Hom}_n(X,Y), \mathrm{Hom}(m,n)) = \mathcal{Z}$$

This proposition will be very important for us later. In the meantime, we will write

$$よ : \mathcal{Z} \to \mathcal{Z}^*$$

for a Yoneda automorphism from the ambient space to the total fiber space.

Remark Here, the relationship $n\mathbf{R}m$ is symmetric, and so it does not induce a preorder on the pro-object $z \in$ **SSets**.

Nothing is stopping us from writing

$$よ_n(\mathcal{Z}) = \lim_{\infty \to n} C^\infty(\mathcal{U})$$

under the condition that the localization functor

$$\mathfrak{n} : n \to n$$

is nilpotent. Notice also that the nilpotent character $\mathfrak{n}$ gives rise to the display block $\phi_{\mathfrak{n}}$.

Definition 2.2.1a Let $\mathscr{P}: x \to y$ be a simply connected path connection. Let it be nilpotent at every step. Then, it is a *locally pointed* connection.

Definition 2.2.1b Let every neighborhood on a manifold M contain locally pointed connections. THen M is said to be a locally pointed manifold. Accordingly, a space S with the same universal property is said to be a locally pointed space.

Proposition 2.2.2 Let Π be a path connected, locally pointed space. Then,

$$\int_0^1 \tau(Hom(x, y)) = f(\tau)$$

gives the complete set of information about a particle at a point τ in time. The よ-link gives the rank p<2 connection on a particle's worldsheet to a standard projection onto a worldline $\mathbb{S}$よ$=\mathfrak{W}$.

A cross-section on $\mathfrak{W}$ is a smash product of Regge trajectories of a particle and an antiparticle. Letting q be particle, and q^- be an antiparticle, we have

$$q \otimes q^- = \mathscr{C}(\mathbb{X})$$
$$q^- \otimes q = -\mathscr{C}(\mathbb{X})$$

Encoding the sub object identifiers as

$$\{q, q^-\} \to \{\alpha \to^n \beta\}$$

gives us a "unique" polymorphism to a span of locally pointed, locally ringed spaces. This polymorphism is co-degenerate with the mutually orthogonal slices of 4-dimensional classical spacetime. This gives rise to a rank n connection between each object, and a representative cocharacter at a designated point about some center on a stratified lightcone. This works, because $\mathbb{L}^4$ has projections from discs which are *covariant* with respect to the orientation of $\mathfrak{b}$, some binomial.

Borrowing again from Emmerson (with some original modification):

$$\prod_{x=0}^{x'} Obj(\mathbb{X}) \propto B_{a_{cent}} \simeq S_k \Rightarrow k = \int_x^{x'} Id_{\mathbb{X}}$$

In the above equation, k is basically a measure of "choiciness;" the entire determination of relata is completely determined by the projectivity of k at a certain slice of spacetime. Spontaneous breaking of symmetry occurs during the natural transformation; syntactically:

No choice → *Choice*;

Notice, there is no morphism

Choice → *No Choice*;

this would violate the arrow of time.

Definition 2.3.1 A stratum is said to be *uniquely determined* if there is only one such restriction $\mathfrak{U}|_S$ from the universal class which yields S.

Proposition 2.3.2 $\mathfrak{U}$ is uniquely determined

Proof Since $\mathfrak{U} \supset k$ for all $k \in \mathfrak{U}$, $\mathfrak{U}$ is the coarsest such space that these elements are comparable with. Thus, $\mathrm{Id}_{\mathfrak{U}}$ is the only operation taking some k to $\mathfrak{U}$.

Because each universe uniquely determines the sup-pole of a compass, a compass

$$\Omega_x^y$$

with $y=\sup(\mathfrak{U})$ has, as a result, a uniquely determined sup-pole.

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