Constant C makes the ABC conjecture hold

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The ABC conjecture in number theory was first proposed by Joseph Oesterl é and David Masser in 1985. Mathematicians declare this conjecture using three related positive integers a, b, and c (satisfying a+b=c). The conjecture states that if there are certain prime powers in the factors of a and b, then c is usually not divisible by the prime powers.

This paper utilizes the fact that the prime factor among all factors in the root number rad (c) can only be a power of 1. Then, analyze all combinations of c that satisfy rad (c)=c, calculate the value of the combination, and find the maximum and minimum values of the root number rad, as well as the maximum exponent between them. Using this maximum index, an equivalent inequality is constructed to prove the ABC conjecture.

keyword

Root/Prime Factor

Positive integers a, b, and c, satisfying the following conditions: $a+b=c$, and $(a,b)=1$ (a, b are mutually prime)

It is not difficult to find that when all factors in $rad(c)$ are prime numbers and the powers of prime numbers are all 1, then $rad(c)=c$

eg: $rad(165)=rad(3^1\cdot 5^1\cdot 11^1)=3\times 5\times 11=165$

Through the prime number theorem, we know that given a positive integer x, the number of prime numbers that do not exceed x is approximately: $\pi(x)\sim x/\ln(x)$

Now let's set the value range of the positive integer c to: $1<c\le x$

We set the number of prime numbers not exceeding x to be a positive integer h, so the value of h is: $h\sim[x/\ln(x)],\ \ h\in N^+$, We use the set $X=\{p_1,p_2\cdots p_h\}$ to represent the set of all prime numbers that do not exceed the integer x

Easy to detect: when $c=p_1^{\ 1}$ or $c=p_1^{\ 1}\cdot p_h^{\ 1}$ or $c=p_1^{\ 1}\cdot p_2^{\ 1}\cdot p_3^{\ 1}\cdots p_h^{\ 1}$, etc,The value of $rad(c)$ is exactly equal to c, that is:

$$c=rad(c)$$

We can calculate the maximum number of combinations in the set of prime numbers where $rad(c)=c$ is:

$$C_h^1 + C_h^2 + C_h^3 + \cdots + C_h^h$$

Because $(a,b)=1$, then $(a,b,c)=1$

Proof:

If a and c are not prime each other, there must be a common divisor k, and because b=c-a, then b and a must also have a common divisor k, which contradicts the prime of a and b, so a, b, and c are also prime each other

If the power of all prime factors in the radical $rad(c)$ is 1, then $c = rad(c)$, then:

$$rad(a \cdot b \cdot c) = rad(a \cdot b) \cdot rad(c)$$

Now let's return to $rad(a \cdot b \cdot c)$ for analysis：

We know that the minimum value of prime factors in $rad(a \cdot b \cdot c)$ is 2, and the minimum number of these prime factors is 1. Therefore, the minimum value of $rad(a \cdot b \cdot c)$ is: $rad(a \cdot b \cdot c)_{\min} = 2^1$

Similarly, when the power of the prime factor in $rad(a \cdot b \cdot c)$ is equal to 1 and its number is the integer $h \sim [x/\ln(x)]$, then the maximum value of $rad(a \cdot b \cdot c)$ is: $rad(a \cdot b \cdot c)_{\max} = \prod_{i=1}^{h} p_i = P$ （$p_i \in X, P \in N^+$）

So we can immediately launch:

$$2 \le rad(a \cdot b \cdot c) \le P \qquad (1)$$

Now let's set $(rad(a \cdot b \cdot c)_{\min})^m = rad(a \cdot b \cdot c)_{\max}, m \in R$, i.e. $2^m = P$, to find the maximum exponent between the minimum and maximum values. by taking the logarithm of both sides of the equation, we can obtain the value of m as:

$$m = \frac{\log p}{\log 2} \qquad (2)$$

Let's analyze the value of c:

We know that the value range of c is: $1 < c \le x$

We know that the set $X = \{p_1, p_2 \cdots p_h\}$ is a set of all prime numbers that does not exceed the integer x, so the construction of the value of the integer c must be: $c=\prod p_i^{\ n}$ $(p_i \in X, i \in N^+, n \in N^+)$, and $c \le x$

If x is an even number, then we can set $x = 2n, n \in N^+$

There must be an odd prime number $pr_1 = n-k, (k < n, k \in N^+)$ and an odd prime number $pr_2 = n+k, (k < n, k \in N^+)$. The relationship between them[2] is $2n = pr_1 + pr_2$, and 2, pr_1, $pr_2 \in X$

So the following two inequalities always hold:

① $P \ge 2 \cdot pr_1 \cdot pr_2$

② $2 \cdot pr_1 \cdot pr_2 - x = 2 \cdot (n^2 - k^2) - 2n = 2n^2 - 2n - k^2 = 2n(n-1) - k^2 > 0$

Immediately available: $c \le x \le P$

If x is an odd number, then we can set $x = 2n\text{-}1, n \in N^+$

Similarly, there must be an odd prime number $pr_1 = n-k, (k < n, k \in N^+)$ and an odd prime number $pr_2 = n+k, (k < n, k \in N^+)$. The relationship between them is $2n = pr_1 + pr_2$, and pr_1, $pr_2 \in X$

① $P \ge pr_1 \cdot pr_2$

② $pr_1 \cdot pr_2 - x = (n^2 - k^2) - (2n\text{-}1) = n^2 - 2n - k^2 + 1 = (n-1)^2 - k^2 \ge 0$

Similarly, immediately available: $c \le x \le P$

So whether x is odd or even, $c \le x \le P$

And because $P = \prod_{i=1}^{h} p_i = rad(a \cdot b \cdot c)_{\max} = 2^m$, we can immediately obtain:

$$c \le P = 2^m \qquad (3)$$

Because $2 \le rad(a \cdot b \cdot c) \le P$, then inequality (3) can be transformed as follows:

$$c \le 2^{m-1} \cdot 2^1 \le 2^{m-1}(rad(a \cdot b \cdot c))^1 < 2^{m-1}(rad(a \cdot b \cdot c))^{1+\varepsilon} \quad \forall \varepsilon > 0$$

$$\Rightarrow c < 2^{m-1}(rad(a \cdot b \cdot c))^{1+\varepsilon}$$

We set $C=2^{m-1}$, and now we have found the constant that always holds the inequality above, namely:

$$C=2^{m-1}$$

Conclusion:

In positive integers, there is equation $a+b=c$, and $(a,b)=1$, when $\forall\varepsilon>0$, $\exists C$ can make these triplets (ABC) satisfy the following inequality, namely:

$$c<C\cdot(rad(a\cdot b\cdot c))^{1+\varepsilon}$$

Example:

a=3,b=5,and c=8,rad(a)=3,rad(b)=5,rad(c)=2,rad(ab)=15,rad(abc)=30,so X={7,5,3,2}

So:

$rad(c)_{\min}=2$, $rad(c)_{\max}=P=7\times5\times3\times2=210$

so:

$$m=\frac{\log p}{\log 2}\approx 7.71$$

so:

$$C=2^{m-1}=2^{7.7143\text{-}1}\approx 105.00$$

The following inequality holds:

$$c=8<C\cdot(rad(a\cdot b\cdot c))^{1+\varepsilon}=105.00\times 2^{1+\varepsilon},\quad \forall\varepsilon>0$$

Conclusion: The ABC conjecture holds.

References

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