The Collatz Conjecture, While F(x) = 1 as X → ∞

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Abstract: The Collatz Conjecture is one of the most famous unsolved problems in mathematics, and the most basic unsolved one. Our contribution is to show that the problem can be split into two dimensions, each dimension can be represented by a single function that can then be shown to not form a loop, other than the established X = 1. The lack of other loops in the collatz conjecture is direct proof of the conjecture itself.

Proof:

The “Collatz Conjecture” is relatively simple.

$$F(n) = \begin{cases} \frac{n}{2} & if\ n\%2\ = 0 \\ \\ 3n+1 & if\ n\%2\ = 1 \end{cases}$$

Or rather: Given an odd number X; 3x + 1 = N, then N = N / 2 until N is odd, when N is odd, go back to step 1 where current N is now X.

More generally you can start with an even number, but all even numbers are to be divided until they are odd as in step two, as all odd numbers must have unique factors of 2 the even numbers end up being trivial. Moving on.

The Conjecture: That no end result N repeats when put back into X for a single linear sequence, except where N/X = 1. EG 3x + 1 where x is 1.

3 + 1 = 4
4 / 2 = 2
2 / 1 = 1
N = 1

See: “The 3x + 1 Problem and Its Generalizations” (Lagarias, 1985 1). Briefly this shows that having no loop in the Collatz Function other than 1 is equivalent to all series converging to 1. We’ll be relying on this for a complete proof but will not be repeating this part here to save on space.

We will now prove there is no loop, by inverting the formula and showing each collatz "sequence" must stretch back towards infinity by contradicting a loop.

First we'll take the standard inverse of the Collatz formula. Specifically, we're going to show a formula where we input an odd number N and 2 to the power Y, apply a bit more, and out will pop an X. That number X will be the X we'd put into 3x + 1 to get an even number, and then reduce it by power 2 to the power Y to get N we put in. The two are the inverse of each other.

First we rewrite the Collatz formula as:

(X3 + 1) / 2 ^ Y = N where X is odd and Y is an integer adjusted such that N is odd. We'll call this the "Forward" formula. This however looks a lot like a step you might take in a factoring algorithm, including Shor's algorithm, which is interesting to note and as we'll see, relevant.

Trivially we can see this is the same as the Collatz formula, we have just compacted it slightly. Moving on.

*((X3 + 1) / 2 ^ Y) * 2 ^ Y = N * 2 ^ Y*
*X3 + 1 = N * 2 Y*
*X3 + 1 - 1 = (N * 2 ^ Y) - 1*
*X3 = (N * 2 ^ Y) -1*
*(X3) /3 = ((N * 2 ^ Y) -1) /3*
*X = (N * 2 ^ Y) / 3 - ⅓*

We'll call this the "inverse" formula. As we have to invert the rules in a way we note that now for a given natural N, we have to adjust Y to give us a natural number. This adjustment to get a natural number is going to be key later. To confirm it works.

*7 = (11 * 2 ^ 1) / 3 - ⅓*

Now that we have the inverse formula, we need to split the right hand side, which we'll move to the left hand side. This is going to split the equation into two "dimensions" which we can solve for separately as we'll see below.

*(N * 2 ^ Y) / 3 - ⅓ = X*
*((2 ^ -Y) / (3 * N)) - ⅓ = X*

This splits us into two dimensions, one is the Y term, one is the N term, but it's still the collatz conjecture, as we'll demonstrate:

*((2 ^ -1) / (3 * 11)) - ⅓ = -7/22*

This gives us an answer wherein the numerator and denominator do not cancel out to an integer, instead our previous integer result is preserved as the numerator of a fraction. The purpose of this is to help visualize the pattern in Collatz by giving us a nice transformation of the equation into two dimensions, once we've done that the denominator will be eliminated and we will be back to our numerator integer only.

I.E. putting in the Y and N gives us not only the input collatz number X, but the intermediate (3x + 1) answer as well. All we're doing is putting out intermediate answer as the denominator, as we'll see this is going to help us find a pattern.

As we now have a convenient form that looks a lot like an irrational ratio such as the golden ratio. Or more closely an infinite series of irrational ratios such as the golden ratio: IE if we take a single value for Y then we'd get something even closer, 2 to a power of a root over a ratio, minus a single value.

Using this inverse equation, we find our "missing" pattern, actually our missing pattern is just that, a ratio, which is why we turned the exponent negative. However the irrational ratio gives us an entirely predictable pattern nonetheless, which we'll represent in the following matrix:

	N = 1	***N = 2***	***N = 3***	***N = 4***	***N = 5***	***N = 6***	***N = 7***	***N = 8***	***N = 9***
Y = 0	*-⅓*	*-⅙*	*-2/9*	*-¼*	*-4/15*	*-5/18*	*-2/7*	*-7/24*	*-8/27*
Y = 1	*-⅙*	*-¼*	*-5/18*	*-7/24*	*-3/10*	*-11/36*	*-13/42*	*-5/16*	*-17/54*
Y = 2	*-¼*	*-7/24*	*-11/36*	*-5/16*	*-19/60*	*-23/72*	*-9/28*	*-31/96*	*-35/108*
Y = 3	*-7/24*	*-5/16*	*-23/72*	*-31/96*	*-13/40*	*-47/144*	*-55/168*	*-21/64*	*-71/216*

Matrix 1

If we look close this gives us our "valid" inverse answers, only every other root Y gives us a valid inverse answer, and only every other third N from even Y, and third N from odd Y, gives us a valid answers. Or rather every sixth odd number. IE

$Y \in$ E gives us {1, 7, 13, 19, …}, called set NE
$Y \in$ O gives us {5, 11, 17, 23, 29…} called set NO

There's a third set that gives all odd numbers in inverse, and that's the one where no forward answer gives us this odd number. In inverse then this is where a Collatz sequence stops, or rather in forward where a Collatz sequence starts. IE

*(21 * (2 * 1) / 3 = 13 - ⅓ = 12⅔*
*(21 * (2 * 2) / 3 = 28- ⅓ = 27⅔*
{3, 9, 15, 21…} Called set NI

We note, NI is the result of 3x+1 not being able to give us this answer for a given valid input. Moving on, we see two directions we can go. One is iterating in N, the other in Y. For iteration in Y the formula is

$$(N, (a/b)), Y+2 = (N, ((a * 4) + 1/(b * 4))$$

For iteration in N we need separate equations for NO and NE. For N1 $\in$ NO, we take N-1 as (N, Y + 1) == (a/b) * 2 then (a + 1/b); we then divide a and b by 3. As such:

$$Y+1(N, (a/b)) = ((a * 2) + 1/(b*2)) / 3$$
$$\text{For } Y = 0, N = 5, a/b = N - 1 / N * 3 = 4/15$$
$$Y(4/15)+1 = ((4 * 2) + 1/(15 * 2))$$
$$(9/30)/3 = 3/10$$

N1 $\in$ NE starts the same however since A and B are divisible by 3 we do that first instead second, and we're moving Y+2 we go back to our previously discussed formula:

$$Y2\ (a/b) = (((N - 1) / 3) * 4) + 1 / N * 4)$$
$$\text{For } Y = 0, N = 7, a/b = (6/3) = 2 / 7$$
$$\text{For } Y = 2 \text{ then}$$
$$(2 * 4) + 1 / 7 * 4 = 9/28$$

For convenience sake we're not going to care about the denominator anymore, as it no longer has an effect on our equation. Remember the denominator is just the intermediate answer, as in:

$$(9 * 3) + 1 = 28$$

However we only care about the numerator here, the input, the output, which is represented as the denominator, will be divided by 2 into another input anyway. However our equation determines the next collatz sequence in/output as it is, making this step unnecessary.

Equation would look like

$$N(a/b) =$$
$$N \in NO\ (a) = (Y1, (((N - 1) * 2) + 1) / 3),(Y3,...)$$
$$N \in NE\ (a) = (Y2, (((N - 1) / 3) * 4) + 1), (Y4,...)$$

$$\text{Where}(,\dots) \text{ is}$$
$$(N, a/b), Y+2 = (N, (a * 4) + 1)$$

To review: We separated the inverse equation out into two dimensions, forming a matrix of answers, by flipping a single sign. This gave us a nice way to accomplish the previous while putting the same valid answer we are looking for in a numerator over a denominator, with the numerator having a flipped sign itself, but as the equation does not go past the 0 line anyway this is inconsequential. Then we eliminated the denominator from the equation as we no longer needed it, while keeping the exact same numerator, which is the answer we are after; briefly we flipped the sign back as well.

Next we're going to finish our proof showing that N can't form a loop. N we'll see is now a well behaved, entirely predictable function. After that we're going to show no loop can form in Y from a simple contradiction. We're going to start our N with inverting the equation again to put us back into the "forward" or standard 3x + 1 Collatz function.

To begin with we'll clean up our formula yet again. In short, we're going to eliminate even numbers as this isn't relevant to us. To do so all we need to do is take our Z+ set and set it to O (odd natural positive integers) as such

$$O = Z+ + (Z+ - 1)$$
$$\text{As such}$$
$$O\{1\} = 1 + 0$$
$$O\{2\} = 2 + 1$$
$$O\{3\} = 3 + 2$$
$$etc.$$

Briefly we'll show our new equations for this new set become more compact, allowing us to make the final proof cleaner and more understandable. First we'll compact NO and NE to

$$(((N - 1) * 2) + 1) / 3)$$
$$((2N - 2) + 1) / 3$$
$$1/3\ (2N - 1)$$
$$\text{Set } NO = N \equiv 0 \pmod 3$$

$$(((N - 1) / 3) * 4) + 1\ )$$
$$((1/3N - \tfrac{1}{3}) * 4)$$
$$(4/3N - 4/3) + 1$$
$$\tfrac{1}{3}\ (4N - 1)$$
$$\text{Set } NE = N \equiv 1 \pmod 3$$

Once we do this it follows that the equations for iterating in N, which we sum down to one function that includes the new variable Q, where Q

$$Q = \lfloor \tfrac{1}{3}(N) \rfloor$$

Which leads to

$$F(n) = \begin{cases} n - q \text{ if } n \equiv 0 \pmod 3 \\ n + q \text{ if } n \equiv 1 \pmod 3 \\ Stop \text{ if } n \equiv 2 \pmod 3 \end{cases}$$

Now, an observation:

$$F(n) = n +/- q \text{ if } n \equiv 1 \text{ or } 0 \pmod 3 \text{ where } q = \tfrac{1}{3}n$$

In order to loop back the sequence must change sign, from negative to positive or positive to negative

$$F(n1) = n1 - q = n2$$
$$n2 \equiv 1 \pmod 3$$
$$F(n2) = n + q (\lfloor(n2 / 3)\rfloor) = n3$$
$$n3 \,\Delta\, n1 = ((2/3n1) * \lfloor 2/3 \rfloor) + (2/3n1)$$

A different way to observe this is to track how much we add or subtract each time, from the perspective of continuing F(n) to F(n). We'll be doing so to disentangle the equation down to a simple "racetrack" of values that correspond to "Q{} =-1, +1, -2…" where the the sign corresponds to the direction in N we travel, and the integer how far in N we travel.

This is just to say, Q-1 means we take N - 1, Q+4 means we go to N +4. To begin:

$$N/3 = Q; \text{ sign} = - \text{ if } \equiv 0 \pmod 3; \text{ sign} = + \text{ if } \equiv 1 \pmod 3; \text{ I/stop if } \equiv 2 \pmod 3$$

Annoyingly this means we'll have to expand/contract Q to N each and every time, so we're going to use a tiny hack and simply shift Q relative to Q as such:

$$Q/3 \equiv 0 \pmod 3 \text{ then } Q = (s)Q + (Q/3)$$
$$\text{If } Q/3 \equiv 1 \pmod 3, \text{ I/Stop}$$
$$\text{If } Q/3 \equiv 2 \pmod 3\ Z = (-s)Q + (+Q/3 + 1/3)$$
$$\text{Where } (s) \text{ is sign}$$

Applied as such.

$$16/3 = 5 \text{ R } 1,\ Q = +5$$
$$5/3 = 1 \text{ R } 2 = (1 + 1, \text{ the second } +1 \text{ is from R2 rounded up})$$
$$5 + (1 + 1) = (-)7$$
$$Q = -7$$
$$7/3 = 2 \text{ R } 1, \text{ I/Stop}$$

Let's see how this relates to the original Collatz

$$N = O\{16\} = 31$$
$$X = (N * 2 \wedge Y) / 3 - \tfrac{1}{3}$$
$$(31 * 2 \wedge 2) / 3 - \tfrac{1}{3} = 41$$
$$41 = O\{21\}$$

$$21/3 = 7$$
$$7 \equiv 0 \pmod 3) \text{ sign} = -$$
$$Q = -7$$
$$\ldots$$

Notice, we have entirely eliminated the Y portion of the equation in the top compared to the bottom, there's no need to check which answer works, the top is now a well behaved function which we are going to prove can't loop. We have also compacted the equation a lot, we didn't even get to the next step showing we reach I/stop, and yet we took less steps to get to our answer.

And one more:

$$15/3 = 5 \text{ R } 0,\ Q = -5$$
$$5/3 = 1 \text{ R } 2 = (1 + 1)$$
$$-5 + (1 + 1) = 3$$
$$Q = 3$$
$$3/3 = 1 \text{ R } 0,$$
$$3 + 1 = 4$$
$$Q = 4$$
$$4/3 = 1 \text{ R } 1,\ I/Stop$$

Now a statement: Changing sign 2 ^ (i) times changes the ratio of the current Q such that no Q may loop back on itself. That is to say, in this framework, we need to go from positive to negative and then back to positive; or from negative to positive than then back to negative in order for a loop to occur.

Now we're going to flip, again, the equation into inverse. Each equation represents a step we can do in inverse, which is to say it represents normal 3x + 1 "forward" direction of collatz, but only our subset that works as a function in reverse. Here Qi is equal to (sign)Q and the operation we'd do to get to the previous Q depending on it's sign.

$$Qi = |(-)Q| + |(-)(Q)/2| = Q^*(3/2)$$
$$Qi = |(-)Q| - |(+)(Q/4) - 1/4)| = (Q * (3/4) - 1/4)$$
$$Qi = (+)Q - (+)Q/4 = Q * (3/4)$$
$$Qi = (+)Q + |((s-)Q/2) + 1/2)| = (Q * (3/2) + 1/2\ )$$

By writing out the inverse and the possibilities we've gotten something useful. We see exactly two possibilities for flipping the sign twice, in between which any number of steps of a ratio Q(3/2) or Q(¾) can occur. We're going to use this. One thing to note, we no longer care about I/Stop, as our proof does not depend on this at all, either for this first part or the second.

First off, a possibility that does result in flipping the sign twice, but in which no loop can occur:

$$Q1 = Q * (3/2) + 1/2$$
$$(Q((3/2) + 1/2) * (3/4)) -1/4 = Q2(9/8) + 1/8$$
$$\text{thus}$$
$$Q2 > Q1$$

Or rather, this only iterates positive. A loop must iterate in both directions, because if there is a loop, eventually Qi = Q1.

This is to say, Qi is just operations of our function after Q1, which can be set arbitrarily. In order for a loop to occur in our function, for a given Q1 (function) Qi we must have Qi = Q1, which we are showing can't happen. In this scenario that's because Qi > Q1, therefore there can't be a loop. With this established we move on.

Next please note, the remainder, non Q part of the equation, adds up to an integer that could be Qi, "the next" Q in the sequence. As such any operation such as Q * (¾) distributes the ratio (ra/ra) to both Q and the remainder (re/re) equally. I.E.

$$(Q(3/2) + 1/2) * (3/4) = Q(9/8) + 3/8$$

We note this to make sense of the fact that for our proof, we're not going to care about the order of operations. Instead our proof rests on the fact that the ratio, Q(ra/ra) and remainder (re/re) can't add up to a Q1 no matter the number of operations we perform. That is to say, we can pick any point to be Q1, apply the allowed operations, and no matter the order we can't get Q1 back. First we'll list the three operations we can use:

$$((Q * (3/2)) + 1/2) * ((Q * (3/4)) -1/4)$$
$$Q * 3/4$$
$$Q * 3/2$$

The first operation is paired, as noted because we don't care about the order of operations, just that some ratio Q(ra/ra) + (re/re) and because, as noted, in order to form a loop we must go "backwards/negative" then "forwards/positive" from a Q1 (or vice versa). Thus, each operation ((Q * (3/2)) + ½) must be paired with an operation ((Q * (¾)) -¼) or vice/versa.

The first thing we'll establish is that we can't get a loop from our first remainder. Re = ⅛. We're going to set Q(equation) = Q to establish the loop. As in, to get a loop, Q must = Q(ra) + (re).

$$(3/4)(3/2)(Q(9/8) + (1/8)) = Q$$
$$Qi(9 * ((3 \wedge i))/De = 8(2 \wedge I)) + re(1 * ((3 \wedge i)/De = 8(2 \wedge I)) = Q$$
$$((re = 3 \wedge i)/De) - ((De) - Qi) = Q$$
$$Q = re/((De) - Qi)$$
$$De = \wedge 2,\ Qi = \wedge 3,\ re = \wedge 3$$
$$De - Qi \neq \wedge 3$$

$$(\hat{}\ 3 / \neq \hat{}\ 3) \neq N$$

So what are variables I and i? “I” is just the fact that we are multiplying these by ¾ or 3/2, and I is some arbitrary N(atural numbers) that gets us there. As the numerator is always 3 * 3, or 9 *3, it must be some i ^ 3. As the denominator is always 8 and always * 2, it must be some I ^ 2.

What we show is prime factorization ends us with a fraction instead of a whole number. That is to say, because Re ends up being a numerator with prime factor 3 and only 3, as it must be 3 to a power, over a denominator that is not prime factor 3, as subtracting a number i with prime factor 3 from i with prime factor 2 cannot produce a number with prime factor 3, we get a fraction that does not equal a natural number. But to be valid we need Q = N, a natural number (integer). So for our “first” Re we can’t get a valid loop.

But what if we increase Re?

Now we note two things. The first is that repeatedly applying the first operation to itself increases (re), but also comes out to a specific ratio:

$$((Q(9/8) + 1/8) \ast 3/2) + \tfrac{1}{2}) = Q(27/16) + 11/16$$
$$((Q(27/16) + 11/16) \ast (\tfrac{3}{4})) - \tfrac{1}{4}) = Q(81/64 = 9/8) + 17/64$$
$$+217/512\ldots \text{ (etc.)}$$

Or rather, cleaned up, and then put into set notation of O (odd) and E(even). Which is about to become relevant.

$$((re) \ast (9/8)) + \tfrac{1}{8}$$
$$Re = (O \ast 9 = O) + ((1 \ast E) = E) = O$$
$$Re = re(N)/re(D)$$
$$re(D) = 2\ \hat{}\ I$$

Quick note, re(N) = remainder Numerator, re(D) = remainder Denominator. re(D) will now be 2 ^ I, while ratio denominator will be i ^ 2, as the remainder will have a larger denominator than ratio.

Moving on, Re > Re1 (⅛) must always be odd, as we always take an odd number, times it by nine, which is odd, then add an even number; the even number comes from needing to match denominators between re and 1/8, which is (re)2 ^ I > (⅛)2 ^ i, thus 1 must be multiplied by (2 ^ i), thus being even; even + odd = odd.

$$(Q = (9/8) \ast (3/2)\ldots(\tfrac{3}{4})\ldots) + (re = (re) \ast (9/8 + \tfrac{1}{8}) \ast (3/2)\ldots(\tfrac{3}{4})\ldots)) = Q$$
$$Q(9 \ast (3\ \hat{}\ i))/(8 \ast (2\ \hat{}\ i)) + re(re \ast (3\ \hat{}\ i))/(2\ \hat{}\ I) = Q$$
$$\text{Shortened: } Qi/(2\ \hat{}\ i) + re/(2\ \hat{}\ I) = Q$$
$$re/(2\ \hat{}\ I) - Q((2\ \hat{}\ i) - Qi = O)/(2\ \hat{}\ i) = Q$$
$$re(O)/2\ \hat{}\ I(E) - Q(O)/2\ \hat{}\ i = Q$$

$$I > i$$
$$re/I \wedge 2 - ((Q/2 \wedge i) * (2 \text{ until } i = I) = QI(E)/2 \wedge I) = Q$$
$$Re(O)/I \wedge 2 - QI(E)/2 \wedge I = Q$$
$$Q = Re(O)/QI(E)$$
$$Q \neq N$$

In short, we showed that given the operations available to us, if Q is taken as a loop (where re > ⅛) then it must equal a ratio where an (O)dd numerator is over an (E)ven denominator. Note that the denominator must be even because, as above, the (O)dd Q(ra) must be multiplied by an even number such that its denominator matches (reD). Trivially, an odd numerator over an even denominator does not equal N(atural numbers). Remember we already covered Q = 0, the existing loop, earlier. However for Q to be valid it must be an integer. Thus no loop can exist for iterating in N.

Now onto the second part, which follows from the first. Note we are still in our new set O. The other way to map each number is with our variable Y, the other axis on our matrix. Please refer back to the original equation F(N), in terms of iterating in Y. For reference:

$$(F(n) * 4) - 1 = Y1, (Y1 * 4) - 1,...$$

This is equivalent to iterating Y+2 into Z+

$$(O * 4) - 1 = (Z+ * 4) + 1$$
$$((2 * 4) - 1 = O(7) = (Z+)13) = (3 * 4) + 1 = 13$$

As well this does not equal F(n), as that must be +/- X/3. Remember, one step in the inverse equation is one application of the Collatz function, 3x + 1 then /2 repeated.

Thus we get to our proof. In order to loop back on itself this way, F(n)... = N, that is to say the two N's are in the same inverse collatz sequence, would have to

$$F(F(N1) * 4) - 1)... = (F(N2) * 4) - 1)$$

That is to say, we would have to take two numbers that equal each other in F(n), and put them both into (F(n) * 4) - 1, and then have them both equal each other in F(n) again. However that's a contradiction. As observed:

$$(F(n) * 4 - 1) = F(E)$$
$$F(n1)... = F(n2)$$
$$-F(E\ n2) = F(n2)$$
$$-F(n2)... = (Fn1)$$
$$\text{Thus}$$
$$-F(n(F(E\ n2)... \neq (F(E\ n1)$$

This is all to say, running our equation backwards can produce multiple results. But “forward” it’s still the Collatz function, still a function that has 1 input and 1 results. Thus for a loop to form, we’d need F(n)1 to = F(n)2 in the normal reverse function. Then we’d need to take both backwards in this second function, to F(E)1 and F(E)2. Then we would need to run the normal reverse function again until F(E)1 = F(E)2. But that can’t happen, we already know that in the “forward” collatz function these two inputs F(E)1 = F(n)1 and F(E)2 = F(n)2. Thus they can’t equal each other here, they must go “forward” and equal each other further on, meaning no loop can form here.

Importantly this goes backwards towards infinity. No matter how many times we connect this way, they have to map forwards to F(n), and since we established the equivalent of F(n) not having any loops except the already established (1) loop, we have proven there are no other loops.

Bibliography

1. The 3x + 1 Problem and Its Generalizations, Jeffery C. Lagarias, 1985