The Collatz Conjecture, While F(x) = 1 as X → ∞

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Abstract: We show the "Collatz Conjecture" is true by showing no loop can form. We do this by transforming the function into two parts. One part shows that to form a loop any output would have to be a fraction, while any valid input an integer, however as a loop the output and input must be the same, thus we show a contradiction. The second part simply shows that to form a loop the collatz conjecture would need to not be a function, and instead need to map to more than 1 output for 1 input, another contradiction.

Proof:

The "Collatz Conjecture" is relatively simple.

$$F(n) = \begin{cases} \frac{n}{2} & if\ n\%2\ = 0 \\ 3n+1 & if\ n\%2\ = 1 \end{cases}$$

Or rather: Given an odd number X; 3x + 1 = N, then N = N / 2 until N is odd, when N is odd, go back to step 1 where current N is now X.

More generally you can start with an even number, but all even numbers are to be divided until they are odd as in step two, as all odd numbers must have unique factors of 2 the even numbers end up being trivial. Moving on.

The Conjecture: That no end result N repeats when put back into X for a single linear sequence, except where N/X = 1.
3x + 1 where x is 1.
3 + 1 = 4
4 / 2 = 2
2 / 1 = 1
N = 1
See: An Analysis of the Collatz Conjecture (1). Briefly this shows that having no loop in the Collatz Function other than 1 is equivalent to all series converging to 1.

We will now prove this is true, by inverting the formula and showing each collatz "sequence" must stretch back towards infinity

First we'll take the inverse of the Collatz formula. Specifically, we're going to show a formula where we input an odd number N and 2 to the power Y, apply a bit more, and out will pop an X.

That number X will be the X we'd put into 3x + 1 to get an even number, and then reduce it by power 2 to the power Y to get N we put in. The two are the inverse of each other.

First we rewrite the Collatz formula as:

(X3 + 1) / 2 ^ Y = N where X is odd and Y is an integer adjusted such that N is odd. We'll call this the "Forward" formula. This however looks a lot like a step you might take in a factoring algorithm, including Shor's algorithm, which is interesting to note.

Trivially we can see this is the same as the Collatz formula, we have just compacted it slightly. Moving on.

$$((X3 + 1) / 2 ^ Y) * 2 ^ Y = N * 2 ^ Y$$
$$X3 + 1 = N * 2 Y$$
$$X3 + 1 - 1 = (N * 2 ^ Y) - 1$$
$$X3 = (N * 2 ^ Y) -1$$
$$(X3) /3 = ((N * 2 ^ Y) -1) /3$$
$$X = (N * 2 ^ Y) / 3 - ⅓$$

We'll call this the "inverse" formula. As we have to invert the rules in a way we note that now for a given natural N, we have to adjust Y to give us a natural number. To confirm it works.

$$7 = (11 * 2 ^ 1) / 3 - ⅓$$

Now that we have the inverse formula, in order to get our ratio all we have to do is invert part of the equation, this is a bit odd but will be explained.

$$(N * 2 ^ Y) / 3 - ⅓$$
$$((2 ^ -Y) / (3 * N)) - ⅓$$

Wait, what does this get us?

$$((2 ^ -1) / (3 * 11)) - ⅓ = -7/22$$

So why are we doing this seemingly arbitrary operation? One, we are going to undo this. Two, this gives us an interesting answer wherein the numerator and denominator do not cancel out to an integer, instead our previous integer result is preserved as the numerator of a fraction. The purpose of this is to help visualize the pattern in Collatz, once we've done that the denominator will be eliminated and we will be back to our numerator integer only.

Finally, you'll notice that the entire operation is validly related to Collatz, as

$$(7 * 3) = 21 + 1 = 22$$
$$22/2 = 11$$

I.E. putting in the Y and N gives us not only the input collatz number X, but the intermediate (3x + 1) answer as well.

As we now have a convenient form that looks a lot like an irrational ratio such as the golden ratio. Or more closely an infinite series of irrational ratios such as the golden ratio: IE if we take a single value for Y then we'd get something even closer, 2 to a power of a root over a ratio, minus a single value.

Using this inverse equation, we find our “missing” pattern, actually our missing pattern is just that, a ratio, which is why we turned the exponent negative. However the irrational ratio gives us an entirely predictable pattern nonetheless.

Y = 0;
N = (1, -⅓);(2, -⅙);(3, -2/9);(4, -¼);(5, -4/15);(6, -5/18);(7, -2/7);(8, -7/24);(9, -8/27)
Y = 1
N = (1, -⅙);(2, -¼);(3, -5/18);(4, -7/24;);(5, -3/10);(6, -11/36);(7, -13/42);(8, -5/16);(9, -17/54)
Y = 2
N = (1, -¼);(2, -7/24);(3, -11/36);(4, -5/16);(5, -19/60);(6, -23/72);(7, -9/28)
Y = 3
N = (1, -7/24);(2, -5/16);(3, -23/72);(4, -31/96);(5, 13/40);(6, 47/144)

This previous pattern gives us our “valid” inverse answers, only every other root Y gives us a valid inverse answer, and only every other third N from even Y, and third N from odd Y, gives us a valid answers. Or rather every sixth odd number. IE

Y ∈ E gives us {1, 7, 13, 19, …}, called set NE
Y ∈ O gives us {5, 11, 17, 23, 29…} called set NO

There's a third set that gives all odd numbers in inverse, and that's the one where no forward answer gives us this odd number. In inverse then this is where a Collatz sequence stops, or rather in forward where a Collatz sequence starts. IE

*(21 * (2 * 1) / 3 = 13 - ⅓ = 12⅔*
*(21 * (2 * 2) / 3 = 28- ⅓ = 27⅔*
{3, 9, 15, 21…} Called set NI

Moving on, we see two directions we can go. One is iterating in N, the other in Y. For iteration in Y the formula is

*(N, (a/b)), Y+2 = (N, ((a * 4) + 1/(b * 4))*

For iteration in N we need separate equations for NO and NE. For N1 ∈ NO, we take N-1 as (N, Y + 1) == (a/b) * 2 then (a + 1/b); we then divide a and b by 3. As such:

$$Y+1(N, (a/b)) = ((a * 2) + 1/(b*2)) / 3$$
$$\text{For } Y = 0,\ N = 5,\ a/b = N - 1 / N * 3 = 4/15$$
$$Y(4/15)+1 = ((4 * 2) + 1/(15 * 2))$$
$$(9/30)/3 = 3/10$$

N1 $\in$ NE starts the same however since A and B are divisible by 3 we do that first instead second, and we're moving Y+2 we go back to our previously discussed formula:

$$Y2\ (a/b) = (((N - 1) / 3) * 4) + 1 / N * 4)$$
$$\text{For } Y = 0,\ N = 7,\ a/b = (6/3) = 2 / 7$$
$$\text{For } Y = 2 \text{ then}$$
$$(2 * 4) + 1 / 7 *4 = 9/28$$

For convenience sake we're not going to care about the denominator anymore, as promised. Equation would look like

$$N(a/b) =$$
$$N \in NO\ (a) = (Y1, (((N - 1) * 2) + 1) / 3),(Y3,...)$$
$$N \in NE\ (a) = (Y2, (((N - 1) / 3) * 4) + 1), (Y4,...)$$

$$\text{Where}(,\ldots) \text{ is}$$
$$(N, a/b),\ Y+2 = (N, (a * 4) + 1)$$

While we can find any amount of patterns here, we're just going to invert.

To begin with we'll clean up our formula yet again. In short, we're going to eliminate even numbers as this isn't relevant to us. To do so all we need to do is take our Z+ set and set it to O (odd natural positive integers) as such

$$O = Z+ + (Z+ - 1)$$
$$\text{As such}$$
$$O\{1\} = 1 + 0$$
$$O\{2\} = 2 + 1$$
$$O\{3\} = 3 + 2$$
$$\text{etc.}$$

Briefly we'll show our new equations for this new set become more compact, allowing us to make the final proof cleaner and more understandable. First we'll compact NO and NE to

$$(((N - 1) * 2) + 1) / 3)$$
$$((2N - 2) + 1) / 3$$
$$1/3\ (2N - 1)$$
$$\text{Set } NO = N \equiv 0 \pmod 3$$

$$(((N - 1) / 3) * 4) + 1)$$
$$((1/3N - ⅓) * 4)$$
$$(4/3N - 4/3) + 1$$
$$⅓ (4N - 1)$$
$$\text{Set } NE = N \equiv 1 \pmod 3$$

Once we do this it follows that the equations for iterating in N, which we sum down to one function F(n) become

$$n - q \text{ if } n \equiv 0 \pmod 3$$
$$F(n) = n + q \text{ if } n \equiv 1 \pmod 3$$
$$\text{Stop if } n \equiv 2 \pmod 3$$
$$\text{Where } q = \lfloor(n / 3)\rfloor$$

Now, an observation:

$$F(n) = n +/- q \text{ if } n \equiv 1 \text{ or } 0 \pmod 3 \text{ where } q = ⅓n$$

In order to loop back the sequence must change sign, from negative to positive or positive to negative

$$F(n1) = n1 - q = n2$$
$$n2 \equiv 1 \pmod 3$$
$$F(n2) = n + q (\lfloor(n2 / 3)\rfloor) = n3$$
$$n3 \Delta n1 = ((2/3n1) * \lfloor 2/3\rfloor) + (2/3n1)$$

That is to say there's a change from switching the sign twice that shows the original number isn't recovered. A different way to observe this is to track how much we add or subtract each time, from the perspective of continuing F(n) to F(n). Or rather, to put it another way, we're going to track Q. We'll be doing so to disentangle the equation further, all the way down to a single specific ratio.

$$N/3 = Q1; \text{ sign} = - \text{ if } 0 \pmod 3); \text{ sign} = + \text{ if } \equiv 1 \pmod 3$$
$$\text{Next } F(n);$$
$$Q \equiv 0 \pmod 3 \text{ then } Q = (s)Q + (Q/3)$$
$$\text{If } Q \equiv 1 \pmod 3, \text{ I/Stop}$$
$$\text{If } Q \equiv 2 \pmod 3 \; Q = (-s)Q + (+Q/3 + 1/3)$$
$$\text{Where (s) is sign}$$

Applied as such.

16/3 = 5 R 1, Q1 = +5

5/3 = 1 R 2 = (1 + 1, the second +1 is from R2 rounded up)
5 + (1 + 1) = (-)7
Q = -7
7/3 = 2 R 1, I/Stop

And

15/3 = 5 R 0, Q1 = -5
5/3 = 1 R 2 = (1 + 1)
-5 + (1 + 1) = 3
Q = 3
3/3 = 1 R 0,
3 + 1 = 4
Q = 4
4/3 = 1 R 1, I/Stop

Now a statement: Changing sign 2 ^ (i) times changes the ratio of the current Q such that no Q may loop back on itself. That is to say, in this framework, we need to go from positive to negative and then back to positive; or from negative to positive than then back to negative in order for a loop to occur.

Now we're going to flip, again, the equation into inverse. Each equation represents a step we can do in inverse, where Qi is equal to (sign)Q and the operation we'd do to get to the previous Q depending on it's sign.

$$Qi = |(-)Q| + |(-)(Q)/2| = Q*(3/2)$$
$$Qi = |(-)Q| - |(+)(Q/4) - 1/4)| = (Q * (3/4) - 1/4)$$
$$Qi = (+)Q - (+)Q/4 = Q * (3/4)$$
$$Qi = (+)Q + |((s-)Q/2) + 1/2)| = (Q * (3/2) + 1/2)$$

By writing out the inverse and the possibilities we've gotten something useful. We see exactly two possibilities for flipping the sign twice, in between which any number of steps of a ratio Q(3/2) or Q(¾) can occur. We're going to use this.

First off, a possibility that does result in flipping the sign twice, but in which no loop can occur:

$$Q1 = Q * (3/2) + 1/2$$
$$(Q((3/2) + 1/2) * (3/4)) -1/4 = Q2(9/8) + 1/8$$
$$thus$$
$$Q2 > Q1$$

Or rather, this only iterates positive. A loop must iterate in both directions, because if there is a loop, eventually Qi = Q1, however in this equation Qi > Q1, therefore there can't be a loop. With this established we move on.

Next please note, the remainder, non Q part of the equation, adds up to an integer that could be Qi, "the next" Q in the sequence. As such any operation such as Q * (¾) distributes the ratio (ra/ra) to both Q and the remainder (re/re) equally. I.E.

$$(Q(3/2) + \tfrac{1}{2}) * (\tfrac{3}{4}) = Q(9/8) + 3/8$$

We note this to make sense of the fact that for our proof, we're not going to care about the order of operations. Instead our proof rests on the fact that the ratio, Q(ra/ra) and remainder (re/re) can't add up to a Q1 no matter the number of operations we perform. That is to say, we can pick any point to be Q1, apply the allowed operations, and no matter the order we can't get Q1 back, because we're going to show (re) ≡ 2 (mod 3). First we'll list the three operations we can use:

$$((Q * (3/2)) + \tfrac{1}{2}) * ((Q * (\tfrac{3}{4})) - \tfrac{1}{4})$$
$$Q * \tfrac{3}{4}$$
$$Q * 3/2$$

The first operation is paired, as noted because we don't care about the order of operations, just that some ratio Q(ra/ra) + (re/re) and because, as noted, in order to form a loop we must go "backwards/negative" then "forwards/positive" from a Q1. Thus, each operation ((Q * (3/2)) + ½) must be paired with an operation ((Q * (¾)) -¼) or vice/versa.

The first thing we'll establish is that we can't get a loop from our first remainder. Re = ⅛. We're going to set Q(equation) = Q to establish the loop.

$$(\tfrac{3}{4})(3/2)(Q(9/8) + (\tfrac{1}{8})) = Q$$
$$Qi(9 * ((3^{I}))/2^{I}) + re(1 * ((3^{I})/2^{I}) = Q$$
$$(re/2^{I}) - Q((2^{I}) - Qi) = Q$$
$$Q = re/((2^{I}) - Qi)$$
$$Re < ((2^{I}) - Qi)$$
$$Q \neq N$$

So what is I? I is just the fact that we are multiplying these parts be ¾ or 3/2, and I is some arbitrary N(atural numbers) that gets us there. As the numerator is always 3 * 3, or 9 *3, it must be some I ^ 3. As the denominator is always 8 and always * 2, it must be some I ^ 2. In this case all is I, rather that I and i (I > i) because the ratio, Q(9/8) re(⅛).

What we show with this is that because Re ends up being the numerator over a larger denominator, it must be some fraction. But to be valid we need Q = N, or a natural number (integer). We'll have to increase Re.

Now we note two things. The first is that repeatedly applying the first operation to itself increases (re), but also comes out to a specific ratio:

$$((Q(9/8) + 1/8) * 3/2) + \tfrac{1}{2}) = Q(27/16) + 11/16$$
$$((Q(27/16) + 11/16) * (\tfrac{3}{4})) - \tfrac{1}{4}) = Q(81/64 = 9/8) + 17/64$$
$$+217/512\ldots \text{ (etc.)}$$

Or rather, cleaned up, and then put into set notation of O (odd) and E(even). Which is about to become relevant.

$$((re) * (9/8)) + \tfrac{1}{8}$$
$$Re = (O * 9 = O) + ((1 * E) = E) = O$$
$$Re = re(N)/re(D)$$
$$re(D) = 2^{I}$$

Quick note, re(N) = remainder Numerator, re(D) = remainder Denominator. re(D) will now be $2 ^ I$, while ratio denominator will be $i ^ 2$, as the remainder will have a larger denominator than ratio.

Moving on, Re > Re1 ($\frac{1}{8}$) must always be odd, as we always take an odd number, times it by nine, which is odd, then add an even number; the even number comes from needing to match denominators between re and 1/8, which is $(re)2 ^ I > (\frac{1}{8})2 ^ i$, thus 1 must be multiplied by $(2 ^ i)$, thus being even; even + odd = odd.

$$(Q = (9/8) * (3/2)\ldots(\tfrac{3}{4})\ldots) + (re = (re) * (9/8 + \tfrac{1}{8}) * (3/2)\ldots(\tfrac{3}{4})\ldots)) = Q$$
$$Q(9 * (3^{i}))/(8 * (2^{i})) + re(re * (3^{i}))/(2^{I}) = Q$$
$$\text{Shortened: } Qi/(2^{i}) + re/(2^{I}) = Q$$
$$re/(2^{I}) - Q((2^{i}) - Qi = O)/(2^{i}) = Q$$
$$re(O)/2^{I}(E) - Q(O)/2^{i} = Q$$
$$I > i$$
$$re/I^{2} - ((Q/2^{i}) * (2 \text{ until } i = I) = QI(E)/2^{I}) = Q$$
$$Re(O)/I^{2} - QI(E)/2^{I} = Q$$
$$Q = Re(O)/QI(E)$$
$$Q \neq N$$

In short, we showed that given the operations available to us, if Q is taken as a loop (where re > $\frac{1}{8}$) then it must equal a ratio where an (O)dd numerator is over an (E)ven denominator. Trivially, this does not equal N, natural numbers above 0 (remember we already covered Q = 0, the existing loop, earlier). However for Q to be valid it must be an integer. Thus no loop can exist.

Now onto the second part, which follows from the first. Note we are still in our new set O. The other way to map each number is to take

$$(F(n) * 4) - 1 = Y1, (Y1 * 4) - 1, \ldots$$

This is equivalent to iterating Y+2 into Z+

$$(O * 4) - 1 = (Z+ * 4) + 1$$
$$((2 * 4) - 1 = O(7) = (Z+)13) = (3 * 4) + 1 = 13$$

As well this does not equal F(n), as that must be +/- X/3. Remember, one step in the inverse equation is one application of the Collatz function, 3x + 1 then /2 repeated.

Thus we get to our proof. In order to loop back on itself this way, F(n)... = N, that is to say the two N's are in the same inverse collatz sequence, would have to

*F(F(N1) * 4) - 1)... = (F(N2) * 4) - 1)*

That is to say, we would have to take two numbers that equal each other in F(n), and put them both into (F(n) * 4) - 1, and then have them both equal each other in F(n) again. However that's a contradiction. As observed:

*(F(n) * 4 - 1) = F(E)*
F(n1)... = F(n2)
-F(E n2) = F(n2)
-F(n2)... = (Fn1)
Thus
-F(n(F(E n2)... ≠ (F(E n1)

This is all to say, running our equation backwards can produce multiple results. But "forward" it's still the Collatz function, still a function that has 1 input and 1 results. Thus for a loop to form, we'd need F(n)1 to = F(n)2 in the normal reverse function. Then we'd need to take both backwards in this second function, to F(E)1 and F(E)2. Then we would need to run the normal reverse function again until F(E)1 = F(E)2. But that can't happen, we already know that in "collatz" function these two inputs F(E)1 = F(n)1 and F(E)2 = F(n)2. Thus they can't equal each other, meaning no loop can form here.

Importantly this goes backwards towards infinity. No matter how many times we connect this way, they have map forwards back to F(n), and since we established the equivalent of F(n) not having any loops except the already established (1) loop, we have proven there are not other loops.