

Beyond the Riemann hypothesis

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Abstract: The functional equation of real variable that Riemann used in his paper was subjected to elementary operations. And I obtained a lot of complex functional equations that the Riemann zeta function follows respectively. Here, functional equation transformations were the main methods for obtaining the complex functional equations.

Half of those are equivalent to the complete symmetric functional equation that the Riemann Xi function follows, and one of those has an origin symmetry with correction terms. From the origin symmetric functional equation including correction terms, the representation containing the leading term of the zeta function for any complex number was obtained. And the Riemann hypothesis was proved by applying reduction to absurdity. Moreover the general representation containing the leading term of the zeta function for any odd number of 3 or more was also obtained.

By suitably combining those functional equations, I observed a new explicit formula for the zeta function. The Riemann hypothesis was again proven using the deductive method. And two types of general representations for the zeta function for any odd number of either 3 or 7, or more, were also obtained from the explicit formula.

In total, three types of general representations for the zeta function for any odd number of either 3 or 7, or more, were discovered.

Conversely, I defined a new function, named the Chi function, for the left side of the origin symmetric functional equation that includes corrective terms. The Chi function is similar to the Riemann Xi function and exhibits origin symmetry. Furthermore, I defined a new function, the eta function, which is similar to the zeta function. The eta function's pole and trivial zeros are the same as those of the zeta function. Furthermore, the Chi and the eta functions have the same non-trivial zeros on the imaginary axis. Here, the imaginary axis corresponds to the critical line of the eta function. And I proposed a generalized Riemann hypothesis for the eta function that states that all non-trivial zeros lie on the imaginary axis. Since I was able to discover the explicit formula for the eta function, the deductive method was used to prove the generalized Riemann hypothesis for the eta function.

As you know, there are different types of transformations between the prime numbers and the non-trivial zeros of the zeta function. I discovered that there are comparable transformations between the prime numbers and the non-trivial zeros of the eta function. Based on the results of numerical experiments, I proposed some conjectures referring to relationships between the prime numbers and non-trivial zeros of the eta function.

1. Introduction

When considering the representations for the Riemann zeta function $\zeta(s)$ for any odd number of 3 or more, especially $\zeta(3)$, $\zeta(5)$, and $\zeta(7)$, a lot of series representations are known. Conversely, each representation for the Riemann zeta function for any odd number of 9 or more, which gives 1 as remainder when divided by 4 (even any series representation) is unknown. This situation seems to be quite similar to the situation at the end era when the Fermat's last theorem was once called the Fermat's conjecture. This was my impression when I first started research.

The initial research motive was to obtain a unified representation for the Riemann zeta function for any odd number of 3 or more, excluding the classification of remainders when divided by 4. I discovered the new functional equation that the Riemann zeta function follows using a technique of functional equation transforma-

tion, which is newly devised in my research process.

This resulted in the unified representation, which does not require classification for any odd number of 3 or more. After solving the initial research problem and three years later, I observed that the newly discovered functional equation has origin symmetry, including correction terms, when the domain of definition is expanded to the whole complex plane.

Moreover, I had a flash of inspiration that the Riemann hypothesis would be solved by creating simultaneous equations consisting of the complete and the origin symmetric functional equations with correction terms.

Here, the complete symmetric functional equation that the Riemann zeta function follows is shown.

$$(1-1) \quad \pi^{-\frac{S}{2}} \Gamma\left(\frac{S}{2}\right) \zeta(S) = \pi^{-\frac{1-S}{2}} \Gamma\left(\frac{1-S}{2}\right) \zeta(1-S) \quad (S \in \mathbb{C} \setminus \{0, 1\}).$$

Riemann used the following functional equation derived from the transformation formula (automorphic) for the theta function to show the complete symmetric functional equation:

$$(1-2) \quad \sum_{n=1}^{\infty} e^{-\pi n^2 x} + \frac{1}{2} = \frac{1}{\sqrt{x}} \left(\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{x}} + \frac{1}{2} \right) \quad (x > 0).$$

Following Riemann, I approached the Riemann hypothesis using the functional equation (1-2) as a starting point “[5].” In this article, I assumed that the complex variable of the Riemann zeta function was θ , excluded referring to pioneering research results. In addition, when both sides are infinite at the same time and the equality is established, as in functional equation (1-1), to show poles explicitly for other functional equations as well, the condition is excluded from the conditions for the equality to be established.

Moreover, I considered developing arguments as elementarily as possible.

Preliminary knowledge is described in Section 3, and additional knowledge is described in Section 19 as a supporting information, with the results used in Section 4 and after if necessary.

2. Preparations

I develop arguments by assuming that the theta function’s functional equation derived from the transformation formula (automorphic) is known:

$$(2-1) \quad \sum_{n=1}^{\infty} e^{-\pi n^2 x} + \frac{1}{2} = \frac{1}{\sqrt{x}} \left(\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{x}} + \frac{1}{2} \right) \quad (x > 0).$$

By multiplying both sides by x after substituting x with x^2 for the positive continuous variable x , the following functional equation is obtained:

$$(2-2) \quad \sum_{n=1}^{\infty} x e^{-\pi n^2 x^2} + \frac{x}{2} = \sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{x^2}} + \frac{1}{2} \quad (x > 0).$$

Both sides of the functional equation (2-2) can be differentiated infinitely many times, indicating that the functional equation (2-2) belongs to the C^∞ class.

Although it is equivalent to the functional equation (2-2), the following functional equation is constructed as a means to find truth:

$$(2-3) \quad \frac{1}{x} = \left(1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 x^2} \right) / \left(1 + 2 \sum_{m=1}^{\infty} e^{-\frac{\pi m^2}{x^2}} \right) \quad (x > 0).$$

For convenience, I define the denominator of the right side of the functional equation (2-3) as the function $\Theta(x)$.

$$(2-4) \quad \Theta(x) := 1 + 2 \sum_{m=1}^{\infty} e^{-\frac{\pi m^2}{x^2}} \quad (x > 0).$$

The functional equation of first-order differentiation is obtained by differentiating both sides of the functional equation (2-2) by x .

$$(2-5) \quad \sum_{n=1}^{\infty} e^{-\pi n^2 x^2} - 2\pi \sum_{n=1}^{\infty} n^2 x^2 e^{-\pi n^2 x^2} + \frac{1}{2} = 2\pi \sum_{n=1}^{\infty} \frac{n^2}{x^3} e^{-\frac{\pi n^2}{x^2}} \quad (x > 0).$$

Differentiation of both sides of the functional equation (2-5) by x provides the following functional equation of second-order differentiation with difference constant multiplication:

$$(2-6) \quad l^{(2)}(x) = r^{(2)}(x) \quad (x > 0).$$

Where

$$(2-7) \quad l^{(2)}(x) = \sum_{n=1}^{\infty} (-3n^2 x + 2\pi n^4 x^3) e^{-\pi n^2 x^2}.$$

$$(2-8) \quad r^{(2)}(x) = \sum_{n=1}^{\infty} (-3n^2 x^{-4} + 2\pi n^4 x^{-6}) e^{-\frac{\pi n^2}{x^2}}.$$

The order of differentiation is indicated by the numeral in the right-shoulder parentheses. This rule will now be followed hereafter.

I decide that the functional equation (2-6) is called the second-order Ir type functional equation. Suffix r means "real." This rule will be followed in the future. Formally, the functional equation of the second-order Ir type is

$$(2-9) \quad l_{[2]}(x) = r_{[2]}(x) \quad (x > 0).$$

Where

$$(2-10) \quad l_{[2]}(x) := l^{(2)}(x) = \sum_{n=1}^{\infty} (-3n^2 x + 2\pi n^4 x^3) e^{-\pi n^2 x^2}.$$

$$(2-11) \quad r_{[2]}(x) := r^{(2)}(x) = \sum_{n=1}^{\infty} (-3n^2 x^{-4} + 2\pi n^4 x^{-6}) e^{-\frac{\pi n^2}{x^2}}.$$

The numeral in the suffix [] shows the order of the functional equation. This rule will also apply in the future. The right side of the second-order Ir type functional equation is considered to be as follows:

$$(2-12) \quad r_{[2]}(x) = \sum_{n=1}^{\infty} \left(\frac{1}{x}\right) \times (-3n^2 x^{-3} + 2\pi n^4 x^{-5}) e^{-\frac{\pi n^2}{x^2}}.$$

The functional equation's relation (2-3) is introduced into the second-order Ir type functional equation, and both sides are multiplied by $\Theta(x)$ to obtain the following functional equation:

$$(2-13) \quad \mathbf{a}_{[2]}(x) + \mathbf{f}_{[2]}(x) = \mathbf{b}_{[2]}(x) + \mathbf{g}_{[2]}(x) \quad (x > 0).$$

Where

$$(2-14) \quad \mathbf{a}_{[2]}(x) = \sum_{n=1}^{\infty} (-3n^2 x + 2\pi n^4 x^3) e^{-\pi n^2 x^2}.$$

$$(2-15) \quad \mathbf{f}_{[2]}(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2 x + 4\pi n^4 x^3) e^{-\pi \left(n^2 x^2 + \frac{m^2}{x^2}\right)}.$$

$$(2-16) \quad b_{[2]}(x) = \sum_{n=1}^{\infty} (-3n^2x^{-3} + 2\pi n^4x^{-5}) e^{-\frac{\pi n^2}{x^2}}.$$

$$(2-17) \quad g_{[2]}(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2x^{-3} + 4\pi n^4x^{-5}) e^{-\pi\left(m^2x^2 + \frac{n^2}{x^2}\right)}.$$

I decide that the functional equation (2-13) is called the second-order IIR type functional equation.

Differentiation of both sides of the functional equation (2-6) by x provides the following third-order differentiation functional equation:

$$(2-18) \quad l^{(3)}(x) = r^{(3)}(x) \quad (x > 0).$$

Where

$$(2-19) \quad l^{(3)}(x) = \sum_{n=1}^{\infty} (-3n^2 + 12\pi n^4x^2 - 4\pi^2 n^6x^4) e^{-\pi n^2x^2}.$$

$$(2-20) \quad r^{(3)}(x) = \sum_{n=1}^{\infty} (12n^2x^{-5} - 18\pi n^4x^{-7} + 4\pi^2 n^6x^{-9}) e^{-\frac{\pi n^2}{x^2}}.$$

To obtain the following third-order IR type functional equation, multiply both sides of the functional equation (2-18) with x :

$$(2-21) \quad l_{[3]}(x) = r_{[3]}(x) \quad (x > 0).$$

Where

$$(2-22) \quad l_{[3]}(x) := x l^{(3)}(x) = \sum_{n=1}^{\infty} (-3n^2x + 12\pi n^4x^3 - 4\pi^2 n^6x^5) e^{-\pi n^2x^2}.$$

$$(2-23) \quad r_{[3]}(x) := x r^{(3)}(x) = \sum_{n=1}^{\infty} (12n^2x^{-4} - 18\pi n^4x^{-6} + 4\pi^2 n^6x^{-8}) e^{-\frac{\pi n^2}{x^2}}.$$

The functional equation's relation (2-3) is introduced into the third-order IR type functional equation, and both sides are multiplied by $\Theta(x)$ to obtain the following functional equation:

$$(2-24) \quad a_{[3]}(x) + f_{[3]}(x) = b_{[3]}(x) + g_{[3]}(x) \quad (x > 0).$$

Where

$$(2-25) \quad a_{[3]}(x) = \sum_{n=1}^{\infty} (-3n^2x + 12\pi n^4x^3 - 4\pi^2 n^6x^5) e^{-\pi n^2x^2}.$$

$$(2-26) \quad f_{[3]}(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2x + 24\pi n^4x^3 - 8\pi^2 n^6x^5) e^{-\pi\left(n^2x^2 + \frac{m^2}{x^2}\right)}.$$

$$(2-27) \quad b_{[3]}(x) = \sum_{n=1}^{\infty} (12n^2x^{-3} - 18\pi n^4x^{-5} + 4\pi^2 n^6x^{-7}) e^{-\frac{\pi n^2}{x^2}}.$$

$$(2-28) \quad g_{[3]}(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (24n^2x^{-3} - 36\pi n^4x^{-5} + 8\pi^2 n^6x^{-7}) e^{-\pi\left(m^2x^2 + \frac{n^2}{x^2}\right)}.$$

I decide that the functional equation (2-24) is called the third-order IIR type functional equation.

The differentiation of both sides of functional equation (2-18) by x provides the fourth-order differentiation functional equation:

$$(2-29) \quad l^{(4)}(x) = r^{(4)}(x) \quad (x > 0).$$

Where

$$(2-30) \quad I^{(4)}(x) = \sum_{n=1}^{\infty} (30\pi n^4 x - 40\pi^2 n^6 x^3 + 8\pi^3 n^8 x^5) e^{-\pi n^2 x^2}.$$

$$(2-31) \quad r^{(4)}(x) = \sum_{n=1}^{\infty} (-60n^2 x^{-6} + 150\pi n^4 x^{-8} - 72\pi^2 n^6 x^{-10} + 8\pi^3 n^8 x^{-12}) e^{-\frac{\pi n^2}{x^2}}.$$

Both sides of the functional equation (2-29) are multiplied by x^2 and both sides of the third-order Ir type functional equation are added to each other to obtain the following fourth-order Ir type functional equation:

$$(2-32) \quad l_{[4]}(x) = r_{[4]}(x) \quad (x > 0).$$

Where

$$(2-33) \quad l_{[4]}(x) := x^2 I^{(4)}(x) + l_{[3]}(x) \\ = \sum_{n=1}^{\infty} (-3n^2 x + 42\pi n^4 x^3 - 44\pi^2 n^6 x^5 + 8\pi^3 n^8 x^7) e^{-\pi n^2 x^2}.$$

$$(2-34) \quad r_{[4]}(x) := x^2 r^{(4)}(x) + r_{[3]}(x) \\ = \sum_{n=1}^{\infty} (-48n^2 x^{-4} + 132\pi n^4 x^{-6} - 68\pi^2 n^6 x^{-8} + 8\pi^3 n^8 x^{-10}) e^{-\frac{\pi n^2}{x^2}}.$$

The functional equation's relation (2-3) is introduced into the fourth-order Ir type functional equation, and both sides are multiplied by $\Theta(x)$ to obtain the following functional equation:

$$(2-35) \quad a_{[4]}(x) + f_{[4]}(x) = b_{[4]}(x) + g_{[4]}(x) \quad (x > 0).$$

Where

$$(2-36) \quad a_{[4]}(x) = \sum_{n=1}^{\infty} (-3n^2 x + 42\pi n^4 x^3 - 44\pi^2 n^6 x^5 + 8\pi^3 n^8 x^7) e^{-\pi n^2 x^2}.$$

$$(2-37) \quad f_{[4]}(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2 x + 84\pi n^4 x^3 - 88\pi^2 n^6 x^5 + 16\pi^3 n^8 x^7) e^{-\pi \left(n^2 x^2 + \frac{m^2}{x^2} \right)}.$$

$$(2-38) \quad b_{[4]}(x) = \sum_{n=1}^{\infty} (-48n^2 x^{-3} + 132\pi n^4 x^{-5} - 68\pi^2 n^6 x^{-7} + 8\pi^3 n^8 x^{-9}) e^{-\frac{\pi n^2}{x^2}}.$$

$$(2-39) \quad g_{[4]}(x) =$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-96n^2 x^{-3} + 264\pi n^4 x^{-5} - 136\pi^2 n^6 x^{-7} + 16\pi^3 n^8 x^{-9}) e^{-\pi \left(m^2 x^2 + \frac{n^2}{x^2} \right)}.$$

I decide that the functional equation (2-35) is called the fourth-order IIr type functional equation.

Differentiation of both sides of the functional equation (2-29) by x provides the fifth-order differentiation functional equation:

$$(2-40) \quad I^{(5)}(x) = r^{(5)}(x) \quad (x > 0).$$

Where

$$(2-41) \quad I^{(5)}(x) = \sum_{n=1}^{\infty} (30\pi n^4 - 180\pi^2 n^6 x^2 + 120\pi^3 n^8 x^4 - 16\pi^4 n^{10} x^6) e^{-\pi n^2 x^2}.$$

$$(2-42) \quad r^{(5)}(x) =$$

$$\sum_{n=1}^{\infty} (360n^2 x^{-7} - 1320\pi n^4 x^{-9} + 1020\pi^2 n^6 x^{-11} - 240\pi^3 n^8 x^{-13} + 16\pi^4 n^{10} x^{-15}) e^{-\frac{\pi n^2}{x^2}}.$$

Both sides of the functional equation (2-40) are multiplied by x^3 and both sides of the fourth-order Ir type functional equation are added to each other to obtain the following fifth-order Ir type functional equation:

$$(2-43) \quad l_{[5]}(x) = r_{[5]}(x) \quad (x > 0).$$

Where

$$(2-44) \quad l_{[5]}(x) := x^3 l^{(5)}(x) + l_{[4]}(x) \\ = \sum_{n=1}^{\infty} (-3n^2x + 72\pi n^4x^3 - 224\pi^2 n^6x^5 + 128\pi^3 n^8x^7 - 16\pi^4 n^{10}x^9) e^{-\pi n^2x^2}.$$

$$(2-45) \quad r_{[5]}(x) := x^3 r^{(5)}(x) + r_{[4]}(x) \\ = \sum_{n=1}^{\infty} (312n^2x^{-4} - 1188\pi n^4x^{-6} + 952\pi^2 n^6x^{-8} - 232\pi^3 n^8x^{-10} + 16\pi^4 n^{10}x^{-12}) e^{-\frac{\pi n^2}{x^2}}.$$

The functional equation' relation (2-3) is introduced into the fifth-order Ir type functional equation, and both sides are multiplied by $\Theta(x)$ to obtain the following functional equation:

$$(2-46) \quad a_{[5]}(x) + f_{[5]}(x) = b_{[5]}(x) + g_{[5]}(x) \quad (x > 0).$$

Where

$$(2-47) \quad a_{[5]}(x) = \sum_{n=1}^{\infty} (-3n^2x + 72\pi n^4x^3 - 224\pi^2 n^6x^5 + 128\pi^3 n^8x^7 - 16\pi^4 n^{10}x^9) e^{-\pi n^2x^2}.$$

$$(2-48) \quad f_{[5]}(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2x + 144\pi n^4x^3 \\ - 448\pi^2 n^6x^5 + 256\pi^3 n^8x^7 - 32\pi^4 n^{10}x^9) e^{-\pi \left(n^2x^2 + \frac{m^2}{x^2} \right)}.$$

$$(2-49) \quad b_{[5]}(x) = \\ \sum_{n=1}^{\infty} (312n^2x^{-3} - 1188\pi n^4x^{-5} + 952\pi^2 n^6x^{-7} - 232\pi^3 n^8x^{-9} + 16\pi^4 n^{10}x^{-11}) e^{-\frac{\pi n^2}{x^2}}.$$

$$(2-50) \quad g_{[5]}(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (624n^2x^{-3} - 2376\pi n^4x^{-5} \\ + 1904\pi^2 n^6x^{-7} - 464\pi^3 n^8x^{-9} + 32\pi^4 n^{10}x^{-11}) e^{-\pi \left(m^2x^2 + \frac{n^2}{x^2} \right)}.$$

I decide that the functional equation (2-46) is called the fifth-order Iir type functional equation.

3. Preliminary knowledge

3-1. The gamma function $\Gamma(Z)$

The gamma function $\Gamma(Z)$ is an absolutely convergent function under the condition of $\text{Re}(Z) > 0$ “[2].”

The following equation provides its definition:

$$(3-1) \quad \Gamma(Z) := \int_0^{\infty} x^{Z-1} e^{-x} dx \quad (\text{Re}(Z) > 0).$$

The formula can be used to calculate the value of the gamma function for any positive integer.

$$(3-2) \quad \Gamma(n) = (n-1)! \quad (n \in \mathbb{N}).$$

The two formulae can be used to calculate value of the gamma function for any positive half-integer.

$$(3-3) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$(3-4) \Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!\sqrt{\pi}}{2^n} \quad (n \in \mathbb{N}).$$

Table shows the values for the gamma function for positive half-integers

n	0	1	2	3	4	5	6	7	8
$\frac{2n+1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$
$\Gamma\left(\frac{2n+1}{2}\right)$	$\sqrt{\pi}$	$\frac{\sqrt{\pi}}{2}$	$\frac{3\sqrt{\pi}}{4}$	$\frac{15\sqrt{\pi}}{8}$	$\frac{105\sqrt{\pi}}{16}$	$\frac{945\sqrt{\pi}}{32}$	$\frac{10395\sqrt{\pi}}{64}$	$\frac{135135\sqrt{\pi}}{128}$	$\frac{2027025\sqrt{\pi}}{256}$

Fig. 3-1

Analytic continuation extends the domain of definition of the gamma function $\Gamma(Z)$ to the whole complex plane. The gamma function has simple poles at $Z = 0, -1, -2, \dots$, i.e.,

$$(3-5) \frac{1}{\Gamma(1-n)} = 0 \quad (n \in \mathbb{N}).$$

Residue of the gamma function at $Z = 0$ is shown as follows:

$$(3-6) \text{Res}(0; \Gamma(Z)) = 1.$$

Residue of the gamma function at $Z = -n$ is shown as follows:

$$(3-7) \text{Res}(-n; \Gamma(Z)) = \frac{(-1)^n}{n} \quad (n \in \mathbb{N}).$$

Except for the poles, the gamma function is regular in the whole complex plane.

In the entire complex plane, the gamma function has no zero, i.e.,

$$(3-8) \Gamma(Z) \neq 0 \quad (Z \in \mathbb{C}).$$

The value of the gamma function for any negative half-integer can be obtained by the formula.

$$(3-9) \Gamma\left(-\frac{2n-1}{2}\right) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!} \quad (n \in \mathbb{N}).$$

Table shows the values for the gamma function for negative half-integers

n	1	2	3	4	5	6	7	8	9
$-\frac{2n-1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$	$-\frac{9}{2}$	$-\frac{11}{2}$	$-\frac{13}{2}$	$-\frac{15}{2}$	$-\frac{17}{2}$
$\Gamma\left(-\frac{2n-1}{2}\right)$	$-2\sqrt{\pi}$	$\frac{4\sqrt{\pi}}{3}$	$-\frac{8\sqrt{\pi}}{15}$	$\frac{16\sqrt{\pi}}{105}$	$-\frac{32\sqrt{\pi}}{945}$	$\frac{64\sqrt{\pi}}{10395}$	$-\frac{128\sqrt{\pi}}{135135}$	$\frac{256\sqrt{\pi}}{2027025}$	$-\frac{512\sqrt{\pi}}{34459425}$

Fig. 3-2

The difference formula for the gamma function is shown as follows:

$$(3-10) \Gamma(1+Z) = Z\Gamma(Z) \quad (Z \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}).$$

The reciprocal formula for the gamma function is shown as follows:

$$(3-11) \Gamma(Z)\Gamma(1-Z) = \frac{\pi}{\sin(\pi Z)} \quad (Z \in \mathbb{C} \setminus \mathbb{Z}).$$

3-2. The Riemann zeta function $\zeta(S)$

The Riemann zeta function $\zeta(S)$ is an absolutely convergent function under the condition of $\text{Re}(S) > 1$ “[2].”

Its definition is provided by the following equation, and it has the Euler product representation:

$$(3 - 12) \quad \zeta (S) := \sum_{n=1}^{\infty} \frac{1}{n^S} = \prod_{p: \text{prime}} (1 - p^{-S})^{-1} \quad (\text{Re}(S) > 1).$$

In this case, the product of the right side crosses all prime numbers.

The Bernoulli numbers B_n are provided by the following equation of definition:

$$(3 - 13) \quad \frac{x}{e^x - 1} := \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

Table shows the values for the Bernoulli numbers B_n

k	0	1	2	3	4	5	6	7	8	9	10
B_{2k}	1	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$
B_{2k+1}	$-\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0

Fig. 3-3

The formula can be used to calculate the exact value of the Riemann zeta function for any even number of 2 or more.

$$(3 - 14) \quad \zeta (2n) = \frac{(-1)^{n-1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!} \quad (n \in \mathbb{N}).$$

Table shows the exact values for the Riemann zeta function for even numbers of 2 or more

n	1	2	3	4	5	6	7	8
2n	2	4	6	8	10	12	14	16
$\zeta (2n)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9450}$	$\frac{\pi^{10}}{93555}$	$\frac{691 \pi^{12}}{638512875}$	$\frac{2 \pi^{14}}{18243225}$	$\frac{3617 \pi^{16}}{325641566250}$

Fig. 3-4

In 1799, R. Apery proved that $\zeta (3)$ is an irrational number using the following series representation “[4]”:

$$(3 - 15) \quad \zeta (3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}}.$$

The following series representation for $\zeta (5)$ is adopted from Ramanujan’s note book:

$$(3 - 16) \quad \zeta (5) = \frac{\pi^5}{294} - \frac{72}{35} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{2\pi n} - 1)} - \frac{2}{35} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{2\pi n} + 1)}.$$

The following series representations for $\zeta (3)$ and $\zeta (7)$ are adopted from Ramanujan’s note book, too:

$$(3 - 17) \quad \zeta (3) = \frac{7\pi^3}{180} - 2 \sum_{n=1}^{\infty} \frac{1}{n^3 (e^{2\pi n} - 1)}, \quad \zeta (7) = \frac{19\pi^7}{56700} - 2 \sum_{n=1}^{\infty} \frac{1}{n^7 (e^{2\pi n} - 1)}, \text{ etc.}$$

By analytic continuation, the domain of definition of the Riemann zeta function $\zeta (S)$ is expanded to the whole complex plane. The Riemann zeta function $\zeta (S)$ has only a single simple pole at $S = 1$, i.e.,

$$(3 - 18) \quad \frac{1}{\zeta (1)} = 0.$$

The Riemann zeta function is regular in the whole complex plane except the pole.

The residue of the Riemann zeta function at $S = 1$ is as follows:

$$(3 - 19) \operatorname{Res} (1 ; \zeta (S)) = 1 .$$

The value of the Riemann zeta function at $S = 0$ is shown as follows:

$$(3 - 20) \zeta (0) = -1 / 2 .$$

The Riemann zeta function has trivial zeros for any even number of -2 or less, i.e.,

$$(3 - 21) \zeta (-2 n) = 0 \quad (n \in \mathbb{N}) .$$

The value of the Riemann zeta function for any odd number of -1 or less can be obtained using the formula.

$$(3 - 22) \zeta (1 - 2 n) = -\frac{B_{2 n}}{2 n} \quad (n \in \mathbb{N}) .$$

Table shows the exact values for the Riemann zeta function for odd numbers of -1 or less

n	1	2	3	4	5	6	7	8	9
1 - 2 n	-1	-3	-5	-7	-9	-11	-13	-15	-17
$\zeta (1 - 2 n)$	$-\frac{1}{12}$	$\frac{1}{120}$	$-\frac{1}{252}$	$\frac{1}{240}$	$-\frac{1}{132}$	$\frac{691}{32760}$	$-\frac{1}{12}$	$\frac{3617}{8160}$	$-\frac{43867}{14364}$

Fig. 3-5

3-3. The Riemann Xi function $\xi (S)$

The Riemann Xi function $\xi (S)$ is defined as follows:

$$(3 - 23) \xi (S) := \frac{S(S-1)}{2} \pi^{-\frac{S}{2}} \Gamma \left(\frac{S}{2} \right) \zeta (S) \quad (S \in \mathbb{C}) .$$

The Riemann Xi function's functional equation exhibits point symmetry, which at $S = 1/2$.

$$(3 - 24) \xi (S) = \xi (1 - S) \quad (S \in \mathbb{C}) .$$

The complete symmetric functional equation, of course, has the same symmetry.

The value of the Riemann Xi function at $S = 0$ is shown as follows:

$$(3 - 25) \xi (0) = 1 / 2 .$$

The Riemann Xi function has an infinite product representation called the Hadamard product.

$$(3 - 26) \xi (S) := \xi (0) \prod_{\substack{\zeta(\rho)=0 \\ \rho: \text{non-trivial zero}}} \left(1 - \frac{S}{\rho} \right) \quad (S \in \mathbb{C}) .$$

In this case, the infinite product crosses all non-trivial zeros of the Riemann zeta function.

This infinite product is absolutely convergent.

3-4. The Riemann hypothesis for the Riemann zeta function

It is well known that all non-trivial zeros of the Riemann zeta function lie in the domain called the critical strip, excluding both side lines on the boundary.

$$(3 - 27) \text{Critical strip} = \{ S \in \mathbb{C} : 0 < \operatorname{Re} (S) < 1 \} .$$

The Riemann hypothesis asserts that all non-trivial zeros of the Riemann zeta function lie on the critical line.

(3 – 28) Critical line = { S ∈ C : Re (S) = 1 / 2 } .

In this article, the Riemann zeta function is hereafter abbreviated to the zeta function.

3-5. The modified Bessel function of the second kind $K_\nu(Z)$

The modified Bessel function of the second kind $K_\nu(Z)$ is defined as an independent solution of the following modified Bessel differential equation “[2], [3]”:

$$(3 – 29) \quad Z^2 y'' + Zy' - (Z^2 + \nu^2) y = 0 \quad (\nu, Z \in \mathbb{C}) .$$

$K_\nu(Z)$ has a branch cut discontinuity in the complex z plane running from $-\infty$ to 0.

The origin symmetry concerning the index ν is shown as follows:

$$(3 – 30) \quad K_\nu(Z) = K_{-\nu}(Z) \quad (\nu, Z \in \mathbb{C}) .$$

The recurrence formula is shown as follows:

$$(3 – 31) \quad \frac{Z}{2} K_{\nu+1}(Z) = \nu K_\nu(Z) + \frac{Z}{2} K_{\nu-1}(Z) \quad (\nu, Z \in \mathbb{C}) .$$

The integral representation is shown as follows “[7]”:

$$(3 – 32) \quad K_\nu(Z) = \frac{1}{2} \int_0^\infty x^{\nu-1} e^{-\frac{Z}{2}\left(x+\frac{1}{x}\right)} dx \quad (\nu, Z \in \mathbb{C}) .$$

When the index ν is not an integer, the modified Bessel function of the second kind is given by the following equation using the modified Bessel function of the first kind, $I_\nu(Z)$:

$$(3 – 33) \quad K_\nu(Z) = \frac{\pi}{2 \sin(\pi \nu)} (I_{-\nu}(Z) - I_\nu(Z)) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}, Z \in \mathbb{C}) .$$

In this case, the function $I_\nu(Z)$ is given as follows:

$$(3 – 34) \quad I_\nu(Z) = \sum_{\mu=0}^\infty \frac{(Z/2)^{2\mu+\nu}}{\mu! \Gamma(\mu+\nu+1)} \quad (\nu, Z \in \mathbb{C}) .$$

$I_\nu(Z)$ has also a branch cut discontinuity in the complex z plane running from $-\infty$ to 0.

When the index ν is a positive integer, the function $K_\nu(Z)$ is given as follows:

$$(3 – 35) \quad K_n(Z) = \lim_{\nu \rightarrow n} K_\nu(Z) = (-1)^{n+1} I_n(Z) \log\left(\frac{Z}{2}\right) + \frac{1}{2} \sum_{\mu=0}^{n-1} (-1)^\mu \frac{(n-\mu-1)!}{\mu!} \left(\frac{Z}{2}\right)^{2\mu-n} \\ + \frac{(-1)^n}{2} \sum_{\mu=0}^\infty \frac{\psi(\mu+n+1) + \psi(\mu+1)}{(n+\mu)! \mu!} \left(\frac{Z}{2}\right)^{n+2\mu} \quad (n \in \mathbb{N}, Z \in \mathbb{C} \setminus \{0\}) .$$

Here, the Psi function $\psi(Z)$ is defined as the logarithmic differentiation of the gamma function, i.e.,

$$(3 – 36) \quad \psi(Z) := \frac{\Gamma'(Z)}{\Gamma(Z)} = \frac{d}{dZ} \log(\Gamma(Z)) \quad (Z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}) .$$

The Psi function is an analytic function in the whole complex plane except at $Z = 0, -1, -2, \dots$. When Z is any integer of 2 or more,

$$(3 – 37) \quad \psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k} \quad (n \in \mathbb{N}) .$$

When $Z = 1$,

$$(3 – 38) \quad \psi(1) = -\gamma .$$

In this case, the constant γ is called Euler’s constant, and its definition and approximate value are given as follows:

$$(3-39) \quad \gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log(n) \right) \approx 0.57721566490.$$

When the index $\nu = 0$, the asymptotic formula for the function $K_0(Z)$ is given as follows “[17]”:

$$(3-40) \quad K_0(x) = (-\gamma + \log(2) - \log(x)) + \frac{1}{4}(1 - \gamma + \log(2) - \log(x))x^2 + O(x)^4 \quad (x > 0).$$

The formula for the function $K_\nu(Z)$ at the index $\nu = 1/2$ is given as follows:

$$(3-41) \quad K_{\frac{1}{2}}(Z) = \sqrt{\frac{\pi}{2Z}} e^{-Z} \quad (Z \in \mathbb{C}).$$

The formula for the function $K_\nu(Z)$ at the index ν for any positive half-integer is given as follows:

$$(3-42) \quad K_{\frac{2n+1}{2}}(Z) = \sqrt{\frac{\pi}{2Z}} e^{-Z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2Z)^k} \quad (n \in \mathbb{N}, Z \in \mathbb{C}).$$

Table shows the formulae for the function $K_\nu(Z)$ at the index ν for any half-integer

ν	$\pm \frac{1}{2}$	$\pm \frac{3}{2}$	$\pm \frac{5}{2}$	$\pm \frac{7}{2}$	$\pm \frac{9}{2}$
$K_\nu(Z)$	$\sqrt{\frac{\pi}{2Z}} e^{-Z}$	$\sqrt{\frac{\pi}{2Z}} e^{-Z} (1 + \frac{1}{Z})$	$\sqrt{\frac{\pi}{2Z}} e^{-Z} (1 + \frac{3}{Z} + \frac{3}{Z^2})$	$\sqrt{\frac{\pi}{2Z}} e^{-Z} (1 + \frac{6}{Z} + \frac{15}{Z^2} + \frac{15}{Z^3})$	$\sqrt{\frac{\pi}{2Z}} e^{-Z} (1 + \frac{10}{Z} + \frac{45}{Z^2} + \frac{105}{Z^3} + \frac{105}{Z^4})$

Fig. 3-6

3-6. The divisor sigma function $\sigma_\theta(n)$

When d is a divisor of any positive integer n , the sum of the divisor sigma function $\sigma_\theta(n)$ runs all divisors, and the equation defining the divisor sigma function $\sigma_\theta(n)$ is shown as follows:

$$(3-43) \quad \sigma_\theta(n) := \sum_{d|n} d^\theta \quad (\theta \in \mathbb{C}, n \in \mathbb{N}).$$

When $\theta = 0$, the divisor sigma function is equivalent to the number of divisors function $\mu(n)$.

$$(3-44) \quad \sigma_0(n) = \sum_{d|n} 1 = \mu(n) \quad (n \in \mathbb{N}).$$

When $n = 1$, the divisor sigma function takes the constant 1 independent of the complex variable θ .

$$(3-45) \quad \sigma_{-\theta}(1) = 1 \quad (\theta \in \mathbb{C}).$$

When two different arbitrary prime numbers a and b are assumed, examples of calculation are shown as follows:

$$(3-46) \quad \sigma_{-\theta}(a) = 1 + \frac{1}{a^\theta} \quad (\theta \in \mathbb{C}).$$

$$(3-47) \quad \sigma_{-\theta}(a^2) = 1 + \frac{1}{a^\theta} + \frac{1}{a^{2\theta}} \quad (\theta \in \mathbb{C}).$$

$$(3-48) \quad \sigma_{-\theta}(a \cdot b) = 1 + \frac{1}{a^\theta} + \frac{1}{b^\theta} + \frac{1}{(a \cdot b)^\theta} = \left(1 + \frac{1}{a^\theta}\right) \left(1 + \frac{1}{b^\theta}\right) \quad (\theta \in \mathbb{C}).$$

When considering the product of two different giant prime numbers, there is a difficulty in factorizing the product. In addition, the divisor sigma function $\sigma_\theta(n)$ has the same difficulty.

4. Functional Equation Transformation, Part1

I consider multiplying both sides of the second-order I_r type functional equation by the kernel function

$$x^\theta e^{-\pi \alpha^2 x^2} \quad (x, \alpha > 0, \theta \in \mathbb{C})$$

to perform the improper integral on the open interval $(0, \infty)$ and to provide the operation for taking the left-sided limit of $\alpha \rightarrow +0$.

To ensure the validity of the above series of operations, it is necessary to evaluate the convergence of the integral and the boundedness of the limit value.

The operation for taking the left-sided limit is carried out at the proper stage of the calculation.

Moreover, the domain of definition is determined sequentially at appropriate stages.

The statements above also apply in Sections 5 and 10, except that the transformation sources are different.

The following are the operations of the functional equation transformation for the second-order I_Γ type functional equation:

$$(4-1) \quad L_{[2]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (-3 n^2 x^{\theta+1} + 2 \pi n^4 x^{\theta+3}) e^{-\pi (n^2 + \alpha^2) x^2} dx.$$

$$(4-2) \quad R_{[2]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (-3 n^2 x^{\theta-4} + 2 \pi n^4 x^{\theta-6}) e^{-\pi \left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.$$

For the integral of $L_{[2]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{y}{\pi(n^2 + \alpha^2)}\right)^{\frac{1}{2}}.$$

$$(4-3) \quad L_{[2]}(\theta) =$$

$$\begin{aligned} & \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \left(-3 n^2 \left(\frac{y}{\pi(n^2 + \alpha^2)}\right)^{\frac{\theta+1}{2}} + 2 \pi n^4 \left(\frac{y}{\pi(n^2 + \alpha^2)}\right)^{\frac{\theta+3}{2}} \right) e^{-y} \frac{1}{2} \left(\frac{1}{\pi(n^2 + \alpha^2)}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy \\ & = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \lim_{\alpha \rightarrow +0} \int_0^\infty \left(-3 \sum_{n=1}^\infty n^2 \left(\frac{1}{n^2 + \alpha^2}\right)^{\frac{\theta}{2}+1} y^{\frac{\theta}{2}} + 2 \sum_{n=1}^\infty n^4 \left(\frac{1}{n^2 + \alpha^2}\right)^{\frac{\theta}{2}+2} y^{\frac{\theta}{2}+1} \right) e^{-y} dy. \end{aligned}$$

Using the left-sided limit of the positive real variable α ,

$$(4-4) \quad L_{[2]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^\theta} \left(-3 y^{\frac{\theta}{2}} + 2 y^{\frac{\theta}{2}+1} \right) e^{-y} dy$$

$$= \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^\infty \frac{1}{n^\theta} \left(-3 \int_0^\infty y^{\frac{\theta}{2}} e^{-y} dy + 2 \int_0^\infty y^{\frac{\theta}{2}+1} e^{-y} dy \right).$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\text{Re}(\theta) > 1$ can be added to the above result because it is a convergent function under the condition.

$$(4-5) \quad L_{[2]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^\infty \frac{1}{n^\theta} \left(-3 \Gamma\left(1 + \frac{\theta}{2}\right) + 2 \Gamma\left(2 + \frac{\theta}{2}\right) \right) \quad (\text{Re}(\theta) > 1).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(4-6) \quad L_{[2]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-3 \cdot \frac{\theta}{2} \Gamma\left(\frac{\theta}{2}\right) + 2 \cdot \frac{\theta}{2} \left(1 + \frac{\theta}{2}\right) \Gamma\left(\frac{\theta}{2}\right) \right) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \cdot \frac{\theta}{2} (\theta - 1) \Gamma\left(\frac{\theta}{2}\right)$$

$$= \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \cdot \frac{\theta}{2} (\theta - 1) \Gamma\left(\frac{\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

Therefore,

$$(4-7) \quad \mathbf{L}_{[2]}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

The function $\mathbf{L}_{[2]}(\theta)$ is a convergent function in the whole complex plane.

For the integral of $\mathbf{R}_{[2]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$(4-8) \quad \mathbf{R}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \left(-3n^2 \left(\frac{1}{y}\right)^{\theta-4} + 2\pi n^4 \left(\frac{1}{y}\right)^{\theta-6} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} (-y^{-2}) dy$$

$$= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \left(-3n^2 y^{2-\theta} + 2\pi n^4 y^{4-\theta} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} dy.$$

Assuming that the integral and the sum can be interchanged, for the integrals of $\mathbf{R}_{[2]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{\alpha x}{n}\right)^{\frac{1}{2}}.$$

$$(4-9) \quad \mathbf{R}_{[2]}(\theta) =$$

$$\lim_{\alpha \rightarrow +0} \sum_{n=1}^\infty \int_0^\infty \left(-3n^2 \left(\frac{\alpha x}{n}\right)^{\frac{2-\theta}{2}} + 2\pi n^4 \left(\frac{\alpha x}{n}\right)^{\frac{4-\theta}{2}} \right) e^{-\pi n \alpha \left(x + \frac{1}{x}\right)} \frac{1}{2} \left(\frac{\alpha}{n}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx$$

$$= \lim_{\alpha \rightarrow +0} \alpha^{\frac{3-\theta}{2}} \sum_{n=1}^\infty n^{\frac{1+\theta}{2}} \left(-3 \cdot \frac{1}{2} \int_0^\infty x^{\frac{3-\theta}{2}-1} e^{-\frac{2\pi n \alpha}{2} \left(x + \frac{1}{x}\right)} dx \right.$$

$$\left. + 2(\pi n \alpha) \cdot \frac{1}{2} \int_0^\infty x^{\frac{5-\theta}{2}-1} e^{-\frac{2\pi n \alpha}{2} \left(x + \frac{1}{x}\right)} dx \right).$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(4-10) \quad \mathbf{R}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \alpha^{\frac{3-\theta}{2}} \sum_{n=1}^\infty n^{\frac{1+\theta}{2}} \left(-3 K_{\frac{3-\theta}{2}}(2\pi n \alpha) + 2(\pi n \alpha) K_{\frac{5-\theta}{2}}(2\pi n \alpha) \right).$$

Because any modified Bessel function of the second kind of the sum converges absolutely, the assumed exchange is justified.

The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$(4-11) \quad \mathbf{R}_{[2]}(\theta) =$$

$$\lim_{\alpha \rightarrow +0} \alpha^{\frac{3-\theta}{2}} \sum_{n=1}^\infty n^{\frac{1+\theta}{2}} \left(-3 K_{\frac{3-\theta}{2}}(2\pi n \alpha) + 2 \left(\frac{3-\theta}{2} K_{\frac{3-\theta}{2}}(2\pi n \alpha) + (\pi n \alpha) K_{\frac{1-\theta}{2}}(2\pi n \alpha) \right) \right)$$

$$= \frac{1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^\infty n^{\frac{-1+\theta}{2}} \left(-\theta(\pi n \alpha) K_{\frac{3-\theta}{2}}(2\pi n \alpha) + 2(\pi n \alpha)^2 K_{\frac{1-\theta}{2}}(2\pi n \alpha) \right)$$

$$= \frac{1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^\infty n^{\frac{-1+\theta}{2}} \left(-\theta \left(\frac{1-\theta}{2} K_{\frac{1-\theta}{2}}(2\pi n \alpha) + (\pi n \alpha) K_{\frac{-1-\theta}{2}}(2\pi n \alpha) \right) \right.$$

$$\left. + 2(\pi n \alpha)^2 K_{\frac{1-\theta}{2}}(2\pi n \alpha) \right)$$

$$= \frac{1}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^\infty n^{\frac{-1+\theta}{2}} \left(\theta(\theta-1) K_{\frac{1-\theta}{2}}(2\pi n \alpha) \right.$$

$$\left. - 2(\pi n \alpha) \theta K_{\frac{1+\theta}{2}}(2\pi n \alpha) + 4(\pi n \alpha)^2 K_{\frac{1-\theta}{2}}(2\pi n \alpha) \right).$$

For convenience, I separate the result of equation (4-11) into three parts adding the condition $\text{Re}(\theta) < 0$.

$$(4-12) \quad \mathbf{C}_{[2]-1}(\theta) := \frac{\theta(\theta-1)}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^\infty n^{\frac{-1+\theta}{2}} K_{\frac{1-\theta}{2}}(2\pi n \alpha) \quad (\text{Re}(\theta) < 0).$$

$$(4-13) \quad C_{[2]-2}(\theta) := -\frac{\theta}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} (\pi n \alpha) K_{\frac{1+\theta}{2}}(2\pi n \alpha) \quad (\operatorname{Re}(\theta) < 0).$$

$$(4-14) \quad C_{[2]-3}(\theta) := \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} (\pi n \alpha)^2 K_{\frac{1-\theta}{2}}(2\pi n \alpha) \quad (\operatorname{Re}(\theta) < 0).$$

Additionally, when the complex variable θ is not an odd number of -1 or less, the modified Bessel function of the second kind is given by the modified Bessel functions of the first kind. For the first part,

$$\begin{aligned} (4-15) \quad C_{[2]-1}(\theta) &= \\ &= \frac{\theta(\theta-1)}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} \frac{\pi}{2 \sin\left(\frac{1-\theta}{2}\pi\right)} \left(I_{\frac{-1+\theta}{2}}(2\pi n \alpha) - I_{\frac{1-\theta}{2}}(2\pi n \alpha) \right) \\ &= \frac{\theta(\theta-1)}{4 \sin\left(\frac{1-\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu + \frac{-1+\theta}{2}}}{\mu! \Gamma\left(\mu + \frac{-1+\theta}{2} + 1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu + \frac{1-\theta}{2}}}{\mu! \Gamma\left(\mu + \frac{1-\theta}{2} + 1\right)} \right) \\ &= \frac{\theta(\theta-1)\pi^{\frac{-1-\theta}{2}}}{4 \sin\left(\frac{1-\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\theta-1} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu}}{\mu! \Gamma\left(\mu + \frac{-1+\theta}{2} + 1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+1-\theta}}{\mu! \Gamma\left(\mu + \frac{1-\theta}{2} + 1\right)} \right) \\ &= \frac{\theta(\theta-1)\pi^{\frac{-1-\theta}{2}}}{4 \sin\left(\frac{1-\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\theta-1} \left(\frac{(\pi n \alpha)^0}{0! \Gamma\left(1 - \frac{1-\theta}{2}\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+2}}{(\mu+1)! \Gamma\left(\mu + \frac{1+\theta}{2} + 1\right)} \right. \\ &\quad \left. - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+1-\theta}}{\mu! \Gamma\left(\mu + \frac{1-\theta}{2} + 1\right)} \right) \quad (\operatorname{Re}(\theta) < 0, \theta \in \mathbb{C} \setminus \{1-2p : p \in \mathbb{N}\}). \end{aligned}$$

Now, using the following fact for calculation:

$$(4-16) \quad \lim_{\alpha \rightarrow +0} (\pi n \alpha)^\beta = \begin{cases} 1 & (\beta = 0, n \in \mathbb{N}), \\ 0 & (\operatorname{Re}(\beta) > 0, n \in \mathbb{N}). \end{cases}$$

Therefore

$$(4-17) \quad C_{[2]-1}(\theta) = \frac{\theta(\theta-1)\pi^{\frac{-1-\theta}{2}}}{4 \sin\left(\frac{1-\theta}{2}\pi\right) \Gamma\left(1 - \frac{1-\theta}{2}\right)} \sum_{n=1}^{\infty} n^{\theta-1} \quad (\operatorname{Re}(\theta) < 0, \theta \in \mathbb{C} \setminus \{1-2p : p \in \mathbb{N}\}).$$

By applying the reciprocal formula for the gamma function,

$$(4-18) \quad C_{[2]-1}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{\frac{-1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \quad (\operatorname{Re}(\theta) < 0, \theta \in \mathbb{C} \setminus \{1-2p : p \in \mathbb{N}\}).$$

For the second part,

$$\begin{aligned} (4-19) \quad C_{[2]-2}(\theta) &= \\ &= -\frac{\theta}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} (\pi n \alpha) \frac{\pi}{2 \sin\left(\frac{1+\theta}{2}\pi\right)} \left(I_{\frac{-1-\theta}{2}}(2\pi n \alpha) - I_{\frac{1+\theta}{2}}(2\pi n \alpha) \right) \\ &= -\frac{\theta}{2 \sin\left(\frac{1+\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} (\pi n \alpha) \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu - \frac{1+\theta}{2}}}{\mu! \Gamma\left(\mu - \frac{1+\theta}{2} + 1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu + \frac{1+\theta}{2}}}{\mu! \Gamma\left(\mu + \frac{1+\theta}{2} + 1\right)} \right) \\ &= -\frac{\theta\pi^{\frac{-1-\theta}{2}}}{2 \sin\left(\frac{1+\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\theta-1} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+1-\theta}}{\mu! \Gamma\left(\mu - \frac{1+\theta}{2} + 1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+2}}{\mu! \Gamma\left(\mu + \frac{1+\theta}{2} + 1\right)} \right) \\ &= 0 \quad (\operatorname{Re}(\theta) < 0, \theta \in \mathbb{C} \setminus \{1-2p : p \in \mathbb{N}\}). \end{aligned}$$

For the third part,

$$\begin{aligned}
 (4-20) \quad C_{[2]-3}(\theta) &= \\
 &= \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} (\pi n \alpha)^2 \frac{\pi}{2 \sin\left(\frac{1-\theta}{2} \pi\right)} \left(I^{\frac{-1+\theta}{2}}(2\pi n \alpha) - I^{\frac{1-\theta}{2}}(2\pi n \alpha) \right) \\
 &= \frac{1}{\sin\left(\frac{1-\theta}{2} \pi\right)} \lim_{\alpha \rightarrow +0} \alpha^{\frac{1-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{-1+\theta}{2}} (\pi n \alpha)^2 \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu + \frac{-1+\theta}{2}}}{\mu! \Gamma\left(\mu + \frac{-1+\theta}{2} + 1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu + \frac{1-\theta}{2}}}{\mu! \Gamma\left(\mu + \frac{1-\theta}{2} + 1\right)} \right) \\
 &= \frac{\pi^{-\frac{1-\theta}{2}}}{\sin\left(\frac{1-\theta}{2} \pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\theta-1} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+2}}{\mu! \Gamma\left(\mu + \frac{-1+\theta}{2} + 1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+3-\theta}}{\mu! \Gamma\left(\mu + \frac{1-\theta}{2} + 1\right)} \right) \\
 &= 0 \quad (\operatorname{Re}(\theta) < 0, \theta \in \mathbb{C} \setminus \{1-2p : p \in \mathbb{N}\}).
 \end{aligned}$$

By combining the above results,

$$(4-21) \quad R_{[2]}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \quad (\operatorname{Re}(\theta) < 0, \theta \in \mathbb{C} \setminus \{1-2p : p \in \mathbb{N}\}).$$

When the complex variable θ is any odd number of -1 or less, for the first part,

$$\begin{aligned}
 (4-22) \quad C_{[2]-1}(1-2p) &= \frac{(1-2p)(-2p)}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} K_p(2\pi n \alpha) \\
 &= \frac{p(2p-1)}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} \left((-1)^{p+1} I_p(2\pi n \alpha) \log(\pi n \alpha) \right. \\
 &\quad \left. + \frac{1}{2} \sum_{\mu=0}^{p-1} (-1)^\mu \frac{(p-\mu-1)!}{\mu!} (\pi n \alpha)^{2\mu-p} + \frac{(-1)^p}{2} \sum_{\mu=0}^{\infty} \frac{\psi(\mu+p+1) + \psi(\mu+1)}{(p+\mu)! \mu!} (\pi n \alpha)^{p+2\mu} \right) \\
 &= \frac{p(2p-1)}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} \left((-1)^{p+1} \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+p}}{\mu! \Gamma(\mu+p+1)} \log(\pi n \alpha) \right. \\
 &\quad \left. + \frac{1}{2} \sum_{\mu=0}^{p-1} (-1)^\mu \frac{(p-\mu-1)!}{\mu!} (\pi n \alpha)^{2\mu-p} + \frac{(-1)^p}{2} \sum_{\mu=0}^{\infty} \frac{\psi(\mu+p+1) + \psi(\mu+1)}{(p+\mu)! \mu!} (\pi n \alpha)^{p+2\mu} \right) \\
 &= \frac{p(2p-1)\pi^{-p}}{2\pi} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{-2p} \left(\sum_{\mu=0}^{p-1} (-1)^\mu \frac{(p-\mu-1)!}{\mu!} (\pi n \alpha)^{2\mu} \right. \\
 &\quad \left. + (-1)^{p+1} \sum_{\mu=0}^{\infty} \frac{2 \log(\pi n \alpha) - \psi(\mu+p+1) - \psi(\mu+1)}{\mu! \Gamma(\mu+p+1)} (\pi n \alpha)^{2\mu+2p} \right) \quad (p \in \mathbb{N}).
 \end{aligned}$$

When the complex variable θ is any odd number of -3 or less,

$$\begin{aligned}
 (4-23) \quad C_{[2]-1}(1-2p) &= \\
 &= \frac{p(2p-1)\pi^{-p}}{2\pi} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{-2p} \left((-1)^0 \frac{(p-1)!}{0!} (\pi n \alpha)^0 + \sum_{\mu=1}^{p-1} (-1)^\mu \frac{(p-\mu-1)!}{\mu!} (\pi n \alpha)^{2\mu} \right) \\
 &= \frac{p!(2p-1)\pi^{-p}}{2\pi} \zeta(2p) \quad (p \in \mathbb{N} \setminus \{1\}).
 \end{aligned}$$

When the complex variable θ is equal to -1 ,

$$(4-24) \quad C_{[2]-1}(-1) = \frac{\pi^{-2}}{2} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{-2} \sum_{\mu=0}^0 (-1)^\mu \frac{(1-\mu-1)!}{\mu!} (\pi n \alpha)^{2\mu} = \frac{\pi^{-2}}{2} \zeta(2) = \frac{1}{12}.$$

For the result of equation (4-23), 1 is substituted directly for p .

$$(4-25) \quad \frac{p!(2p-1)\pi^{-p}}{2\pi} \zeta(2p) \Big|_{p=1} = \frac{1!\pi^{-1}}{2\pi} \zeta(2) = \frac{1}{12}.$$

This result is the same to the result of equation (4-24); hence, they can be combined.

$$(4-26) \quad C_{[2]-1}(1-2p) = \frac{p!(2p-1)\pi^{-p}}{2\pi} \zeta(2p) \quad (p \in \mathbb{N}).$$

For the second part,

$$(4-27) \quad C_{[2]-2}(1-2p) = -\frac{1-2p}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} (\pi n \alpha) K_{p-1}(2\pi n \alpha) \quad (p \in \mathbb{N}).$$

When the complex variable θ is any odd number of -3 or less,

$$\begin{aligned} (4-28) \quad C_{[2]-2}(1-2p) &= \frac{2p-1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} (\pi n \alpha) \left((-1)^p I_{p-1}(2\pi n \alpha) \log(\pi n \alpha) \right. \\ &\quad \left. + \frac{1}{2} \sum_{\mu=0}^{p-2} (-1)^\mu \frac{(p-\mu-2)!}{\mu!} (\pi n \alpha)^{2\mu-p+1} + \frac{(-1)^{p-1}}{2} \sum_{\mu=0}^{\infty} \frac{\psi(\mu+p)+\psi(\mu+1)}{(p+\mu-1)!\mu!} (\pi n \alpha)^{p+2\mu-1} \right) \\ &= \frac{2p-1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} (\pi n \alpha) \left((-1)^p \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+p-1}}{\mu! \Gamma(\mu+p)} \log(\pi n \alpha) \right. \\ &\quad \left. + \frac{1}{2} \sum_{\mu=0}^{p-2} (-1)^\mu \frac{(p-\mu-2)!}{\mu!} (\pi n \alpha)^{2\mu-p+1} + \frac{(-1)^{p-1}}{2} \sum_{\mu=0}^{\infty} \frac{\psi(\mu+p)+\psi(\mu+1)}{(p+\mu-1)!\mu!} (\pi n \alpha)^{p+2\mu-1} \right) \\ &= \frac{(2p-1)\pi^{-p}}{2\pi} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{-2p} \left((-1)^p \sum_{\mu=0}^{\infty} \frac{2 \log(\pi n \alpha) - \psi(\mu+p) - \psi(\mu+1)}{\mu! \Gamma(\mu+p)} (\pi n \alpha)^{2\mu+2p} \right. \\ &\quad \left. + \sum_{\mu=0}^{p-2} (-1)^\mu \frac{(p-\mu-2)!}{\mu!} (\pi n \alpha)^{2\mu+2} \right) = 0 \quad (p \in \mathbb{N} \setminus \{1\}). \end{aligned}$$

When the complex variable θ is equal to -1 ,

$$\begin{aligned} (4-29) \quad C_{[2]-2}(-1) &= \pi^{-2} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{-2} (\pi n \alpha)^2 K_0(2\pi n \alpha) \\ &= \pi^{-2} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{-2} (\pi n \alpha)^2 \left((-\gamma + \log(2) - \log(2\pi n \alpha)) \right. \\ &\quad \left. + \frac{1}{4} (1 - \gamma + \log(2) - \log(2\pi n \alpha)) (2\pi n \alpha)^2 + O(2\pi n \alpha)^4 \right) = 0. \end{aligned}$$

Here, I used the asymptotic formula for the function $K_0(Z)$.

For the third part,

$$\begin{aligned} (4-30) \quad C_{[2]-3}(1-2p) &= \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} (\pi n \alpha)^2 K_p(2\pi n \alpha) \\ &= \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} (\pi n \alpha)^2 \left((-1)^{p+1} \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+p}}{\mu! \Gamma(\mu+p+1)} \log(\pi n \alpha) \right. \\ &\quad \left. + \frac{1}{2} \sum_{\mu=0}^{p-1} (-1)^\mu \frac{(p-\mu-1)!}{\mu!} (\pi n \alpha)^{2\mu-p} + \frac{(-1)^p}{2} \sum_{\mu=0}^{\infty} \frac{\psi(\mu+p+1)+\psi(\mu+1)}{(p+\mu)!\mu!} (\pi n \alpha)^{p+2\mu} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{\pi^{-p}}{\pi} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{-2p} \left((-1)^{p+1} \sum_{\mu=0}^{\infty} \frac{2 \log(\pi n \alpha) - \psi(\mu+p+1) - \psi(\mu+1)}{\mu! \Gamma(\mu+p+1)} (\pi n \alpha)^{2\mu+2p+2} \right. \\
 &\quad \left. + \sum_{\mu=0}^{p-1} (-1)^{\mu} \frac{(p-\mu-1)!}{\mu!} (\pi n \alpha)^{2\mu+2} \right) = 0 \quad (p \in \mathbb{N}).
 \end{aligned}$$

For the right side of equation (4-21), $(1-2p)$ is substituted directly for θ .

$$\begin{aligned}
 (4-31) \quad &\frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \Big|_{\theta=1-2p} = \frac{(1-2p)(-2p)}{4\pi} \pi^{-p} \Gamma(p) \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \\
 &= \frac{p!(2p-1)\pi^{-p}}{2\pi} \zeta(2p) \quad (p \in \mathbb{N}).
 \end{aligned}$$

This result is the same to the result of equation (4-26); hence, they can be combined.

$$(4-32) \quad \mathbf{R}_{[2]}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \quad (\operatorname{Re}(\theta) < 0).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(4-33) \quad \mathbf{R}_{[2]}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

The function $\mathbf{R}_{[2]}(\theta)$ is also a convergent function in the whole complex plane. Thus, the following equation holds true:

$$(4-34) \quad \mathbf{L}_{[2]}(\theta) = \mathbf{R}_{[2]}(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(4-35) \quad \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

I decide that the functional equation (4-35) is called the second-order I_c type functional equation. Suffix c means “complex.” This rule is also hereafter applicable.

It is equivalent to the functional equation that the Riemann Xi function follows.

5. Functional Equation Transformation, Part2

The following are the operations of the functional equation transformation for the second-order Π_r type functional equation:

$$(5-1) \quad \mathbf{A}_{[2]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (-3n^2 x^{\theta+1} + 2\pi n^4 x^{\theta+3}) e^{-\pi(n^2 + \alpha^2)x^2} dx.$$

$$(5-2) \quad \mathbf{F}_{[2]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2 x^{\theta+1} + 4\pi n^4 x^{\theta+3}) e^{-\pi\left((n^2 + \alpha^2)x^2 + \frac{m^2}{x^2}\right)} dx.$$

$$(5-3) \quad \mathbf{B}_{[2]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (-3n^2 x^{\theta-3} + 2\pi n^4 x^{\theta-5}) e^{-\pi\left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.$$

$$(5-4) \quad \mathbf{G}_{[2]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2 x^{\theta-3} + 4\pi n^4 x^{\theta-5}) e^{-\pi\left((m^2 + \alpha^2)x^2 + \frac{n^2}{x^2}\right)} dx.$$

Immediately, the following obvious result is obtained:

$$(5-5) \mathbf{A}_{[2]}(\theta) = \mathbf{L}_{[2]}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

For the integral of $\mathbf{B}_{[2]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$(5-6) \mathbf{B}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-3n^2 \left(\frac{1}{y}\right)^{\theta-3} + 2\pi n^4 \left(\frac{1}{y}\right)^{\theta-5} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} (-y^{-2}) dy$$

$$= \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-3n^2 y^{1-\theta} + 2\pi n^4 y^{3-\theta} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} dy.$$

Assuming that the integral and the sum can be interchanged, for the integrals of $\mathbf{B}_{[2]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{\alpha x}{n}\right)^{\frac{1}{2}}.$$

$$(5-7) \mathbf{B}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \int_0^{\infty} \left(-3n^2 \left(\frac{\alpha x}{n}\right)^{\frac{1-\theta}{2}} + 2\pi n^4 \left(\frac{\alpha x}{n}\right)^{\frac{3-\theta}{2}} \right) e^{-\pi n \alpha \left(x + \frac{1}{x}\right)} \frac{1}{2} \left(\frac{\alpha}{n}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx$$

$$= \lim_{\alpha \rightarrow +0} \alpha^{\frac{2-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{2+\theta}{2}} \left(-3 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi n \alpha}{2} \left(x + \frac{1}{x}\right)} dx \right.$$

$$\left. + 2(\pi n \alpha) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi n \alpha}{2} \left(x + \frac{1}{x}\right)} dx \right).$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(5-8) \mathbf{B}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \alpha^{\frac{2-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{2+\theta}{2}} \left(-3 K_{\frac{2-\theta}{2}}(2\pi n \alpha) + 2(\pi n \alpha) K_{\frac{4-\theta}{2}}(2\pi n \alpha) \right).$$

Because any modified Bessel function of the second kind of the sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$(5-9) \mathbf{B}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \alpha^{\frac{2-\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{2+\theta}{2}} \left(-3 K_{\frac{2-\theta}{2}}(2\pi n \alpha) \right.$$

$$\left. + 2 \left(\frac{2-\theta}{2} K_{\frac{2-\theta}{2}}(2\pi n \alpha) + (\pi n \alpha) K_{\frac{-\theta}{2}}(2\pi n \alpha) \right) \right)$$

$$= \frac{1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \left(-(\theta+1)(\pi n \alpha) K_{\frac{2-\theta}{2}}(2\pi n \alpha) + 2(\pi n \alpha)^2 K_{\frac{\theta}{2}}(2\pi n \alpha) \right)$$

$$= \frac{1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \left(-(\theta+1) \left(\frac{-\theta}{2} K_{\frac{-\theta}{2}}(2\pi n \alpha) + (\pi n \alpha) K_{\frac{-2-\theta}{2}}(2\pi n \alpha) \right) \right.$$

$$\left. + 2(\pi n \alpha)^2 K_{\frac{\theta}{2}}(2\pi n \alpha) \right)$$

$$= \frac{1}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \left(\theta(\theta+1) K_{\frac{\theta}{2}}(2\pi n \alpha) \right.$$

$$\left. - 2(\theta+1)(\pi n \alpha) K_{\frac{2+\theta}{2}}(2\pi n \alpha) + 4(\pi n \alpha)^2 K_{\frac{\theta}{2}}(2\pi n \alpha) \right).$$

For convenience, I separate the result of equation (5-9) into three parts adding the condition $\text{Re}(\theta) < -1$.

$$(5-10) \mathbf{D}_{[2]-1}(\theta) := \frac{\theta(\theta+1)}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi n \alpha) \quad (\text{Re}(\theta) < -1).$$

$$(5-11) \mathbf{D}_{[2]-2}(\theta) := -\frac{\theta+1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} (\pi n \alpha) K_{\frac{2+\theta}{2}}(2\pi n \alpha) \quad (\text{Re}(\theta) < -1).$$

$$(5-12) \quad \mathbf{D}_{[2]-3}(\theta) := \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} (\pi n \alpha)^2 \mathbf{K}_{\frac{\theta}{2}}(2\pi n \alpha) \quad (\operatorname{Re}(\theta) < -1).$$

Furthermore, when the complex variable θ is not an even number of -2 or less, the modified Bessel function of the second kind is provided using the modified Bessel functions of the first kind. For the first part,

$$\begin{aligned} (5-13) \quad \mathbf{D}_{[2]-1}(\theta) &= \frac{\theta(\theta+1)}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \frac{\pi}{2 \sin\left(\frac{\theta}{2}\pi\right)} \left(\mathbf{I}_{-\frac{\theta}{2}}(2\pi n \alpha) - \mathbf{I}_{\frac{\theta}{2}}(2\pi n \alpha) \right) \\ &= \frac{\theta(\theta+1)}{4 \sin\left(\frac{\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu-\frac{\theta}{2}}}{\mu! \Gamma\left(\mu-\frac{\theta}{2}+1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+\frac{\theta}{2}}}{\mu! \Gamma\left(\mu+\frac{\theta}{2}+1\right)} \right) \\ &= \frac{\theta(\theta+1)\pi^{\frac{\theta}{2}}}{4 \sin\left(\frac{\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\theta} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu-\theta}}{\mu! \Gamma\left(\mu-\frac{\theta}{2}+1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu}}{\mu! \Gamma\left(\mu+\frac{\theta}{2}+1\right)} \right) \\ &= \frac{\theta(\theta+1)\pi^{\frac{\theta}{2}}}{4 \sin\left(\frac{\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\theta} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu-\theta}}{\mu! \Gamma\left(\mu-\frac{\theta}{2}+1\right)} - \frac{(\pi n \alpha)^0}{0! \Gamma\left(1+\frac{\theta}{2}\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+2}}{(\mu+1)! \Gamma\left(\mu+\frac{\theta}{2}+2\right)} \right) \\ &= \frac{\theta(\theta+1)\pi^{\frac{\theta}{2}}}{4 \sin\left(-\frac{\theta}{2}\pi\right) \Gamma\left(1-\left(-\frac{\theta}{2}\right)\right)} \sum_{n=1}^{\infty} n^{\theta} \quad (\operatorname{Re}(\theta) < -1, \theta \in \mathbb{C} \setminus \{-2p : p \in \mathbb{N}\}). \end{aligned}$$

By applying the reciprocal formula for the gamma function,

$$(5-14) \quad \mathbf{D}_{[2]-1}(\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \quad (\operatorname{Re}(\theta) < -1, \theta \in \mathbb{C} \setminus \{-2p : p \in \mathbb{N}\}).$$

For the second part,

$$\begin{aligned} (5-15) \quad \mathbf{D}_{[2]-2}(\theta) &= \\ &= -\frac{\theta+1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} (\pi n \alpha) \frac{\pi}{2 \sin\left(\frac{2+\theta}{2}\pi\right)} \left(\mathbf{I}_{-\frac{2+\theta}{2}}(2\pi n \alpha) - \mathbf{I}_{\frac{2+\theta}{2}}(2\pi n \alpha) \right) \\ &= -\frac{\theta+1}{2 \sin\left(\frac{2+\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} (\pi n \alpha) \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu-\frac{2+\theta}{2}}}{\mu! \Gamma\left(\mu-\frac{2+\theta}{2}+1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+\frac{2+\theta}{2}}}{\mu! \Gamma\left(\mu+\frac{2+\theta}{2}+1\right)} \right) \\ &= -\frac{(\theta+1)\pi^{\frac{\theta}{2}}}{2 \sin\left(\frac{2+\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\theta} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu-\theta}}{\mu! \Gamma\left(\mu-\frac{\theta}{2}\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+2}}{\mu! \Gamma\left(\mu+\frac{\theta}{2}+2\right)} \right) \\ &= 0 \quad (\operatorname{Re}(\theta) < -1, \theta \in \mathbb{C} \setminus \{-2p : p \in \mathbb{N}\}). \end{aligned}$$

For the third part,

$$\begin{aligned} (5-16) \quad \mathbf{D}_{[2]-3}(\theta) &= \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \alpha^{-\frac{\theta}{2}} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} (\pi n \alpha)^2 \frac{\pi}{2 \sin\left(\frac{\theta}{2}\pi\right)} \left(\mathbf{I}_{-\frac{\theta}{2}}(2\pi n \alpha) - \mathbf{I}_{\frac{\theta}{2}}(2\pi n \alpha) \right) \\ &= \frac{\pi^{\frac{\theta}{2}}}{\sin\left(\frac{\theta}{2}\pi\right)} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^{\frac{\theta}{2}} \left(\sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu-\theta+2}}{\mu! \Gamma\left(\mu-\frac{\theta}{2}+1\right)} - \sum_{\mu=0}^{\infty} \frac{(\pi n \alpha)^{2\mu+2}}{\mu! \Gamma\left(\mu+\frac{\theta}{2}+1\right)} \right) \\ &= 0 \quad (\operatorname{Re}(\theta) < -1, \theta \in \mathbb{C} \setminus \{-2p : p \in \mathbb{N}\}). \end{aligned}$$

By combining the above results,

$$(5-17) \quad \mathbf{B}_{[2]}(\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \quad (\operatorname{Re}(\theta) < -1, \theta \in \mathbb{C} \setminus \{-2p : p \in \mathbb{N}\}).$$

When the complex variable θ is any even number of -2 or less, for the first part,

$$(5-18) \quad \mathbf{D}_{[2]-1}(-2p) = \frac{(-2p)(-2p+1)}{2\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} K_{-p}(2\pi n \alpha)$$

$$= \mathbf{C}_{[2]-1}(1-2p) = \frac{p!(2p-1)\pi^{-p}}{2\pi} \zeta(2p) \quad (p \in \mathbb{N}).$$

For the second part,

$$(5-19) \quad \mathbf{D}_{[2]-2}(-2p) = -\frac{-2p+1}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} (\pi n \alpha) K_{1-p}(2\pi n \alpha)$$

$$= \mathbf{C}_{[2]-2}(1-2p) = 0 \quad (p \in \mathbb{N}).$$

For the third part,

$$(5-20) \quad \mathbf{D}_{[2]-3}(-2p) = \frac{2}{\pi} \lim_{\alpha \rightarrow +0} \alpha^p \sum_{n=1}^{\infty} n^{-p} (\pi n \alpha)^2 K_{-p}(2\pi n \alpha)$$

$$= \mathbf{C}_{[2]-3}(1-2p) = 0 \quad (p \in \mathbb{N}).$$

For the right side of equation (5-17), $(-2p)$ is substituted directly for θ .

$$(5-21) \quad \frac{\theta(\theta+1)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \Big|_{\theta=-2p} = \frac{(-2p)(-2p+1)}{4\pi} \pi^{-p} \Gamma(p) \sum_{n=1}^{\infty} \frac{1}{n^{2p}}$$

$$= \frac{p!(2p-1)\pi^{-p}}{2\pi} \zeta(2p) \quad (p \in \mathbb{N}).$$

This result is the same to the result of equation (5-18), so one equation can be presented.

$$(5-22) \quad \mathbf{B}_{[2]}(\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \quad (\operatorname{Re}(\theta) < -1).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(5-23) \quad \mathbf{B}_{[2]}(\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) \quad (\theta \in \mathbb{C}).$$

The function $\mathbf{B}_{[2]}(\theta)$ is also a convergent function in the whole complex plane. And the following origin symmetry holds:

$$(5-24) \quad \mathbf{A}_{[2]}(-\theta) = \mathbf{B}_{[2]}(\theta) \quad (\theta \in \mathbb{C}).$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $\mathbf{F}_{[2]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{m y}{\sqrt{n^2 + \alpha^2}} \right)^{\frac{1}{2}}.$$

$$(5-25) \quad \mathbf{F}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{\infty} \left[-6n^2 \left(\frac{m y}{\sqrt{n^2 + \alpha^2}} \right)^{\frac{\theta+1}{2}} \right.$$

$$\left. + 4\pi n^4 \left(\frac{m y}{\sqrt{n^2 + \alpha^2}} \right)^{\frac{\theta+3}{2}} \right] e^{-\pi m \sqrt{n^2 + \alpha^2} \left(y + \frac{1}{y} \right)} \frac{1}{2} \left(\frac{m}{\sqrt{n^2 + \alpha^2}} \right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy$$

$$\begin{aligned}
 &= \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-6 n^2 \left(\frac{m}{\sqrt{n^2 + \alpha^2}} \right)^{\frac{\theta+2}{2}} \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta}{2}} e^{-\pi m \sqrt{n^2 + \alpha^2} \left(y + \frac{1}{y} \right)} dy \right. \\
 &\quad \left. + 4 \pi n^4 \left(\frac{m}{\sqrt{n^2 + \alpha^2}} \right)^{\frac{\theta+4}{2}} \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+2}{2}} e^{-\pi m \sqrt{n^2 + \alpha^2} \left(y + \frac{1}{y} \right)} dy \right).
 \end{aligned}$$

Using the left-sided limit of the positive real variable α ,

$$\begin{aligned}
 (5-26) \quad F_{[2]}(\theta) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-6 n^2 \left(\frac{m}{n} \right)^{\frac{\theta+2}{2}} \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta}{2}} e^{-\pi m n \left(y + \frac{1}{y} \right)} dy \right. \\
 &\quad \left. + 4 \pi n^4 \left(\frac{m}{n} \right)^{\frac{\theta+4}{2}} \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+2}{2}} e^{-\pi m n \left(y + \frac{1}{y} \right)} dy \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n} \right)^{\frac{\theta}{2}} \left(-3 \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+2}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y} \right)} dy \right. \\
 &\quad \left. + 2 (\pi m n) \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+4}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y} \right)} dy \right).
 \end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(5-27) \quad F_{[2]}(\theta) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n} \right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) + 2 (\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) \right).$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$\begin{aligned}
 (5-28) \quad F_{[2]}(\theta) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n} \right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) \right. \\
 &\quad \left. + 2 \left(\frac{\theta+2}{2} K_{\frac{\theta+2}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta}{2}}(2\pi m n) \right) \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n} \right)^{\frac{\theta}{2}} \left((\theta-1) (\pi m n) K_{\frac{\theta+2}{2}}(2\pi m n) + 2 (\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n) \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n} \right)^{\frac{\theta}{2}} \left((\theta-1) \left(\frac{\theta}{2} K_{\frac{\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta-2}{2}}(2\pi m n) \right) + 2 (\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n) \right) \\
 &= \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n} \right)^{\frac{\theta}{2}} \left(\theta (\theta-1) K_{\frac{\theta}{2}}(2\pi m n) \right. \\
 &\quad \left. + (\theta-1) (2\pi m n) K_{\frac{\theta-2}{2}}(2\pi m n) + (2\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n) \right).
 \end{aligned}$$

For the integral of $G_{[2]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned}
 (5-29) \quad G_{[2]}(\theta) &= \lim_{\alpha \rightarrow +0} \int_{\infty}^0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-6 n^2 \left(\frac{1}{y} \right)^{\theta-3} \right. \\
 &\quad \left. + 4 \pi n^4 \left(\frac{1}{y} \right)^{\theta-5} \right) e^{-\pi \left(\frac{m^2 + \alpha^2}{y^2} + n^2 y^2 \right)} (-y^{-2}) dy
 \end{aligned}$$

$$= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty (-6 n^2 y^{1-\theta} + 4 \pi n^4 y^{3-\theta}) e^{-\pi \left(n^2 y^2 + \frac{m^2 + \alpha^2}{y^2} \right)} dy.$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $G_{[2]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{\sqrt{m^2 + \alpha^2} x}{n} \right)^{\frac{1}{2}}.$$

$$\begin{aligned} (5-30) \quad G_{[2]}(\theta) &= \lim_{\alpha \rightarrow +0} \sum_{n=1}^\infty \sum_{m=1}^\infty \int_0^\infty \left(-6 n^2 \left(\frac{\sqrt{m^2 + \alpha^2} x}{n} \right)^{\frac{1-\theta}{2}} \right. \\ &\quad \left. + 4 \pi n^4 \left(\frac{\sqrt{m^2 + \alpha^2} x}{n} \right)^{\frac{3-\theta}{2}} \right) e^{-\pi \sqrt{m^2 + \alpha^2} n \left(x + \frac{1}{x} \right)} \frac{1}{2} \left(\frac{\sqrt{m^2 + \alpha^2}}{n} \right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx \\ &= \lim_{\alpha \rightarrow +0} \sum_{n=1}^\infty \sum_{m=1}^\infty \left(-6 n^2 \left(\frac{\sqrt{m^2 + \alpha^2}}{n} \right)^{\frac{2-\theta}{2}} \cdot \frac{1}{2} \int_0^\infty x^{-\frac{\theta}{2}} e^{-\pi \sqrt{m^2 + \alpha^2} n \left(x + \frac{1}{x} \right)} dx \right. \\ &\quad \left. + 4 \pi n^4 \left(\frac{\sqrt{m^2 + \alpha^2}}{n} \right)^{\frac{4-\theta}{2}} \cdot \frac{1}{2} \int_0^\infty x^{\frac{2-\theta}{2}} e^{-\pi \sqrt{m^2 + \alpha^2} n \left(x + \frac{1}{x} \right)} dx \right). \end{aligned}$$

Using the left-sided limit of the positive real variable α ,

$$\begin{aligned} (5-31) \quad G_{[2]}(\theta) &= \sum_{n=1}^\infty \sum_{m=1}^\infty \left(-6 n^2 \left(\frac{m}{n} \right)^{\frac{2-\theta}{2}} \cdot \frac{1}{2} \int_0^\infty x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right. \\ &\quad \left. + 4 \pi n^4 \left(\frac{m}{n} \right)^{\frac{4-\theta}{2}} \cdot \frac{1}{2} \int_0^\infty x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right) \\ &= \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n} \right)^{-\frac{\theta}{2}} \left(-3 \cdot \frac{1}{2} \int_0^\infty x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right. \\ &\quad \left. + 2 (\pi m n) \cdot \frac{1}{2} \int_0^\infty x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right). \end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(5-32) \quad G_{[2]}(\theta) = \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n} \right)^{-\frac{\theta}{2}} \left(-3 K_{\frac{2-\theta}{2}}(2\pi m n) + 2 (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n) \right).$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$\begin{aligned} (5-33) \quad G_{[2]}(\theta) &= \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n} \right)^{-\frac{\theta}{2}} \left(-3 K_{\frac{2-\theta}{2}}(2\pi m n) \right. \\ &\quad \left. + 2 \left(\frac{2-\theta}{2} K_{\frac{2-\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{-\theta}{2}}(2\pi m n) \right) \right) \\ &= \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty \left(\frac{m}{n} \right)^{-\frac{\theta}{2}} \left(-(\theta+1) (\pi m n) K_{\frac{2-\theta}{2}}(2\pi m n) + 2 (\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n) \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-(\theta+1)\left(\frac{-\theta}{2} K_{\frac{-\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{-\theta-2}{2}}(2\pi m n)\right)\right. \\
 &\quad \left.+ 2(\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n)\right) \\
 &= \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(\theta(\theta+1) K_{\frac{\theta}{2}}(2\pi m n) - (\theta+1)(2\pi m n) K_{\frac{\theta+2}{2}}(2\pi m n)\right. \\
 &\quad \left.+ (2\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n)\right).
 \end{aligned}$$

The following origin symmetry holds:

$$(5-34) \quad \mathbf{F}_{[2]}(\pm\theta) = \mathbf{G}_{[2]}(\mp\theta) \quad (\text{the double sign is in same order}).$$

Now, I define the set of lattice points on the hyperbola $xy = p$ in the first quadrant as follows:

$$(5-35) \quad \Lambda_p := \{(m, n) : m, n \in \mathbb{N}, mn = p\}.$$

From the definition,

$$\begin{aligned}
 (5-36) \quad &\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty} \sum_{(m,n) \in \Lambda_p} \left(\frac{n^2}{p}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty} \sum_{(m,n) \in \Lambda_p} n^{\theta} p^{-\frac{\theta}{2}} \\
 &= \sum_{p=1}^{\infty} \sigma_{\theta}(p) p^{-\frac{\theta}{2}} \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

Here, $\sigma_{\theta}(p)$ is the divisor sigma function. Conversely,

$$\begin{aligned}
 (5-37) \quad &\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty} \sum_{(m,n) \in \Lambda_p} \left(\frac{p}{n^2}\right)^{\frac{\theta}{2}} = \sum_{p=1}^{\infty} \sum_{(m,n) \in \Lambda_p} n^{-\theta} p^{\frac{\theta}{2}} \\
 &= \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

$$\begin{aligned}
 (5-38) \quad &\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} = \sum_{p=1}^{\infty} \sum_{(m,n) \in \Lambda_p} \left(\frac{m^2}{p}\right)^{-\frac{\theta}{2}} = \sum_{p=1}^{\infty} \sum_{(m,n) \in \Lambda_p} m^{-\theta} p^{\frac{\theta}{2}} \\
 &= \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

The left sides of equations (5-36) and (5-37) are equivalent due to axial symmetry between the points (m, n) and (n, m) .

And I obtain the following formula with the origin symmetry for the complex variable θ :

$$(5-39) \quad \sigma_{-\theta}(p) p^{\frac{\theta}{2}} = \sigma_{\theta}(p) p^{-\frac{\theta}{2}} \quad (p \in \mathbb{N}, \theta \in \mathbb{C}).$$

From the equations (5-28) and (5-37),

$$\begin{aligned}
 (5-40) \quad &\mathbf{F}_{[2]}(\theta) = \\
 &\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(-\theta(1-\theta) + (2\pi p)^2\right) K_{\frac{\theta}{2}}(2\pi p) - (1-\theta)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p).
 \end{aligned}$$

From the equations (5-33) and (5-38),

$$(5-41) \quad \mathbf{G}_{[2]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta(1+\theta) + (2\pi p)^2\right) K_{\frac{\theta}{2}}(2\pi p) - (1+\theta)(2\pi p) K_{\frac{2+\theta}{2}}(2\pi p).$$

The same term appears on both the right sides of equations (5-40) and (5-41). I define the following equations to reduce it:

$$(5-42) \quad \hat{F}_{[2]}(\theta) := F_{[2]}(\theta) - \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} (2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p).$$

$$(5-43) \quad \hat{G}_{[2]}(\theta) := G_{[2]}(\theta) - \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} (2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p).$$

Therefore,

$$(5-44) \quad \hat{F}_{[2]}(\theta) = \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right).$$

$$(5-45) \quad \hat{G}_{[2]}(\theta) = \frac{-\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(-\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2+\theta}{2}}(2\pi p) \right).$$

For the right side of equation (5-44), the recurrence formula for the modified Bessel function of the second kind is applied.

$$(5-46) \quad \begin{aligned} \hat{F}_{[2]}(\theta) &= \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta K_{\frac{\theta}{2}}(2\pi p) + 2 \left(\frac{-\theta}{2} K_{\frac{-\theta}{2}}(2\pi p) + (\pi p) K_{\frac{-\theta-2}{2}}(2\pi p) \right) \right) \\ &= \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} (2\pi p) K_{\frac{2+\theta}{2}}(2\pi p). \end{aligned}$$

By changing to the integral representation,

$$(5-47) \quad \hat{F}_{[2]}(\theta) = \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} (2\pi p) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{\theta+2}{2}-1} e^{-\frac{2\pi p}{2} \left(x + \frac{1}{x}\right)} dx.$$

For the integral of $\hat{F}_{[2]}(\theta)$, I perform the variable transformation

$$x = \frac{y^2}{\pi p}.$$

$$(5-48) \quad \begin{aligned} \hat{F}_{[2]}(\theta) &= \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} (2\pi p) \cdot \frac{1}{2} \int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta}{2}} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \\ &= \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} (2\pi p) \left(\frac{1}{\pi p}\right)^{\frac{\theta+2}{2}} \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy. \end{aligned}$$

Therefore,

$$(5-49) \quad \hat{F}_{[2]}(\theta) = \frac{2(\theta-1)\pi^{-\frac{\theta}{2}}}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy.$$

For the right side of equation (5-45), I change to the integral representations.

$$(5-50) \quad \begin{aligned} \hat{G}_{[2]}(\theta) &= \frac{\theta+1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{\theta-2}{2}} e^{-\pi p \left(x + \frac{1}{x}\right)} dx \right. \\ &\quad \left. - (2\pi p) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{\theta}{2}} e^{-\pi p \left(x + \frac{1}{x}\right)} dx \right). \end{aligned}$$

For the integrals of $\hat{G}_{[2]}(\theta)$, I perform the variable transformation

$$x = \frac{y^2}{\pi p}.$$

$$(5-51) \quad \hat{G}_{[2]}(\theta) = \frac{\theta+1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta \cdot \frac{1}{2} \int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta-2}{2}} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \right. \\ \left. - (2\pi p) \cdot \frac{1}{2} \int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta}{2}} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \right)$$

Therefore,

$$(5-52) \quad \hat{G}_{[2]}(\theta) = \frac{(\theta+1)\pi^{-\frac{\theta}{2}}}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(\theta \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy - 2 \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right).$$

The following equations are defined to evaluate the absolute convergence of the functions $F_{[2]}-\hat{\text{hat}}(\theta)$ and $G_{[2]}-\hat{\text{hat}}(\theta)$:

$$(5-53) \quad S_{\pm}(\theta) :=$$

$$\sum_{p=1}^{\infty} \sigma_{-\theta}(p) \int_0^{\infty} y^{\pm 1 + \text{Re}(\theta)} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (\text{the double sign is in same order}).$$

For both sides of equation (5-49), the absolute value is taken.

$$(5-54) \quad |\hat{F}_{[2]}(\theta)| \leq \frac{2|\theta-1|\pi^{-\frac{\text{Re}(\theta)}{2}}}{\pi} |S_{+}(\theta)|.$$

Similarly, for both sides of equation (5-52), the absolute value is taken.

$$(5-55) \quad |\hat{G}_{[2]}(\theta)| \leq \frac{|\theta+1|\pi^{-\frac{\text{Re}(\theta)}{2}}}{\pi} (|\theta| \cdot |S_{-}(\theta)| + 2 |S_{+}(\theta)|).$$

Furthermore, for both sides of equations (5-53), the absolute values are taken.

$$(5-56) \quad |S_{\pm}(\theta)| \leq$$

$$\sum_{p=1}^{\infty} \sigma_{-\text{Re}(\theta)}(p) \int_0^{\infty} y^{\pm 1 + \text{Re}(\theta)} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (\text{the double sign is in same order}).$$

Now, the following obvious fact is introduced into the inequalities (5-56):

$$(5-57) \quad \sigma_{-\text{Re}(\theta)}(p) < p^2 \zeta(2 + \text{Re}(\theta)) \quad (\text{Re}(\theta) > -1, p \in \mathbb{N}).$$

The following proposition decides that the condition should be $\text{Re}(\theta) > -1$:

$$(5-58) \quad 2 + \text{Re}(\theta) > 1 \implies \zeta(2 + \text{Re}(\theta)) > 1 \quad (\theta \in \mathbb{C}).$$

Under the condition of $\text{Re}(\theta) > -1$, the absolute value of $S_{+}(\theta)$ is written using the absolutely convergent integrals.

$$(5-59) \quad |S_{+}(\theta)| < \zeta(2 + \text{Re}(\theta)) \sum_{p=1}^{\infty} p^2 \int_0^{\infty} y^{1 + \text{Re}(\theta)} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (\text{Re}(\theta) > -1).$$

Because the variable of the zeta function is irrelevant to variable p , the zeta function of the infinite sum is shifted outside.

Under the condition of $\text{Re}(\theta) > 0$, the absolute value of $S_{-}(\theta)$ is also written using the absolutely convergent integrals.

$$(5-60) \quad |S_-(\theta)| < \zeta(2 + \operatorname{Re}(\theta)) \sum_{p=1}^{\infty} p^2 \int_0^{\infty} y^{-1+\operatorname{Re}(\theta)} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (\operatorname{Re}(\theta) > 0).$$

Here, the condition $\operatorname{Re}(\theta) > 0$ is applied to the gamma function described later.

Because each integral converges absolutely, I can change the order of relation between the infinite sum and the integrals for the function $S_+(\theta)$. And the infinite sum is written as the limit of the L -th partial sum.

$$(5-61) \quad |S_+(\theta)| < \zeta(2 + \operatorname{Re}(\theta)) \int_0^{\infty} y^{1+\operatorname{Re}(\theta)} e^{-y^2} \lim_{L \rightarrow \infty} \sum_{p=1}^L p^2 e^{-\frac{(\pi p)^2}{y^2}} dy \quad (\operatorname{Re}(\theta) > -1).$$

The end point of the partial sum is expanded to L^2 from L after correcting the function in the finite sum.

$$(5-62) \quad |S_+(\theta)| < \zeta(2 + \operatorname{Re}(\theta)) \int_0^{\infty} y^{1+\operatorname{Re}(\theta)} e^{-y^2} \lim_{L \rightarrow \infty} \sum_{p=1}^{L^2} p e^{-\frac{\pi^2 p}{y^2}} dy \quad (\operatorname{Re}(\theta) > -1).$$

From the result of equation (19-92), the limit value of the sum of L^2 terms is described using the hyperbolic sine.

$$(5-63) \quad |S_+(\theta)| < \frac{\zeta(2 + \operatorname{Re}(\theta))}{4} \int_0^{\infty} y^{1+\operatorname{Re}(\theta)} \frac{e^{-y^2}}{\sinh^2\left(\frac{\pi^2}{2y^2}\right)} dy \quad (\operatorname{Re}(\theta) > -1).$$

By the same way,

$$(5-64) \quad |S_-(\theta)| < \frac{\zeta(2 + \operatorname{Re}(\theta))}{4} \int_0^{\infty} y^{-1+\operatorname{Re}(\theta)} \frac{e^{-y^2}}{\sinh^2\left(\frac{\pi^2}{2y^2}\right)} dy \quad (\operatorname{Re}(\theta) > 0).$$

To evaluate the integrals of inequalities (5-63) and (5-64), I prepare the function $f(x)$. The equation of definition is shown as follows:

$$(5-65) \quad f(x) := \frac{1}{\pi^3} e^{-\frac{x^2}{\pi}} - \frac{e^{-x^2}}{\sinh^2\left(\frac{\pi^2}{2x^2}\right)} \quad (x > 0).$$

Function values at both sides of the open interval are given by taking the limits.

$$(5-66) \quad \lim_{x \rightarrow +0} f(x) = \frac{1}{\pi^3}.$$

$$(5-67) \quad \lim_{x \rightarrow \infty} f(x) = 0.$$

The function $f(x)$ is differentiated so as to understand its behavior.

$$(5-68) \quad f'(x) = -\frac{2x}{\pi^4} e^{-\frac{x^2}{\pi}} + \frac{2xe^{-x^2}}{\sinh^2\left(\frac{\pi^2}{2x^2}\right)} - \frac{2\pi^2 e^{-x^2} \cosh\left(\frac{\pi^2}{2x^2}\right)}{x^3 \sinh^3\left(\frac{\pi^2}{2x^2}\right)}$$

$$= \left(-x^4 e^{-\frac{x^2}{\pi}} \sinh^3\left(\frac{\pi^2}{2x^2}\right) + \pi^4 x^4 e^{-x^2} \sinh\left(\frac{\pi^2}{2x^2}\right) - \pi^6 e^{-x^2} \cosh\left(\frac{\pi^2}{2x^2}\right) \right) / \left(\frac{\pi^4}{2} x^3 \sinh^3\left(\frac{\pi^2}{2x^2}\right) \right)$$

$$(x > 0).$$

α_1 and β_1 are presumably the solutions to the equation $f'(x) = 0$. The numerical solutions are shown as follows:

$$(5-69) \quad f'(\alpha_1) = f'(\beta_1) = 0 \quad (\alpha_1 \simeq 2.12138, \beta_1 \simeq 2.59993).$$

$$(5-70) \quad f(\alpha_1) \simeq 0.00141964 > 0.$$

Table shows the increase and decrease in the function

x	+0	α_1	β_1	∞
$f'(x)$		-	0	+	0	-	
$f(x)$	$\frac{1}{\pi^3}$	\searrow	minimum 0.0014196...	\nearrow	maximum 0.0019232...	\searrow	0

Fig. 5-1

From the table of the increase and decrease,

$$(5-71) \quad f'(x) > 0 \quad (x > 0).$$

Therefore

$$(5-72) \quad \frac{e^{-x^2}}{\sinh^2\left(\frac{\pi^2}{2x^2}\right)} < \frac{1}{\pi^3} e^{-\frac{x^2}{\pi}} \quad (x > 0).$$

This result is introduced into the right side of inequality (5-63).

And for the integral, I perform the variable transformation

$$y = (\pi x)^{\frac{1}{2}}.$$

$$(5-73) \quad |S_+(\theta)| < \frac{\zeta(2+\text{Re}(\theta))}{4\pi^3} \int_0^\infty y^{1+\text{Re}(\theta)} e^{-\frac{y^2}{\pi}} dy =$$

$$\frac{\zeta(2+\text{Re}(\theta))}{4\pi^3} \int_0^\infty (\pi x)^{\frac{1+\text{Re}(\theta)}{2}} e^{-x} \frac{1}{2} \pi^{\frac{1}{2}} x^{-\frac{1}{2}} dx =$$

$$\frac{\zeta(2+\text{Re}(\theta))}{8\pi^3} \pi^{\frac{2+\text{Re}(\theta)}{2}} \int_0^\infty x^{\frac{2+\text{Re}(\theta)}{2}-1} e^{-x} dx \quad (\text{Re}(\theta) > -1).$$

Therefore

$$(5-74) \quad |S_+(\theta)| < \frac{1}{8\pi^3} \pi^{\frac{2+\text{Re}(\theta)}{2}} \Gamma\left(\frac{2+\text{Re}(\theta)}{2}\right) \zeta(2+\text{Re}(\theta)) \quad (\text{Re}(\theta) > -1).$$

By the same way,

$$(5-75) \quad |S_-(\theta)| < \frac{\zeta(2+\text{Re}(\theta))}{4\pi^3} \int_0^\infty y^{-1+\text{Re}(\theta)} e^{-\frac{y^2}{\pi}} dy =$$

$$\frac{\zeta(2+\text{Re}(\theta))}{4\pi^3} \int_0^\infty (\pi x)^{\frac{-1+\text{Re}(\theta)}{2}} e^{-x} \frac{1}{2} \pi^{\frac{1}{2}} x^{-\frac{1}{2}} dx =$$

$$\frac{\zeta(2+\text{Re}(\theta))}{8\pi^3} \pi^{\frac{\text{Re}(\theta)}{2}} \int_0^\infty x^{\frac{\text{Re}(\theta)}{2}-1} e^{-x} dx \quad (\text{Re}(\theta) > 0).$$

Therefore

$$(5-76) \quad |S_-(\theta)| < \frac{1}{8\pi^3} \pi^{\frac{\text{Re}(\theta)}{2}} \Gamma\left(\frac{\text{Re}(\theta)}{2}\right) \zeta(2+\text{Re}(\theta)) \quad (\text{Re}(\theta) > 0).$$

From the inequalities (5-54) and (5-74),

$$(5-77) \quad \left| \hat{F}_{[2]}(\theta) \right| \leq \frac{2|\theta-1|\pi^{-\frac{\text{Re}(\theta)}{2}}}{\pi} \cdot \frac{1}{8\pi^3} \pi^{\frac{2+\text{Re}(\theta)}{2}} \Gamma\left(\frac{2+\text{Re}(\theta)}{2}\right) \zeta(2+\text{Re}(\theta)) =$$

$$\frac{|\theta-1|}{4\pi^3} \Gamma\left(\frac{2+\text{Re}(\theta)}{2}\right) \zeta(2+\text{Re}(\theta)) \quad (\text{Re}(\theta) > -1, \text{ the equality sign holds at } \theta = 1).$$

From the inequalities (5-55), (5-74), and (5-76),

$$\begin{aligned}
 (5-78) \quad & \left| \hat{G}_{[2]}(\theta) \right| < \frac{|\theta+1|\pi^{-\frac{\text{Re}(\theta)}{2}}}{\pi} \left(|\theta| \cdot \frac{1}{8\pi^3} \pi^{\frac{\text{Re}(\theta)}{2}} \Gamma\left(\frac{\text{Re}(\theta)}{2}\right) \zeta(2+\text{Re}(\theta)) \right. \\
 & \left. + 2 \cdot \frac{1}{8\pi^3} \pi^{\frac{2+\text{Re}(\theta)}{2}} \Gamma\left(\frac{2+\text{Re}(\theta)}{2}\right) \zeta(2+\text{Re}(\theta)) \right) \\
 & = \frac{|\theta+1|(|\theta|+\pi \text{Re}(\theta))}{8\pi^4} \Gamma\left(\frac{\text{Re}(\theta)}{2}\right) \zeta(2+\text{Re}(\theta)) \quad (\text{Re}(\theta) > 0).
 \end{aligned}$$

From the origin symmetry between the functions $F_{[2]}-\text{hat}(\theta)$ and $G_{[2]}-\text{hat}(\theta)$,

$$(5-79) \quad \left| \hat{F}_{[2]}(\theta) \right| = \left| \hat{G}_{[2]}(-\theta) \right| < \frac{|\theta+1|(|\theta|-\pi \text{Re}(\theta))}{8\pi^4} \Gamma\left(\frac{-\text{Re}(\theta)}{2}\right) \zeta(2-\text{Re}(\theta)) \quad (\text{Re}(\theta) < 0).$$

$$\begin{aligned}
 (5-80) \quad & \left| \hat{G}_{[2]}(\theta) \right| = \\
 & \left| \hat{F}_{[2]}(-\theta) \right| \leq \frac{|\theta+1|}{4\pi^3} \Gamma\left(\frac{2-\text{Re}(\theta)}{2}\right) \zeta(2-\text{Re}(\theta)) \quad (\text{Re}(\theta) < 1, \text{ the equality sign holds at } \theta = -1).
 \end{aligned}$$

The function $F_{[2]}-\text{hat}(\theta)$ is an absolutely convergent function in the whole complex plane, based on the inequalities (5-77) and (5-79).

The function $G_{[2]}-\text{hat}(\theta)$ is also an absolutely convergent function in the whole complex plane, based on the inequalities (5-78) and (5-80).

Combining the above results yields the following functional equation:

$$(5-81) \quad \mathbf{A}_{[2]}(\theta) + \hat{\mathbf{F}}_{[2]}(\theta) = \mathbf{B}_{[2]}(\theta) + \hat{\mathbf{G}}_{[2]}(\theta) \quad (\theta \in \mathbb{C}).$$

Where

$$(5-82) \quad \mathbf{A}_{[2]}(\theta) = \frac{-\theta(1-\theta)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta).$$

$$(5-83) \quad \mathbf{B}_{[2]}(\theta) = \frac{\theta(1+\theta)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(\frac{-\theta}{2}\right) \zeta(-\theta).$$

$$(5-84) \quad \hat{\mathbf{F}}_{[2]}(\theta) = \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right).$$

$$(5-85) \quad \hat{\mathbf{G}}_{[2]}(\theta) = \frac{-\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{\theta}(p) p^{-\frac{\theta}{2}} \left(-\theta K_{\frac{-\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2+\theta}{2}}(2\pi p) \right).$$

In the whole complex plane, all functions on both sides of the functional equation (5-81) are convergent.

It is the origin symmetric functional equation with correction terms, and in this case the functions $F_{[2]}-\text{hat}(\theta)$ and $G_{[2]}-\text{hat}(\theta)$ are applicable to the correction terms “[8].” I decide that it is called the second-order Π_c type functional equation.

6. The first proof of the Riemann hypothesis

6-1. Derivation of a representation containing the leading term of the zeta function for any complex number

The second-order Π_c type functional equation is shown again, but with a bit of difference.

$$(6-1) \quad \mathbf{A}_{[2]}(\theta) = \mathbf{B}_{[2]}(\theta) + \mathbf{H}_{[2]}(\theta) \quad (\theta \in \mathbb{C}).$$

Where

$$(6-2) \quad \mathbf{A}_{[2]}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta).$$

$$(6-3) \quad \mathbf{B}_{[2]}(\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta).$$

$$(6-4) \quad \mathbf{H}_{[2]}(\theta) := \hat{\mathbf{G}}_{[2]}(\theta) - \hat{\mathbf{F}}_{[2]}(\theta).$$

The equations (5-84) and (5-85) are introduced into the equation (6-4).

$$(6-5) \quad \mathbf{H}_{[2]}(\theta) = \frac{-\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{\theta}(p) p^{-\frac{\theta}{2}} \left(-\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2+\theta}{2}}(2\pi p) \right) \\ - \frac{\theta-1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

The relations of the equations (3-30) and (5-39) are applied to the equation (6-5).

$$(6-6) \quad \mathbf{H}_{[2]}(\theta) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(-2\theta K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + (\theta-1)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) + (\theta+1)(2\pi p) K_{\frac{2+\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

The recurrence formula for the modified Bessel function of the second kind is applied to the equation (6-6).

$$(6-7) \quad \mathbf{H}_{[2]}(\theta) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(-2\theta K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + (\theta-1)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) + 2(\theta+1) \left(\frac{\theta}{2} K_{\frac{\theta}{2}}(2\pi p) + (\pi p) K_{\frac{\theta-2}{2}}(2\pi p) \right) \right) \\ = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta(\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 2\theta(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

Therefore,

$$(6-8) \quad \mathbf{H}_{[2]}(\theta) = -\frac{\theta}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

By combining the equations (6-2), (6-3), and (6-8),

$$(6-9) \quad \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) \\ - \frac{\theta}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

Both sides of the equation (6-9) are multiplied by

$$\frac{4\pi}{\theta(\theta-1)\Gamma\left(\frac{\theta}{2}\right)} \pi^{\frac{\theta}{2}}$$

to obtain the zeta-hat function.

$$(6-10) \quad \hat{\zeta}(\theta) = \frac{\theta+1}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\sqrt{\pi}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) \\ - \frac{4\pi^{\frac{\theta}{2}}}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

This is the representation containing the leading term of the zeta function for any complex number. The above equation also gives the analytic continuation of the zeta function explicitly.

6-2. The first proof of the Riemann hypothesis

The Riemann hypothesis is demonstrated using the following equation:

$$(6-11) \quad (\theta-1) \Gamma\left(\frac{\theta}{2}\right) \pi^{-\frac{\theta}{2}} \hat{\zeta}(\theta) = (\theta+1) \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) \\ + 4 \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((1-\theta) K_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

Here, the above equation is obtained from

$$(\theta-1) \Gamma\left(\frac{\theta}{2}\right) \pi^{-\frac{\theta}{2}} \times (\text{equation (6-10)}).$$

Pioneers' previous research established the existence of non-trivial zeros on the critical strip.

It is complicated to describe and insignificant to examine by focusing on this region.

The proof of the reduction to absurdity will be performed by assuming the whole complex plane as the target region, excluding the critical line, and assuming the existence of at least one non-trivial zero $(x + iy)$.

Under the assumption,

$$(6-12) \quad \hat{\zeta}(x + iy) = 0 \quad (x \in \mathbb{R} \setminus \{1/2\}, y \in \mathbb{R}).$$

Under the assumption, the point $(1 - x - iy)$ is also the non-trivial zero of the zeta function due to the point symmetry of the Riemann Xi function.

$$(6-13) \quad \hat{\zeta}(1 - x - iy) = 0 \quad (x \in \mathbb{R} \setminus \{1/2\}, y \in \mathbb{R}).$$

For equation (6-11), $(x + iy)$ is substituted for θ .

$$(6-14) \quad (x + iy - 1) \pi^{-\frac{x + iy}{2}} \Gamma\left(\frac{x + iy}{2}\right) \hat{\zeta}(x + iy) = \\ (1 + x + iy) \pi^{-\frac{1 + x + iy}{2}} \Gamma\left(\frac{1 + x + iy}{2}\right) \zeta(1 + x + iy) + 4 \sum_{p=1}^{\infty} \sigma_{-x - iy}(p) p^{\frac{x + iy}{2}} \\ \left((1 - x - iy) K_{\frac{x + iy}{2}}(2\pi p) - 2(2\pi p) K_{\frac{2 - x - iy}{2}}(2\pi p) \right) \quad (x \in \mathbb{R} \setminus \{1/2\}, y \in \mathbb{R}).$$

For equation (6-11), $(1 - x - iy)$ is substituted for θ .

$$(6-15) \quad (-x - iy) \pi^{-\frac{1 - x - iy}{2}} \Gamma\left(\frac{1 - x - iy}{2}\right) \hat{\zeta}(1 - x - iy) = \\ (2 - x - iy) \pi^{-\frac{2 - x - iy}{2}} \Gamma\left(\frac{2 - x - iy}{2}\right) \zeta(2 - x - iy) + 4 \sum_{p=1}^{\infty} \sigma_{-1 + x + iy}(p) p^{\frac{1 - x - iy}{2}} \\ \left((x + iy) K_{\frac{1 - x - iy}{2}}(2\pi p) - 2(2\pi p) K_{\frac{1 + x + iy}{2}}(2\pi p) \right) \quad (x \in \mathbb{R} \setminus \{1/2\}, y \in \mathbb{R}).$$

According to the assumption, the left side of equation (6-14) equals zero.

$$(6-16) \quad 0 = (1 + x + iy) \pi^{-\frac{1 + x + iy}{2}} \Gamma\left(\frac{1 + x + iy}{2}\right) \zeta(1 + x + iy) + 4 \sum_{p=1}^{\infty} \sigma_{-x - iy}(p) p^{\frac{x + iy}{2}} \\ \left((1 - x - iy) K_{\frac{x + iy}{2}}(2\pi p) - 2(2\pi p) K_{\frac{2 - x - iy}{2}}(2\pi p) \right) \quad (x \in \mathbb{R} \setminus \{1/2\}, y \in \mathbb{R}).$$

Based on this assumption, the left side of equation (6-15) is also zero.

$$(6-17) \quad 0 = (2 - x - iy) \pi^{-\frac{2 - x - iy}{2}} \Gamma\left(\frac{2 - x - iy}{2}\right) \zeta(2 - x - iy) + 4 \sum_{p=1}^{\infty} \sigma_{-1 + x + iy}(p) p^{\frac{1 - x - iy}{2}} \\ \left((x + iy) K_{\frac{1 - x - iy}{2}}(2\pi p) - 2(2\pi p) K_{\frac{1 + x + iy}{2}}(2\pi p) \right) \quad (x \in \mathbb{R} \setminus \{1/2\}, y \in \mathbb{R}).$$

The non-trivial zeros of the zeta function off the critical line are given by the solutions of the simultaneous equations with the qualification consisting of the equations (6-16) and (6-17). To solve the simultaneous

equations with the qualification, I consider the simultaneous equations excluding the qualification.

The following are the simultaneous equations without qualification:

$$(6-18) \quad (1+x+iy) \pi^{-\frac{1+x+iy}{2}} \Gamma\left(\frac{1+x+iy}{2}\right) \zeta(1+x+iy) + 4 \sum_{p=1}^{\infty} \sigma_{-x-iy}(p) p^{\frac{x+iy}{2}}$$

$$\left((1-x-iy) K_{\frac{x+iy}{2}}(2\pi p) - 2(2\pi p) K_{\frac{2-x-iy}{2}}(2\pi p)\right) = 0 \quad (x, y \in \mathbb{R}).$$

$$(6-19) \quad (2-x-iy) \pi^{-\frac{2-x-iy}{2}} \Gamma\left(\frac{2-x-iy}{2}\right) \zeta(2-x-iy) + 4 \sum_{p=1}^{\infty} \sigma_{-1+x+iy}(p) p^{\frac{1-x-iy}{2}}$$

$$\left((x+iy) K_{\frac{1-x-iy}{2}}(2\pi p) - 2(2\pi p) K_{\frac{1+x+iy}{2}}(2\pi p)\right) = 0 \quad (x, y \in \mathbb{R}).$$

The non-trivial zeros are determined by solving the simultaneous equations (6-18) and (6-19).

Because the equations are symmetrical,

$$(6-20) \quad x = 1/2.$$

is immediately determined.

For equation (6-18), 1/2 is substituted for x.

$$(6-21) \quad \left(\frac{3}{2}+iy\right) \pi^{-\frac{3+iy}{4}} \Gamma\left(\frac{3+iy}{4}\right) \zeta\left(\frac{3+iy}{2}\right) + 4 \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+iy}{4}}$$

$$\left(\left(\frac{1}{2}-iy\right) K_{\frac{1+iy}{4}}(2\pi p) - 2(2\pi p) K_{\frac{3-iy}{4}}(2\pi p)\right) = 0 \quad (y \in \mathbb{R}).$$

For equation (6-19), 1/2 is substituted for x.

$$(6-22) \quad \left(\frac{3}{2}-iy\right) \pi^{-\frac{3-iy}{4}} \Gamma\left(\frac{3-iy}{4}\right) \zeta\left(\frac{3-iy}{2}\right) + 4 \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}+iy}(p) p^{\frac{1-iy}{4}}$$

$$\left(\left(\frac{1}{2}+iy\right) K_{\frac{1-iy}{4}}(2\pi p) - 2(2\pi p) K_{\frac{3+iy}{4}}(2\pi p)\right) = 0 \quad (y \in \mathbb{R}).$$

The equations (6-21) and (6-22) are complex conjugates of one another. And I can freely choose either of the equations (6-21) or (6-22) as the determining equation of the real variable y. The existence proof of the real solutions for the determining function is not given. As a result, for the simultaneous equations (6-16) and (6-17), there are no solutions off the critical line. This fact contradicts the assumption of reduction to absurdity. Therefore, the assumption ‘‘At least one non-trivial zero lies in the whole complex plane excluding the critical line’’ is denied, and ‘‘There is no non-trivial zero in the whole complex plane excluding the critical line at all’’ is confirmed. Incidentally, in collaboration with J.E. Littlewood, G.H. Hardy proved in 1914 that an infinite number of non-trivial zeros lie on the critical line.

By combining the time result and Hardy’s result, I concluded that all non-trivial zeros of the zeta function lie on the critical line. Therefore, the Riemann hypothesis for the zeta function is proved correct.

6-3. Visualization of the zeta-hat function

Currently, it is known that all real parts of the non-trivial zeros of the zeta function are 1/2.

I define that ρ_m is the positive imaginary part of the zeta function’s m-th non-trivial zero, to which the number is allocated in order that is closer to the real axis.

$$(6-23) \quad \hat{\zeta}\left(\frac{1}{2} \pm i \rho_m\right) := 0 \quad (0 < \rho_1 \leq \rho_2 \leq \rho_3 \leq \dots).$$

$$(6-24) \quad \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\} \simeq$$

$$\{14.134725142, 21.022039639, 25.010857580, 30.424876126, 32.935061588\}.$$

The infinite sum that is terminated at a finite term λ is used for the approximate calculation for the zeta function.

Now, I decide that the positive integer λ will be referred to as the degree of approximation and this rule is applied hereafter.

The approximate formula for the zeta-hat function is written as $(\text{zeta-hat})_\lambda(\theta)$.

$$(6-25) \quad \hat{\zeta}_\lambda(\theta) := \frac{\theta+1}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\sqrt{\pi}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) - \frac{4\pi^{\frac{\theta}{2}}}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)} \sum_{p=1}^{\lambda} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

Graph of $\text{Re}\left(\hat{\zeta}_5\left(\frac{1}{2} + iy\right)\right)$ and $\text{Im}\left(\hat{\zeta}_5\left(\frac{1}{2} + iy\right)\right)$, $y \in [0, 35]$

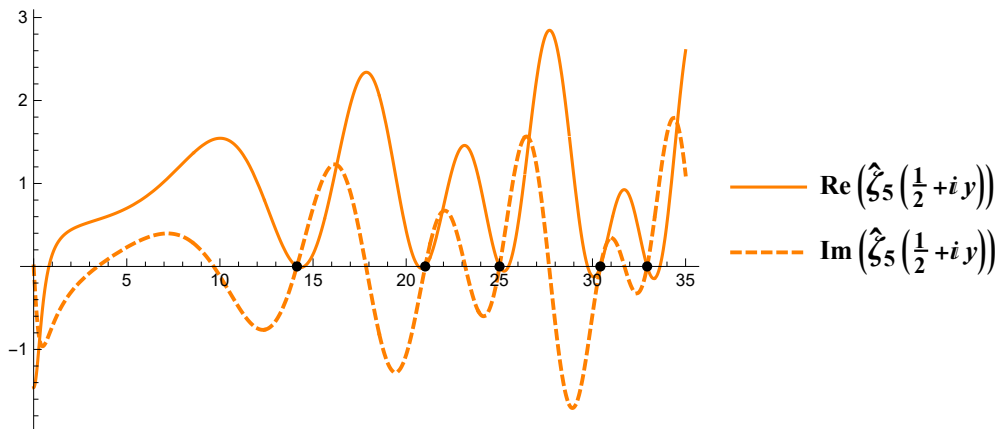


Fig. 6-1

The dots on the horizontal axis indicate the positive imaginary parts of the non-trivial zeros of the zeta function.

Five non-trivial zeros of the zeta function can be found on this interval.

Here are the details.

$$(6-26) \quad \left\{ \text{Re}\left(\hat{\zeta}_7\left(\frac{1}{2} + i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\hat{\zeta}_7\left(\frac{1}{2} + i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{9.16162 \times 10^{-11}, -1.57783 \times 10^{-10}\},$$

$$\left\{ \text{Im}\left(\hat{\zeta}_7\left(\frac{1}{2} + i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Im}\left(\hat{\zeta}_7\left(\frac{1}{2} + i\left(\frac{14\,134\,725\,142}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-5.75483 \times 10^{-10}, 9.91110 \times 10^{-10}\}.$$

$$(6-27) \quad \left\{ \text{Re}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-1.91909 \times 10^{-10}, 3.05551 \times 10^{-10}\},$$

$$\left\{ \text{Im}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Im}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{21\,022\,039\,639}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-8.55883 \times 10^{-10}, 1.36271 \times 10^{-9}\}.$$

$$(6-28) \quad \left\{ \text{Re}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{5.15602 \times 10^{-10}, -3.84471 \times 10^{-10}\},$$

$$\left\{ \text{Im}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Im}\left(\hat{\zeta}_8\left(\frac{1}{2} + i\left(\frac{25\,010\,857\,580}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-1.48458 \times 10^{-9}, 1.10701 \times 10^{-9}\}.$$

$$(6-29) \quad \left\{ \text{Re}\left(\hat{\zeta}_9\left(\frac{1}{2} + i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\hat{\zeta}_9\left(\frac{1}{2} + i\left(\frac{30\,424\,876\,126}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-5.73733 \times 10^{-10}, 7.61286 \times 10^{-10}\},$$

$$\begin{aligned}
 & \left\{ \operatorname{Im} \left(\hat{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{30\,424\,876\,126}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Im} \left(\hat{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{30\,424\,876\,126}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \\
 & \quad \left\{ -9.62767 \times 10^{-10}, 1.27749 \times 10^{-9} \right\}. \\
 (6-30) \quad & \left\{ \operatorname{Re} \left(\hat{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{32\,935\,061\,588}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\hat{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{32\,935\,061\,588}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \\
 & \quad \left\{ 5.54802 \times 10^{-10}, -9.46306 \times 10^{-10} \right\}, \\
 & \left\{ \operatorname{Im} \left(\hat{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{32\,935\,061\,588}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Im} \left(\hat{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{32\,935\,061\,588}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \\
 & \quad \left\{ -8.57881 \times 10^{-10}, 1.46326 \times 10^{-9} \right\}.
 \end{aligned}$$

7. The general representation containing the leading term of the zeta function for any odd number of 3 or more

In this section, the exact values of the zeta function for any even number of 4 or more will be treated as the known values for convenience.

7-1. Derivation of a general representation containing the leading term of the zeta function for any odd number of 3 or more

Equations (5-49) and (5-52) are introduced on the right side of equation (6-4).

$$\begin{aligned}
 (7-1) \quad \mathbf{H}_{[2]}(\theta) &= \frac{(\theta+1)\pi^{-\frac{\theta}{2}}}{\pi} \sum_{\mathbf{p}=1}^{\infty} \sigma_{-\theta}(\mathbf{p}) \left(\theta \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi \mathbf{p})^2}{y^2}\right)} dy \right. \\
 &\quad \left. - 2 \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi \mathbf{p})^2}{y^2}\right)} dy \right) - \frac{2(\theta-1)\pi^{-\frac{\theta}{2}}}{\pi} \sum_{\mathbf{p}=1}^{\infty} \sigma_{-\theta}(\mathbf{p}) \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi \mathbf{p})^2}{y^2}\right)} dy \\
 &= -\frac{\theta\pi^{-\frac{\theta}{2}}}{\pi} \sum_{\mathbf{p}=1}^{\infty} \sigma_{-\theta}(\mathbf{p}) \left(4 \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi \mathbf{p})^2}{y^2}\right)} dy \right. \\
 &\quad \left. - (\theta+1) \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi \mathbf{p})^2}{y^2}\right)} dy \right) \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 (7-2) \quad \hat{\zeta}(\theta) &= \frac{\theta+1}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)\sqrt{\pi}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) - \frac{4}{(\theta-1)\Gamma\left(\frac{\theta}{2}\right)} \sum_{\mathbf{p}=1}^{\infty} \sigma_{-\theta}(\mathbf{p}) \\
 &\quad \left(4 \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi \mathbf{p})^2}{y^2}\right)} dy - (\theta+1) \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi \mathbf{p})^2}{y^2}\right)} dy \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).
 \end{aligned}$$

For equation (7-2), any odd number of 3 or more is substituted for θ .

$$(7-3) \quad \hat{\zeta}(2k+1) = \frac{(k+1)\Gamma(k+1)}{k\Gamma\left(\frac{2k+1}{2}\right)\sqrt{\pi}} \zeta(2k+2) - \frac{4}{k\Gamma\left(\frac{2k+1}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \\ \left(2 \int_0^{\infty} y^{2(k+1)} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy - (k+1) \int_0^{\infty} y^{2k} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right) \quad (k \in \mathbb{N}).$$

The description in subsection 19-4 gives the general representation for the two integrals in equation (7-3).

$$(7-4) \quad \int_0^{\infty} x^{2k} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^{\mu} \sqrt{\pi} e^{-2\pi p} \quad (p, k \in \mathbb{N}).$$

Where

$$(7-5) \quad a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \quad (k \in \mathbb{N}, \mu = 0, 1, \dots, k).$$

And the following coefficients deserve special mention:

$$(7-6) \quad a_{k,k} = 1 \quad (k \in \mathbb{N}).$$

$$(7-7) \quad a_{k,1} = a_{k,0} = (2k-1)!! \quad (k \in \mathbb{N}).$$

The above relations are introduced on the right side of equation (7-3).

$$(7-8) \quad \hat{\zeta}(2k+1) = \frac{(k+1)\Gamma(k+1)}{k\Gamma\left(\frac{2k+1}{2}\right)\sqrt{\pi}} \zeta(2k+2) - \frac{4}{k\Gamma\left(\frac{2k+1}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \\ \left(2 \cdot \frac{1}{2^{k+2}} \sum_{\mu=0}^{k+1} a_{k+1,\mu} (2\pi p)^{\mu} \sqrt{\pi} e^{-2\pi p} - (k+1) \cdot \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^{\mu} \sqrt{\pi} e^{-2\pi p} \right) \\ = \frac{(k+1)\Gamma(k+1)}{k \frac{(2k-1)!!}{2^k} \sqrt{\pi} \cdot \sqrt{\pi}} \zeta(2k+2) - \frac{2}{k \frac{(2k-1)!!}{2^k} \sqrt{\pi} 2^k} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \\ \left(a_{k+1,k+1} (2\pi p)^{k+1} + \sum_{\mu=0}^k (a_{k+1,\mu} - (k+1) a_{k,\mu}) (2\pi p)^{\mu} \right) \sqrt{\pi} e^{-2\pi p} \\ = \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2) - \frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \\ \left(a_{k+1,k+1} (2\pi p)^{k+1} + \sum_{\mu=0}^k (a_{k+1,\mu} - (k+1) a_{k,\mu}) (2\pi p)^{\mu} \right) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

The following equation is the obvious from the definition of $a_{k,\mu}$:

$$(7-9) \quad a_{k+1,k+1} = 1 \quad (k \in \mathbb{N}).$$

Therefore

$$(7-10) \quad \hat{\zeta}(2k+1) = \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2) - \frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \\ \left((2\pi p)^{k+1} + \sum_{\mu=0}^k (a_{k+1,\mu} - (k+1) a_{k,\mu}) (2\pi p)^{\mu} \right) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

For the general representation containing the leading term of the zeta function for any odd number of 3 or more, new coefficients are employed.

The coefficients are determined by defining the following equation:

$$(7-11) \quad \mathbf{b}_{k,\mu} := \mathbf{a}_{k+1,\mu} - (k+1) \mathbf{a}_{k,\mu} \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \dots, k).$$

The coefficients are calculated as follows:

$$(7-12) \quad \mathbf{b}_{k,\mu} = \frac{(2k+2-\mu)!}{2^{k+1-\mu} \mu! (k+1-\mu)!} - (k+1) \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!}$$

$$= \frac{\mu(\mu-1)+2k(k+1-\mu)}{2^{k+1-\mu} \mu! (k+1-\mu)!} (2k-\mu)! \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \dots, k).$$

For equation (7-12), 0 and 1 are substituted directly for μ .

$$(7-13) \quad \mathbf{b}_{k,0} = \frac{\mu(\mu-1)+2k(k+1-\mu)}{2^{k+1-\mu} \mu! (k+1-\mu)!} (2k-\mu)! \Big|_{\mu=0} = \frac{2k(k+1)}{2^{k+1} (k+1)!} (2k)!$$

$$= \frac{k}{2^k k!} (2k)! = k(2k-1)!! \quad (k \in \mathbb{N}).$$

$$(7-14) \quad \mathbf{b}_{k,1} = \frac{\mu(\mu-1)+2k(k+1-\mu)}{2^{k+1-\mu} \mu! (k+1-\mu)!} (2k-\mu)! \Big|_{\mu=1} = \frac{2k^2}{2^k k!} (2k-1)!$$

$$= \frac{k}{2^{k-1} (k-1)!} (2k-1)! = k(2k-1)!! \quad (k \in \mathbb{N}).$$

And for equation (7-12), $k+1$ is substituted directly for μ .

$$(7-15) \quad \mathbf{b}_{k,k+1} = \frac{\mu(\mu-1)+2k(k+1-\mu)}{2^{k+1-\mu} \mu! (k+1-\mu)!} (2k-\mu)! \Big|_{\mu=k+1} = \frac{(k+1)k}{2^0 (k+1)! 0!} (k-1)! = 1 \quad (k \in \mathbb{N}).$$

As a result, the general representation containing the leading term of the zeta function for any odd number of 3 or more is obtained. This is also the first general representation for the zeta function for any odd number of 3 or more.

$$(7-16) \quad \hat{\zeta}(2k+1) = \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2)$$

$$- \frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(\sum_{\mu=0}^{k+1} \mathbf{b}_{k,\mu} (2\pi p)^\mu \right) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

The following expression is equivalent to the above equation.

$$(7-17) \quad \hat{\zeta}(2k+1) = \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2)$$

$$- \frac{2(2\pi)^{2k+1}}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{2k+1}(p) \left(\sum_{\mu=0}^{k+1} \frac{\mathbf{b}_{k,\mu}}{(2\pi p)^{2k+1-\mu}} \right) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

Where

$$(7-18) \quad \mathbf{b}_{k,\mu} = \frac{\mu(\mu-1)+2k(k+1-\mu)}{2^{k+1-\mu} \mu! (k+1-\mu)!} (2k-\mu)! \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \dots, k+1).$$

And the coefficients deserving special mention are as follows:

$$(7-19) \quad \mathbf{b}_{k,k+1} = 1 \quad (k \in \mathbb{N}).$$

$$(7-20) \quad \mathbf{b}_{k,1} = \mathbf{b}_{k,0} = k(2k-1)!! \quad (k \in \mathbb{N}).$$

Table of the leading terms and the coefficients of the infinite series

k	1	2	3	4
$\frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \cdot \zeta(2k+2)$	$\frac{2\pi^3}{45}$	$\frac{4\pi^5}{945}$	$\frac{32\pi^7}{70875}$	$\frac{32\pi^9}{654885}$
$-\frac{2}{k(2k-1)!!}$	-2	$-\frac{1}{3}$	$-\frac{2}{45}$	$-\frac{1}{210}$
k	5	6	7	8
$\frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \cdot \zeta(2k+2)$	$\frac{353792\pi^{11}}{67043851875}$	$\frac{1024\pi^{13}}{1806079275}$	$\frac{29630464\pi^{15}}{488950811724375}$	$\frac{179679232\pi^{17}}{27870196268289375}$
$-\frac{2}{k(2k-1)!!}$	$-\frac{2}{4725}$	$-\frac{1}{31185}$	$-\frac{2}{945945}$	$-\frac{1}{8108100}$

Fig. 7-1

Table of the coefficients $b_{k,\mu}$

k	$b_{k,9}$	$b_{k,8}$	$b_{k,7}$	$b_{k,6}$	$b_{k,5}$	$b_{k,4}$	$b_{k,3}$	$b_{k,2}$	$b_{k,1}$	$b_{k,0}$
1								1	1	1
2							1	3	6	6
3						1	6	21	45	45
4					1	10	55	195	420	420
5				1	15	120	630	2205	4725	4725
6			1	21	231	1680	8505	29295	62370	62370
7		1	28	406	3906	26775	131670	446985	945945	945945
8	1	36	666	8190	72765	478170	2297295	7702695	16216200	16216200

Fig. 7-2

7-2. Another viewpoint

The equation (7-16) by changing the start point of the finite sum from 0 to 1 is written.

$$(7-21) \quad \hat{\zeta}(2k+1) = \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2) - \frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(\sum_{\mu=1}^{k+1} b_{k,\mu} (2\pi p)^\mu + k(2k-1)!! \right) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

And the function $E(2k+1)$ that is the separated term by changing the finite sum's starting is defined as follows:

$$(7-22) \quad E(2k+1) := -\frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) k(2k-1)!! e^{-2\pi p} \quad (k \in \mathbb{N}).$$

By simplifying,

$$(7-23) \quad E(2k+1) = -2 \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

There are two other viewpoints on the function $E(2k+1)$.

The second viewpoint is that the function $E(2k+1)$ includes the zeta function in itself for any odd number of 3 or more, as follows:

$$\begin{aligned}
 (7-24) \quad \mathbf{E}(2k+1) &= -2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^{2k+1}} e^{-2\pi m n} = - \sum_{n=1}^{\infty} \frac{\coth(\pi n) - 1}{n^{2k+1}} \\
 &= - \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} + \zeta(2k+1) \quad (k \in \mathbb{N}).
 \end{aligned}$$

When this relation is introduced into equation (7-16), the cancellation of the zeta functions is observed on both sides of the equation.

$$\begin{aligned}
 (7-25) \quad \hat{\zeta}(2k+1) &= \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2) \\
 &- \frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(\sum_{\mu=0}^k \mathbf{b}_{k,\mu+1} (2\pi p)^\mu \right) (2\pi p) e^{-2\pi p} \\
 &- \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} + \zeta(2k+1) \quad (k \in \mathbb{N}).
 \end{aligned}$$

Therefore, the essence of equation (7-16) does not represent the zeta function for any odd number of 3 or more, but the following representation does:

$$\begin{aligned}
 (7-26) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} &= \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2) \\
 &- \frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(\sum_{\mu=0}^k \mathbf{b}_{k,\mu+1} (2\pi p)^\mu \right) (2\pi p) e^{-2\pi p} \quad (k \in \mathbb{N}).
 \end{aligned}$$

The third viewpoint is that the function $\mathbf{E}(2k+1)$ can be written using the infinite series as follows:

$$\begin{aligned}
 (7-27) \quad \mathbf{E}(2k+1) &= - \sum_{n=1}^{\infty} \frac{\coth(\pi n) - 1}{n^{2k+1}} = - \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} \cdot \frac{\cosh(\pi n) - \sinh(\pi n)}{\sinh(\pi n)} \\
 &= - \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} \cdot \frac{2e^{-\pi n}}{e^{\pi n} - e^{-\pi n}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{2k+1} (e^{2\pi n} - 1)} \quad (k \in \mathbb{N}).
 \end{aligned}$$

From the result of equation (7-24),

$$(7-28) \quad \zeta(2k+1) = \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} + \mathbf{E}(2k+1) \quad (k \in \mathbb{N}).$$

Ramanujan discovered the explicit formula for the first term on the right side of equation (7-28), where $2k+1$ is equivalent to 3 modulo 4, as follows “[14]”:

$$(7-29) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} = 2^{2k} \pi^{2k+1} \sum_{\mu=0}^{k+1} (-1)^{\mu+1} \frac{\mathbf{B}_{2\mu}}{(2\mu)!} \cdot \frac{\mathbf{B}_{2k+2-2\mu}}{(2k+2-2\mu)!}$$

$$(k \in \mathbb{N}, 2k+1 \equiv 3 \pmod{4}).$$

By combining the results of equations (7-27), (7-28), and (7-29),

$$(7-30) \quad \zeta(2k+1) = 2^{2k} \pi^{2k+1} \sum_{\mu=0}^{k+1} (-1)^{\mu+1} \frac{\mathbf{B}_{2\mu}}{(2\mu)!} \cdot \frac{\mathbf{B}_{2k+2-2\mu}}{(2k+2-2\mu)!} - 2 \sum_{n=1}^{\infty} \frac{1}{n^{2k+1} (e^{2\pi n} - 1)}$$

$$(k \in \mathbb{N}, 2k+1 \equiv 3 \pmod{4}).$$

I also arrived at the same representation for the zeta function as Ramanujan for 3, 7, and 11, and so on.

7-3. Some Examples of calculation

When $\theta = 3$,

The representations for the zeta-hat(3) are

$$(7-31) \quad \hat{\zeta}(3) = \frac{2\pi^3}{45} - 2 \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p)^2 + (2\pi p) + 1) e^{-2\pi p}.$$

$$(7-32) \quad \hat{\zeta}(3) = \frac{2\pi^3}{45} - 2(2\pi)^3 \sum_{p=1}^{\infty} \sigma_3(p) \left(\frac{1}{2\pi p} + \frac{1}{(2\pi p)^2} + \frac{1}{(2\pi p)^3} \right) e^{-2\pi p}.$$

In contrast, the essence of equation (7-31) is

$$(7-33) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} = \frac{2\pi^3}{45} - 2 \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p) + 1) (2\pi p) e^{-2\pi p}.$$

The description in subsection 8-6 shows the representation of equation (7-31) using hyperbolic functions.

$$(7-34) \quad \hat{\zeta}(3) = \frac{2\pi^3}{45} - 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^3} ((2\pi m n)^2 + (2\pi m n) + 1) e^{-2\pi m n}$$

$$= \frac{2\pi^3}{45} - 2 \sum_{n=1}^{\infty} \left(\frac{(2\pi n)^2}{n^3} \cdot \frac{\coth(\pi n)}{4 \sinh^2(\pi n)} + \frac{(2\pi n)}{n^3} \cdot \frac{1}{4 \sinh^2(\pi n)} + \frac{1}{n^3} \cdot \frac{\coth(\pi n) - 1}{2} \right).$$

Therefore,

$$(7-35) \quad \hat{\zeta}(3) = \frac{2\pi^3}{45} - \sum_{n=1}^{\infty} \left(\frac{2\pi^2 \coth(\pi n)}{n \sinh^2(\pi n)} + \frac{\pi}{n^2 \sinh^2(\pi n)} + \frac{\coth(\pi n) - 1}{n^3} \right).$$

For equation (7-29), 1 is substituted for k.

$$(7-36) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} = 2^2 \pi^3 \sum_{\mu=0}^2 (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{4-2\mu}}{(4-2\mu)!}$$

$$= 2^2 \pi^3 \left(-\frac{2}{4!} \left(-\frac{1}{30} \right) + \left(\frac{1}{2!} \cdot \frac{1}{6} \right)^2 \right) = \frac{7\pi^3}{180}.$$

Also for equation (7-30), 1 is substituted for k.

$$(7-37) \quad \zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{n=1}^{\infty} \frac{1}{n^3 (e^{2\pi n} - 1)}.$$

By combining the results of equations (7-33) and (7-36), I obtained the infinite series that gives the transcendental number “[13].”

$$(7-38) \quad \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p) + 1) (2\pi p) e^{-2\pi p} = \frac{\pi^3}{360}.$$

The equations (7-31), (7-32), (7-33), (7-35), (7-36), and (7-37) are equivalent each other.

When $\theta = 5$,

The representations for the zeta-hat(5) are

$$(7-39) \quad \hat{\zeta}(5) = \frac{4\pi^5}{945} - \frac{1}{3} \sum_{p=1}^{\infty} \sigma_{-5}(p) ((2\pi p)^3 + 3(2\pi p)^2 + 6(2\pi p) + 6) e^{-2\pi p}.$$

$$(7-40) \quad \hat{\zeta}(5) = \frac{4\pi^5}{945} - \frac{(2\pi)^5}{3} \sum_{p=1}^{\infty} \sigma_5(p) \left(\frac{1}{(2\pi p)^2} + \frac{3}{(2\pi p)^3} + \frac{6}{(2\pi p)^4} + \frac{6}{(2\pi p)^5} \right) e^{-2\pi p}.$$

The essence of equation (7-39) is

$$(7-41) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^5} = \frac{4\pi^5}{945} - \frac{1}{3} \sum_{p=1}^{\infty} \sigma_{-5}(p) ((2\pi p)^2 + 3(2\pi p) + 6) (2\pi p) e^{-2\pi p}.$$

The representation of equation (7-39) using hyperbolic functions is as follows:

$$\begin{aligned}
 (7-42) \quad \hat{\zeta}(5) &= \frac{4\pi^5}{945} - \frac{1}{3} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^5} ((2\pi m n)^3 + 3(2\pi m n)^2 + 6(2\pi m n) + 6) e^{-2\pi m n} \\
 &= \frac{4\pi^5}{945} - \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{(2\pi n)^3}{n^5} \cdot \frac{3}{8} \left(\frac{2}{3} + \frac{1}{\sinh^2(\pi n)} \right) \frac{1}{\sinh^2(\pi n)} + \frac{3(2\pi n)^2}{n^5} \cdot \frac{\coth(\pi n)}{4 \sinh^2(\pi n)} \right. \\
 &\quad \left. + \frac{6(2\pi n)}{n^5} \cdot \frac{1}{4 \sinh^2(\pi n)} + \frac{6}{n^5} \cdot \frac{\coth(\pi n) - 1}{2} \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (7-43) \quad \hat{\zeta}(5) &= \frac{4\pi^5}{945} - \sum_{n=1}^{\infty} \left(\left(\frac{2}{3} + \frac{1}{\sinh^2(\pi n)} \right) \frac{\pi^3}{n^2 \sinh^2(\pi n)} + \frac{\pi^2 \coth(\pi n)}{n^3 \sinh^2(\pi n)} \right. \\
 &\quad \left. + \frac{\pi}{n^4 \sinh^2(\pi n)} + \frac{\coth(\pi n) - 1}{n^5} \right).
 \end{aligned}$$

The equations (7-39), (7-40), (7-41), and (7-43) are equivalent each other.

Hereafter, the principal results are displayed in order without explanation. When $\theta = 7$,

$$(7-44) \quad \hat{\zeta}(7) = \frac{32\pi^7}{70875} - \frac{2}{45} \sum_{p=1}^{\infty} \sigma_{-7}(p) ((2\pi p)^4 + 6(2\pi p)^3 + 21(2\pi p)^2 + 45(2\pi p) + 45) e^{-2\pi p}.$$

$$\begin{aligned}
 (7-45) \quad \hat{\zeta}(7) &= \frac{32\pi^7}{70875} \\
 &- \frac{2(2\pi)^7}{45} \sum_{p=1}^{\infty} \sigma_7(p) \left(\frac{1}{(2\pi p)^3} + \frac{6}{(2\pi p)^4} + \frac{21}{(2\pi p)^5} + \frac{45}{(2\pi p)^6} + \frac{45}{(2\pi p)^7} \right) e^{-2\pi p}.
 \end{aligned}$$

$$\begin{aligned}
 (7-46) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} &= \frac{32\pi^7}{70875} \\
 &- \frac{2}{45} \sum_{p=1}^{\infty} \sigma_{-7}(p) ((2\pi p)^3 + 6(2\pi p)^2 + 21(2\pi p) + 45) (2\pi p) e^{-2\pi p}.
 \end{aligned}$$

$$\begin{aligned}
 (7-47) \quad \hat{\zeta}(7) &= \frac{32\pi^7}{70875} - \sum_{n=1}^{\infty} \left(\left(\frac{1}{3} + \frac{1}{\sinh^2(\pi n)} \right) \frac{8\pi^4 \coth(\pi n)}{15 n^3 \sinh^2(\pi n)} \right. \\
 &\quad \left. + \left(\frac{2}{3} + \frac{1}{\sinh^2(\pi n)} \right) \frac{4\pi^3}{5 n^4 \sinh^2(\pi n)} + \frac{14\pi^2 \coth(\pi n)}{15 n^5 \sinh^2(\pi n)} + \frac{\pi}{n^6 \sinh^2(\pi n)} + \frac{\coth(\pi n) - 1}{n^7} \right).
 \end{aligned}$$

$$\begin{aligned}
 (7-48) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} &= 2^6 \pi^7 \sum_{\mu=0}^4 (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{8-2\mu}}{(8-2\mu)!} \\
 &= 2^6 \pi^7 \left(-\frac{2}{8!} \left(-\frac{1}{30} \right) + 2 \left(\frac{1}{2!} \cdot \frac{1}{6} \right) \left(\frac{1}{6!} \cdot \frac{1}{42} \right) - \left(\frac{1}{4!} \left(-\frac{1}{30} \right) \right)^2 \right) = \frac{19\pi^7}{56700}.
 \end{aligned}$$

$$(7-49) \quad \zeta(7) = \frac{19\pi^7}{56700} - 2 \sum_{n=1}^{\infty} \frac{1}{n^7 (e^{2\pi n} - 1)}.$$

$$(7-50) \quad \sum_{p=1}^{\infty} \sigma_{-7}(p) ((2\pi p)^3 + 6(2\pi p)^2 + 21(2\pi p) + 45) (2\pi p) e^{-2\pi p} = \frac{11\pi^7}{4200}.$$

When $\theta = 9$,

$$\begin{aligned}
 (7-51) \quad \hat{\zeta}(9) &= \frac{32\pi^9}{654885} \\
 &- \frac{1}{210} \sum_{p=1}^{\infty} \sigma_{-9}(p) ((2\pi p)^5 + 10(2\pi p)^4 + 55(2\pi p)^3 + 195(2\pi p)^2 + 420(2\pi p) + 420) e^{-2\pi p}.
 \end{aligned}$$

$$(7-52) \quad \hat{\zeta}(9) = \frac{32\pi^9}{654885} - \frac{(2\pi)^9}{210} \sum_{p=1}^{\infty} \sigma_9(p) \left(\frac{1}{(2\pi p)^4} + \frac{10}{(2\pi p)^5} + \frac{55}{(2\pi p)^6} + \frac{195}{(2\pi p)^7} + \frac{420}{(2\pi p)^8} + \frac{420}{(2\pi p)^9} \right) e^{-2\pi p}.$$

$$(7-53) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^9} = \frac{32\pi^9}{654885} - \frac{1}{210} \sum_{p=1}^{\infty} \sigma_{-9}(p) ((2\pi p)^4 + 10(2\pi p)^3 + 55(2\pi p)^2 + 195(2\pi p) + 420) (2\pi p) e^{-2\pi p}.$$

$$(7-54) \quad \hat{\zeta}(9) = \frac{32\pi^9}{654885} - \sum_{n=1}^{\infty} \left(\left(\frac{2}{15} + \frac{1}{\sinh^2(\pi n)} + \frac{1}{\sinh^4(\pi n)} \right) \frac{2\pi^5}{7n^4 \sinh^2(\pi n)} + \left(\frac{1}{3} + \frac{1}{\sinh^2(\pi n)} \right) \frac{4\pi^4 \coth(\pi n)}{7n^5 \sinh^2(\pi n)} + \left(\frac{2}{3} + \frac{1}{\sinh^2(\pi n)} \right) \frac{11\pi^3}{14n^6 \sinh^2(\pi n)} + \frac{13\pi^2 \coth(\pi n)}{14n^7 \sinh^2(\pi n)} + \frac{\pi}{n^8 \sinh^2(\pi n)} + \frac{\coth(\pi n) - 1}{n^9} \right).$$

When $\theta = 11$,

$$(7-55) \quad \hat{\zeta}(11) = \frac{353792\pi^{11}}{67043851875} - \frac{2}{4725} \sum_{p=1}^{\infty} \sigma_{-11}(p) ((2\pi p)^6 + 15(2\pi p)^5 + 120(2\pi p)^4 + 630(2\pi p)^3 + 2205(2\pi p)^2 + 4725(2\pi p) + 4725) e^{-2\pi p}.$$

$$(7-56) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{11}} = \frac{353792\pi^{11}}{67043851875} - \frac{2}{4725} \sum_{p=1}^{\infty} \sigma_{-11}(p) ((2\pi p)^5 + 15(2\pi p)^4 + 120(2\pi p)^3 + 630(2\pi p)^2 + 2205(2\pi p) + 4725) (2\pi p) e^{-2\pi p}.$$

$$(7-57) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{11}} = 2^{10} \pi^{11} \sum_{\mu=0}^6 (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{12-2\mu}}{(12-2\mu)!} = 2^{10} \pi^{11} \left(-\frac{2}{12!} \left(-\frac{691}{2730} \right) + 2 \left(\frac{1}{2!} \cdot \frac{1}{6} \right) \left(\frac{1}{10!} \cdot \frac{5}{66} \right) - 2 \left(\frac{1}{4!} \left(-\frac{1}{30} \right) \cdot \frac{1}{8!} \left(-\frac{1}{30} \right) \right) + \left(\frac{1}{6!} \cdot \frac{1}{42} \right)^2 \right) = \frac{1453\pi^{11}}{425675250}.$$

$$(7-58) \quad \zeta(11) = \frac{1453\pi^{11}}{425675250} - 2 \sum_{n=1}^{\infty} \frac{1}{n^{11} (e^{2\pi n} - 1)}.$$

$$(7-59) \quad \sum_{p=1}^{\infty} \sigma_{-11}(p) ((2\pi p)^5 + 15(2\pi p)^4 + 120(2\pi p)^3 + 630(2\pi p)^2 + 2205(2\pi p) + 4725) (2\pi p) e^{-2\pi p} = \frac{249889\pi^{11}}{56756700}.$$

When $\theta = 13$,

$$(7-60) \quad \hat{\zeta}(13) = \frac{1024\pi^{13}}{1806079275} - \frac{1}{31185} \sum_{p=1}^{\infty} \sigma_{-13}(p) ((2\pi p)^7 + 21(2\pi p)^6 + 231(2\pi p)^5 + 1680(2\pi p)^4 + 8505(2\pi p)^3 + 29295(2\pi p)^2 + 62370(2\pi p) + 62370) e^{-2\pi p}.$$

$$(7-61) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{13}} = \frac{1024\pi^{13}}{1806079275} - \frac{1}{31185} \sum_{p=1}^{\infty} \sigma_{-13}(p) ((2\pi p)^6 + 21(2\pi p)^5 + 231(2\pi p)^4 + 1680(2\pi p)^3 + 8505(2\pi p)^2 + 29295(2\pi p) + 62370) (2\pi p) e^{-2\pi p}.$$

When $\theta = 15$,

$$(7-62) \quad \hat{\zeta}(15) = \frac{29\,630\,464\,\pi^{15}}{488\,950\,811\,724\,375} - \frac{2}{945\,945} \sum_{p=1}^{\infty} \sigma_{-15}(p) ((2\pi p)^8 + 28(2\pi p)^7 \\ + 406(2\pi p)^6 + 3906(2\pi p)^5 + 26\,775(2\pi p)^4 + 131\,670(2\pi p)^3 \\ + 446\,985(2\pi p)^2 + 945\,945(2\pi p) + 945\,945) e^{-2\pi p}.$$

$$(7-63) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{15}} = \frac{29\,630\,464\,\pi^{15}}{488\,950\,811\,724\,375} - \frac{2}{945\,945} \sum_{p=1}^{\infty} \sigma_{-15}(p) ((2\pi p)^7 + 28(2\pi p)^6 \\ + 406(2\pi p)^5 + 3906(2\pi p)^4 + 26\,775(2\pi p)^3 + 131\,670(2\pi p)^2 \\ + 446\,985(2\pi p) + 945\,945) (2\pi p) e^{-2\pi p}.$$

$$(7-64) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{15}} = 2^{14} \pi^{15} \sum_{\mu=0}^8 (-1)^{\mu+1} \frac{B_{2\mu}}{(2\mu)!} \cdot \frac{B_{16-2\mu}}{(16-2\mu)!} \\ = 2^{14} \pi^{15} \left(-\frac{2}{16!} \left(-\frac{3617}{510} \right) + 2 \left(\frac{1}{2!} \cdot \frac{1}{6} \right) \left(\frac{1}{14!} \cdot \frac{7}{6} \right) \right. \\ \left. - 2 \left(\frac{1}{4!} \left(-\frac{1}{30} \right) \cdot \frac{1}{12!} \left(-\frac{691}{2730} \right) \right) + 2 \left(\frac{1}{6!} \cdot \frac{1}{42} \right) \left(\frac{1}{10!} \cdot \frac{5}{66} \right) - \left(\frac{1}{8!} \left(-\frac{1}{30} \right) \right)^2 \right) = \frac{13\,687\,\pi^{15}}{390\,769\,879\,500}.$$

$$(7-65) \quad \zeta(15) = \frac{13\,687\,\pi^{15}}{390\,769\,879\,500} - 2 \sum_{n=1}^{\infty} \frac{1}{n^{15} (e^{2\pi n} - 1)}.$$

$$(7-66) \quad \sum_{p=1}^{\infty} \sigma_{-15}(p) ((2\pi p)^7 + 28(2\pi p)^6 + 406(2\pi p)^5 + 3906(2\pi p)^4 + 26\,775(2\pi p)^3 \\ + 131\,670(2\pi p)^2 + 446\,985(2\pi p) + 945\,945) (2\pi p) e^{-2\pi p} = \frac{16\,672\,807\,\pi^{15}}{1\,378\,377\,000}.$$

When $\theta = 17$,

$$(7-67) \quad \hat{\zeta}(17) = \frac{179\,679\,232\,\pi^{17}}{27\,870\,196\,268\,289\,375} - \frac{1}{8\,108\,100} \sum_{p=1}^{\infty} \sigma_{-17}(p) ((2\pi p)^9 + 36(2\pi p)^8 \\ + 666(2\pi p)^7 + 8190(2\pi p)^6 + 72\,765(2\pi p)^5 + 478\,170(2\pi p)^4 \\ + 2\,297\,295(2\pi p)^3 + 7\,702\,695(2\pi p)^2 + 16\,216\,200(2\pi p) + 16\,216\,200) e^{-2\pi p}.$$

$$(7-68) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{17}} = \frac{179\,679\,232\,\pi^{17}}{27\,870\,196\,268\,289\,375} - \frac{1}{8\,108\,100} \sum_{p=1}^{\infty} \sigma_{-17}(p) ((2\pi p)^8 + 36(2\pi p)^7 \\ + 666(2\pi p)^6 + 8190(2\pi p)^5 + 72\,765(2\pi p)^4 + 478\,170(2\pi p)^3 \\ + 2\,297\,295(2\pi p)^2 + 7\,702\,695(2\pi p) + 16\,216\,200) (2\pi p) e^{-2\pi p}.$$

The infinite sum that is terminated at a finite term λ is used for the approximate calculation for the zeta function.

Now, I decide that the positive integer λ will be referred to as the degree of approximation.

The approximate formula for the zeta function is written as $(\text{zeta} - \text{hat})_{\lambda}(2k+1)$.

$$(7-69) \quad \hat{\zeta}_{\lambda}(2k+1) := \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2) \\ - \frac{2}{k(2k-1)!!} \sum_{p=1}^{\lambda} \sigma_{-(2k+1)}(p) \left(\sum_{\mu=0}^{k+1} b_{k,\mu} (2\pi p)^{\mu} \right) e^{-2\pi p} \quad (k, \lambda \in \mathbb{N}).$$

I use the built-in Mathematica functions to perform numerical calculations, for example, `DivisorSigma[θ , n]` and `Zeta[S]`.

Furthermore, when calculating the values of the function $(\text{zeta} - \text{hat})_{\lambda}(2k+1)$, the Mathematica inputs are shown as follows “[17]”:

$$\mathbf{b}\{\{k_{-}, \mu_{-}\}\} := \frac{\mu(\mu-1) + 2k(k+1-\mu)}{2^{k+1-\mu} \mu! (k+1-\mu)!};$$

$$\text{zh}[\{k_-, \lambda_-\}] := \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \text{Zeta}[2k+2] - \frac{2}{k(2k-1)!!}$$

$$\text{Sum}[\text{DivisorSigma}[-(2k+1), p] (\text{Sum}[\text{b}[\{k, \mu\}] (2\pi p)^\mu, \{\mu, 0, k+1\}]) e^{-2\pi p}, \{p, 1, \lambda\}];$$

$$\text{N}[\{\text{zh}[\{1, 1\}], \text{zh}[\{1, 2\}], \text{zh}[\{1, 3\}], \text{zh}[\{1, 4\}], \text{Zeta}[3], 9]$$

By increasing the degree of approximation, it asymptotes the true value of the zeta function for any odd number of 3 or more.

$$(7-70) \quad \{\hat{\zeta}_1(3), \hat{\zeta}_2(3), \hat{\zeta}_3(3), \hat{\zeta}_4(3), \zeta(3)\}$$

$$\approx \{1.20340751, 1.20206199, 1.20205692, 1.20205690, 1.20205690\}.$$

$$(7-71) \quad \{\hat{\zeta}_1(5), \hat{\zeta}_2(5), \hat{\zeta}_3(5), \hat{\zeta}_4(5), \zeta(5)\}$$

$$\approx \{1.03998935, 1.03694501, 1.03692783, 1.03692776, 1.03692776\}.$$

$$(7-72) \quad \{\hat{\zeta}_1(7), \hat{\zeta}_2(7), \hat{\zeta}_3(7), \hat{\zeta}_4(7), \zeta(7)\}$$

$$\approx \{1.01476857, 1.00840017, 1.00834956, 1.00834928, 1.00834928\}.$$

$$(7-73) \quad \{\hat{\zeta}_1(9), \hat{\zeta}_2(9), \hat{\zeta}_3(9), \hat{\zeta}_4(9), \hat{\zeta}_5(9), \zeta(9)\}$$

$$\approx \{1.01392241, 1.00213605, 1.00200927, 1.00200840, 1.00200839, 1.00200839\}.$$

By choosing the approximation appropriately, for example, a value can be obtained with 30 decimal places of precision.

$$(7-74) \quad \{\hat{\zeta}_{12}(3), \zeta(3)\}$$

$$\approx \{1.202056903159594285399738161511, 1.202056903159594285399738161511\}.$$

$$(7-75) \quad \{\hat{\zeta}_{13}(5), \zeta(5)\}$$

$$\approx \{1.036927755143369926331365486457, 1.036927755143369926331365486457\}.$$

$$(7-76) \quad \{\hat{\zeta}_{13}(7), \zeta(7)\}$$

$$\approx \{1.008349277381922826839797549850, 1.008349277381922826839797549850\}.$$

$$(7-77) \quad \{\hat{\zeta}_{14}(9), \zeta(9)\}$$

$$\approx \{1.002008392826082214417852769232, 1.002008392826082214417852769232\}.$$

8. Study of the zeta function for any even number of 0 or more, and for any odd number of -1 or less

The right side of the second-order I_c type functional equation is replaced with the result of equation (4-8). The equation (4-8) is shown again.

$$(8-1) \quad \mathbf{R}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (-3n^2 y^{2-\theta} + 2\pi n^4 y^{4-\theta}) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} dy.$$

According to the knowledge in Section 4, the integral and the sum can be interchanged. Therefore,

$$(8-2) \quad \mathbf{R}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \sum_{n=1}^\infty \int_0^\infty (-3n^2 y^{2-\theta} + 2\pi n^4 y^{4-\theta}) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} dy.$$

For the integrals of $\mathbf{R}_{[2]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi n}}.$$

$$(8-3) \mathbf{R}_{[2]}(\theta) =$$

$$\begin{aligned} & \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \int_0^{\infty} \left(-3 n^2 \left(\frac{x}{\sqrt{\pi n}} \right)^{2-\theta} + 2 \pi n^4 \left(\frac{x}{\sqrt{\pi n}} \right)^{4-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} \frac{1}{\sqrt{\pi n}} dx \\ &= \frac{\theta-1}{\pi} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(-3 \int_0^{\infty} x^{2-\theta} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx + 2 \int_0^{\infty} x^{4-\theta} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx \right). \end{aligned}$$

Equation (4-7) is shown again.

$$(8-4) \mathbf{L}_{[2]}(\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(8-5) \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{\pi} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(-3 \int_0^{\infty} x^{2-\theta} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx + 2 \int_0^{\infty} x^{4-\theta} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx \right) \quad (\theta \in \mathbb{C}).$$

Both sides of the equation (8-5) are multiplied by

$$\frac{2\pi}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)} \pi^{\frac{\theta}{2}}$$

to obtain the zeta function $\zeta(\theta)$.

$$(8-6) \zeta(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(-3 \int_0^{\infty} x^{2-\theta} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx + 2 \int_0^{\infty} x^{4-\theta} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

When the complex variable θ is set to any even number of 4 or less, the representation type1 for the zeta function with the quadrable integrals is obtained.

$$(8-7) \zeta(2k) = \frac{2\pi^{2k}}{(2k-1)\Gamma(k+1)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n^{1-2k}} \left(-3 \int_0^{\infty} x^{2(1-k)} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx + 2 \int_0^{\infty} x^{2(2-k)} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx \right) \quad (k = 2, 1, 0, -1, -2, \dots).$$

When the complex variable θ is set to any even number of -2 or less, the zeta function has the value zero at these points (the trivial zeros). Furthermore, knowing that no even number of 6 or more can be calculated using equation (8-7), the equation of definition (4-2) is presented again.

$$(8-8) \mathbf{R}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-3 n^2 x^{\theta-4} + 2 \pi n^4 x^{\theta-6} \right) e^{-\pi \left(\frac{n^2}{x^2} + \alpha^2 x^2 \right)} dx.$$

Similarly, the integral and the sum can be interchanged. Therefore,

$$(8-9) \mathbf{R}_{[2]}(\theta) = \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \int_0^{\infty} \left(-3 n^2 x^{\theta-4} + 2 \pi n^4 x^{\theta-6} \right) e^{-\pi \left(\frac{n^2}{x^2} + \alpha^2 x^2 \right)} dx.$$

For the integrals of $R_{[2]}(\theta)$, I perform the variable transformation

$$x = \frac{y}{\sqrt{\pi} \alpha}.$$

$$\begin{aligned} (8-10) \quad R_{[2]}(\theta) &= \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \int_0^{\infty} \left(-3 n^2 \left(\frac{y}{\sqrt{\pi} \alpha} \right)^{\theta-4} \right. \\ &\quad \left. + 2 \pi n^4 \left(\frac{y}{\sqrt{\pi} \alpha} \right)^{\theta-6} \right) e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} \frac{1}{\sqrt{\pi} \alpha} dy \\ &= \frac{\theta-1}{\pi} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^{\theta-3} \sum_{n=1}^{\infty} n^2 \left(-3 \int_0^{\infty} y^{\theta-4} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy \right. \\ &\quad \left. + 2 (\pi n \alpha)^2 \int_0^{\infty} y^{\theta-6} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy \right). \end{aligned}$$

From the equation $L_{[2]}(\theta) = R_{[2]}(\theta)$,

$$\begin{aligned} (8-11) \quad \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) &= \frac{\theta-1}{\pi} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^{\theta-3} \sum_{n=1}^{\infty} n^2 \\ &\quad \left(-3 \int_0^{\infty} y^{\theta-4} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy + 2 (\pi n \alpha)^2 \int_0^{\infty} y^{\theta-6} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy \right) \quad (\theta \in \mathbb{C}). \end{aligned}$$

Both sides of the equation (8-11) are multiplied by

$$\frac{2\pi}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)} \pi^{\frac{\theta}{2}}$$

to obtain the zeta function $\zeta(\theta)$.

$$\begin{aligned} (8-12) \quad \zeta(\theta) &= \frac{2\pi^{\theta}}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^{\theta-3} \sum_{n=1}^{\infty} n^2 \left(-3 \int_0^{\infty} y^{\theta-4} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy \right. \\ &\quad \left. + 2 (\pi n \alpha)^2 \int_0^{\infty} y^{\theta-6} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy \right) \quad (\theta \in \mathbb{C} \setminus \{1\}). \end{aligned}$$

When the complex variable θ is set to any even number of 6 or more, I obtained the representation type2 for the zeta function with the quadrable integrals.

$$\begin{aligned} (8-13) \quad \zeta(2k) &= \frac{2\pi^{2k}}{(2k-1)\Gamma(k+1)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^{2k-3} \sum_{n=1}^{\infty} n^2 \\ &\quad \left(-3 \int_0^{\infty} y^{2(k-2)} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy + 2 (\pi n \alpha)^2 \int_0^{\infty} y^{2(k-3)} e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2} \right)} dy \right) \\ &\quad (k = 3, 4, 5, \dots). \end{aligned}$$

Both sides of the complete symmetric functional equation are multiplied by

$$\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\theta}{2}\right)} \pi^{-\frac{\theta}{2}}$$

to obtain the function $\zeta(1 - \theta)$.

$$(8-14) \quad \zeta(1 - \theta) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\theta}{2}\right)} \pi^{-\theta} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C} \setminus \{0\}).$$

For equation (8-7), 0 is substituted for k.

$$(8-15) \quad \zeta(0) =$$

$$\begin{aligned} & \frac{2\pi^0}{(-1)\Gamma(1)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n} \left(-3 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx + 2 \int_0^{\infty} x^4 e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx \right) \\ &= -\frac{2}{\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n} \left(-3 \cdot \frac{(2\pi n \alpha) + 1}{4} \sqrt{\pi} e^{-2\pi n \alpha} + 2 \cdot \frac{(2\pi n \alpha)^2 + 3(2\pi n \alpha) + 3}{8} \sqrt{\pi} e^{-2\pi n \alpha} \right) \\ &= -2 \lim_{\alpha \rightarrow +0} (\pi \alpha)^2 \sum_{n=1}^{\infty} n e^{-2\pi n \alpha} = -2 \lim_{\alpha \rightarrow +0} (\pi \alpha)^2 \cdot \frac{1}{4} \left(\left(\frac{1}{\pi \alpha}\right)^2 - \frac{1}{3} + \mathbf{O}(\pi \alpha)^2 \right). \end{aligned}$$

Therefore

$$(8-16) \quad \zeta(0) = -\frac{1}{2} \lim_{\alpha \rightarrow +0} (1 - \mathbf{O}(\pi \alpha)^2) = -\frac{1}{2}.$$

For equation (8-7), 1 is substituted for k.

$$(8-17) \quad \zeta(2) =$$

$$\begin{aligned} & \frac{2\pi^2}{\Gamma(2)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n^{-1}} \left(-3 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx + 2 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx \right) \\ &= \frac{2\pi^2}{\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n \left(-3 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi n \alpha} + 2 \cdot \frac{(2\pi n \alpha) + 1}{4} \sqrt{\pi} e^{-2\pi n \alpha} \right) \\ &= 2\pi^2 \lim_{\alpha \rightarrow +0} \left((\pi \alpha) \cdot \frac{1}{4} \left(\left(\frac{1}{\pi \alpha}\right)^3 - \frac{(\pi \alpha)}{15} + \mathbf{O}(\pi \alpha)^3 \right) - \frac{1}{4} \left(\left(\frac{1}{\pi \alpha}\right)^2 - \frac{1}{3} + \frac{(\pi \alpha)^2}{15} - \mathbf{O}(\pi \alpha)^4 \right) \right) \\ &= 2\pi^2 \lim_{\alpha \rightarrow +0} \left(\left(\frac{1}{4} - \frac{1}{4}\right) \left(\frac{1}{\pi \alpha}\right)^2 - \frac{1}{4} \left(-\frac{1}{3}\right) - \mathbf{O}(\pi \alpha)^2 \right). \end{aligned}$$

Therefore

$$(8-18) \quad \zeta(2) = 2\pi^2 \lim_{\alpha \rightarrow +0} \left(0 \cdot \left(\frac{1}{\pi \alpha}\right)^2 + \frac{1}{12} - \mathbf{O}(\pi \alpha)^2 \right) = \frac{\pi^2}{6}.$$

For equation (8-14), 2 is substituted for θ .

$$(8-19) \quad \zeta(-1) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{1}{2}\right)} \pi^{-2} \Gamma(1) \zeta(2) = -\frac{1}{12}.$$

For equation (8-7), 2 is substituted for k.

$$(8-20) \quad \zeta(4) =$$

$$\begin{aligned} & \frac{2\pi^4}{3\Gamma(3)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} \frac{1}{n^{-3}} \left(-3 \int_0^{\infty} x^{-2} e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx + 2 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx \right) \\ &= \frac{\pi^4}{3\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \sum_{n=1}^{\infty} n^3 \left(-3 \cdot \frac{\sqrt{\pi}}{2\pi n \alpha} e^{-2\pi n \alpha} + 2 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi n \alpha} \right) \\ &= \frac{\pi^4}{6} \lim_{\alpha \rightarrow +0} \left(-\frac{3}{\pi \alpha} \sum_{n=1}^{\infty} n^2 e^{-2\pi n \alpha} + 2 \sum_{n=1}^{\infty} n^3 e^{-2\pi n \alpha} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi^4}{6} \lim_{\alpha \rightarrow +0} \left(-\frac{3}{\pi \alpha} \cdot \frac{1}{4} \left(\left(\frac{1}{\pi \alpha} \right)^3 - \frac{(\pi \alpha)}{15} + \frac{4(\pi \alpha)^3}{189} - \mathbf{O}(\pi \alpha)^5 \right) + \right. \\
&\quad \left. 2 \cdot \frac{3}{8} \left(\left(\frac{1}{\pi \alpha} \right)^4 + \frac{1}{45} - \frac{4(\pi \alpha)^2}{189} + \mathbf{O}(\pi \alpha)^4 \right) \right) \\
&= \frac{\pi^4}{6} \lim_{\alpha \rightarrow +0} \left(\left(-3 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} \right) \left(\frac{1}{\pi \alpha} \right)^4 - 3 \cdot \frac{1}{4} \left(-\frac{1}{15} \right) + 2 \cdot \frac{3}{8} \cdot \frac{1}{45} - \mathbf{O}(\pi \alpha)^2 \right).
\end{aligned}$$

Therefore

$$(8-21) \quad \zeta(4) = \frac{\pi^4}{6} \lim_{\alpha \rightarrow +0} \left(0 \cdot \left(\frac{1}{\pi \alpha} \right)^4 + \frac{1}{15} - \mathbf{O}(\pi \alpha)^2 \right) = \frac{\pi^4}{90}.$$

For equation (8-14), 4 is substituted for θ .

$$(8-22) \quad \zeta(-3) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{3}{2}\right)} \pi^{-4} \Gamma(2) \zeta(4) = \frac{1}{120}.$$

For equation (8-13), 3 is substituted for k .

$$\begin{aligned}
(8-23) \quad \zeta(6) &= \frac{2\pi^6}{5\Gamma(4)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^3 \sum_{n=1}^{\infty} n^2 \left(-3 \int_0^{\infty} y^2 e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2}\right)} dy \right. \\
&\quad \left. + 2(\pi n \alpha)^2 \int_0^{\infty} y^0 e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2}\right)} dy \right) \\
&= \frac{\pi^6}{15\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^3 \sum_{n=1}^{\infty} n^2 \left(-3 \cdot \frac{(2\pi n \alpha) + 1}{4} \sqrt{\pi} e^{-2\pi n \alpha} + \frac{(2\pi n \alpha)^2}{2} \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi n \alpha} \right) \\
&= \frac{\pi^6}{60} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^3 \left(4(\pi \alpha)^2 \sum_{n=1}^{\infty} n^4 e^{-2\pi n \alpha} - 6(\pi \alpha) \sum_{n=1}^{\infty} n^3 e^{-2\pi n \alpha} - 3 \sum_{n=1}^{\infty} n^2 e^{-2\pi n \alpha} \right) \\
&= \frac{\pi^6}{60} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi \alpha} \right)^3 \left(4(\pi \alpha)^2 \cdot \frac{3}{4} \left(\left(\frac{1}{\pi \alpha} \right)^5 + \frac{2(\pi \alpha)}{189} - \frac{(\pi \alpha)^3}{135} + \mathbf{O}(\pi \alpha)^5 \right) \right. \\
&\quad \left. - 6(\pi \alpha) \cdot \frac{3}{8} \left(\left(\frac{1}{\pi \alpha} \right)^4 + \frac{1}{45} - \frac{4(\pi \alpha)^2}{189} + \frac{(\pi \alpha)^4}{135} - \mathbf{O}(\pi \alpha)^6 \right) \right. \\
&\quad \left. - 3 \cdot \frac{1}{4} \left(\left(\frac{1}{\pi \alpha} \right)^3 - \frac{(\pi \alpha)}{15} + \frac{4(\pi \alpha)^3}{189} - \frac{(\pi \alpha)^5}{225} + \mathbf{O}(\pi \alpha)^7 \right) \right) \\
&= \frac{\pi^6}{60} \lim_{\alpha \rightarrow +0} \left(\left(4 \cdot \frac{3}{4} - 6 \cdot \frac{3}{8} - 3 \cdot \frac{1}{4} \right) \left(\frac{1}{\pi \alpha} \right)^6 + \left(-6 \cdot \frac{3}{8} \cdot \frac{1}{45} - 3 \cdot \frac{1}{4} \left(-\frac{1}{15} \right) \right) \left(\frac{1}{\pi \alpha} \right)^2 \right. \\
&\quad \left. + 4 \cdot \frac{3}{4} \cdot \frac{2}{189} - 6 \cdot \frac{3}{8} \left(-\frac{4}{189} \right) - 3 \cdot \frac{1}{4} \cdot \frac{4}{189} - \mathbf{O}(\pi \alpha)^2 \right).
\end{aligned}$$

Therefore

$$(8-24) \quad \zeta(6) = \frac{\pi^6}{60} \lim_{\alpha \rightarrow +0} \left(0 \cdot \left(\left(\frac{1}{\pi \alpha} \right)^6 + \left(\frac{1}{\pi \alpha} \right)^2 \right) + \frac{4}{63} - \mathbf{O}(\pi \alpha)^2 \right) = \frac{\pi^6}{945}.$$

For equation (8-14), 6 is substituted for θ .

$$(8-25) \quad \zeta(-5) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{5}{2}\right)} \pi^{-6} \Gamma(3) \zeta(6) = -\frac{1}{252}.$$

For equation (8-13), 4 is substituted for k .

$$\begin{aligned}
 (8-26) \quad \zeta(8) &= \frac{2\pi^8}{7\Gamma(5)\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi\alpha} \right)^5 \sum_{n=1}^{\infty} n^2 \left(-3 \int_0^{\infty} y^4 e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2}\right)} dy \right. \\
 &\quad \left. + 2(\pi n \alpha)^2 \int_0^{\infty} y^2 e^{-\left(y^2 + \frac{(\pi n \alpha)^2}{y^2}\right)} dy \right) \\
 &= \frac{\pi^8}{84\sqrt{\pi}} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi\alpha} \right)^5 \sum_{n=1}^{\infty} n^2 \left(-3 \cdot \frac{(2\pi n \alpha)^2 + 3(2\pi n \alpha) + 3}{8} \sqrt{\pi} e^{-2\pi n \alpha} \right. \\
 &\quad \left. + \frac{(2\pi n \alpha)^2}{2} \cdot \frac{(2\pi n \alpha) + 1}{4} \sqrt{\pi} e^{-2\pi n \alpha} \right) \\
 &= \frac{\pi^8}{672} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi\alpha} \right)^5 \left(8(\pi\alpha)^3 \sum_{n=1}^{\infty} n^5 e^{-2\pi n \alpha} - 8(\pi\alpha)^2 \sum_{n=1}^{\infty} n^4 e^{-2\pi n \alpha} \right. \\
 &\quad \left. - 18(\pi\alpha) \sum_{n=1}^{\infty} n^3 e^{-2\pi n \alpha} - 9 \sum_{n=1}^{\infty} n^2 e^{-2\pi n \alpha} \right) \\
 &= \frac{\pi^8}{672} \lim_{\alpha \rightarrow +0} \left(\frac{1}{\pi\alpha} \right)^5 \left(8(\pi\alpha)^3 \cdot \frac{15}{8} \left(\left(\frac{1}{\pi\alpha} \right)^6 - \frac{2}{945} + \frac{(\pi\alpha)^2}{225} - \frac{4(\pi\alpha)^4}{1485} + \mathbf{O}(\pi\alpha)^6 \right) \right. \\
 &\quad - 8(\pi\alpha)^2 \cdot \frac{3}{4} \left(\left(\frac{1}{\pi\alpha} \right)^5 + \frac{2(\pi\alpha)}{189} - \frac{(\pi\alpha)^3}{135} + \frac{4(\pi\alpha)^5}{1485} - \mathbf{O}(\pi\alpha)^7 \right) \\
 &\quad - 18(\pi\alpha) \cdot \frac{3}{8} \left(\left(\frac{1}{\pi\alpha} \right)^4 + \frac{1}{45} - \frac{4(\pi\alpha)^2}{189} + \frac{(\pi\alpha)^4}{135} - \frac{8(\pi\alpha)^6}{4455} + \mathbf{O}(\pi\alpha)^8 \right) \\
 &\quad \left. - 9 \cdot \frac{1}{4} \left(\left(\frac{1}{\pi\alpha} \right)^3 - \frac{(\pi\alpha)}{15} + \frac{4(\pi\alpha)^3}{189} - \frac{(\pi\alpha)^5}{225} + \frac{8(\pi\alpha)^7}{10395} - \mathbf{O}(\pi\alpha)^9 \right) \right) \\
 &= \frac{\pi^8}{672} \lim_{\alpha \rightarrow +0} \left(\left(8 \cdot \frac{15}{8} - 8 \cdot \frac{3}{4} - 18 \cdot \frac{3}{8} - 9 \cdot \frac{1}{4} \right) \left(\frac{1}{\pi\alpha} \right)^8 + \left(-18 \cdot \frac{3}{8} \cdot \frac{1}{45} - 9 \cdot \frac{1}{4} \left(-\frac{1}{15} \right) \right) \left(\frac{1}{\pi\alpha} \right)^4 \right. \\
 &\quad \left. + \left(8 \cdot \frac{15}{8} \left(-\frac{2}{945} \right) - 8 \cdot \frac{3}{4} \cdot \frac{2}{189} - 18 \cdot \frac{3}{8} \left(-\frac{4}{189} \right) - 9 \cdot \frac{1}{4} \cdot \frac{4}{189} \right) \left(\frac{1}{\pi\alpha} \right)^2 \right. \\
 &\quad \left. + 8 \cdot \frac{15}{8} \cdot \frac{1}{225} - 8 \cdot \frac{3}{4} \left(-\frac{1}{135} \right) - 18 \cdot \frac{3}{8} \cdot \frac{1}{135} - 9 \cdot \frac{1}{4} \left(-\frac{1}{225} \right) - \mathbf{O}(\pi\alpha)^2 \right).
 \end{aligned}$$

Therefore

$$(8-27) \quad \zeta(8) = \frac{\pi^8}{672} \lim_{\alpha \rightarrow +0} \left(0 \cdot \left(\left(\frac{1}{\pi\alpha} \right)^8 + \left(\frac{1}{\pi\alpha} \right)^4 + \left(\frac{1}{\pi\alpha} \right)^2 \right) + \frac{16}{225} - \mathbf{O}(\pi\alpha)^2 \right) = \frac{\pi^8}{9450}.$$

For equation (8-14), 8 is substituted for θ .

$$(8-28) \quad \zeta(-7) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{7}{2}\right)} \pi^{-8} \Gamma(4) \zeta(8) = \frac{1}{240}.$$

For even numbers of 2, 4, 6, and 8, the zeta function has the shape of

(Coefficient) \times {(0 \times (Divergent terms) + (Constant) - (Vanishing terms)} in the stage just before taking the limit.

Let remember equation (7-26), which is shown again.

$$\begin{aligned}
 (8-29) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{2k+1}} &= \frac{(k+1)(k-1)! 2^k}{(2k-1)!! \pi} \zeta(2k+2) \\
 - \frac{2}{k(2k-1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) &\left(\sum_{\mu=0}^k b_{k,\mu+1}(2\pi p)^{\mu} \right) (2\pi p) e^{-2\pi p} \quad (k \in \mathbb{N}).
 \end{aligned}$$

The series representation of the zeta function is demonstrated for every even number of 4 or more.

The following are some calculation examples:

$$(8-30) \zeta(4) = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} + \frac{\pi}{2} \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p) + 1) (2\pi p) e^{-2\pi p}$$

$$= \frac{7\pi^4}{720} + \frac{\pi}{2} \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p) + 1) (2\pi p) e^{-2\pi p}.$$

$$(8-31) \zeta(6) = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^5} + \frac{\pi}{12} \sum_{p=1}^{\infty} \sigma_{-5}(p) ((2\pi p)^2 + 3(2\pi p) + 6) (2\pi p) e^{-2\pi p}.$$

$$(8-32) \zeta(8) =$$

$$\frac{15\pi}{64} \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} + \frac{\pi}{96} \sum_{p=1}^{\infty} \sigma_{-7}(p) ((2\pi p)^3 + 6(2\pi p)^2 + 21(2\pi p) + 45) (2\pi p) e^{-2\pi p}$$

$$= \frac{19\pi^8}{241920} + \frac{\pi}{96} \sum_{p=1}^{\infty} \sigma_{-7}(p) ((2\pi p)^3 + 6(2\pi p)^2 + 21(2\pi p) + 45) (2\pi p) e^{-2\pi p}.$$

$$(8-33) \zeta(10) = \frac{7\pi}{32} \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^9}$$

$$+ \frac{\pi}{960} \sum_{p=1}^{\infty} \sigma_{-9}(p) ((2\pi p)^4 + 10(2\pi p)^3 + 55(2\pi p)^2 + 195(2\pi p) + 420) (2\pi p) e^{-2\pi p}.$$

The zeta function has the shape {(Constant or infinite series to be considered as constant) + (Coefficient) × (Infinite series)} for any even number of 4 or more.

To understand the kind of series representation $\zeta(2)$ have, recall equation (7-6), which is shown again.

$$(8-34) \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\theta(\theta+1)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta)$$

$$- \frac{\theta}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

After exchanging both sides of the equation by moving the second term from the left to the right side,

$$(8-35) \frac{\theta(\theta+1)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta)$$

$$+ \frac{\theta}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 2(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

For the second term on the right side of equation (8-35), the recurrence formula for the modified Bessel function of the second kind is applied.

$$(8-36) \frac{\theta}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 4(\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right)$$

$$= \frac{\theta}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta-1) K_{\frac{\theta}{2}}(2\pi p) + 4 \left((\pi p) K_{\frac{-\theta-2}{2}}(2\pi p) + \left(\frac{-\theta}{2}\right) K_{\frac{-\theta}{2}}(2\pi p) \right) \right)$$

$$= \frac{\theta}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(4(\pi p) K_{\frac{\theta+2}{2}}(2\pi p) - (\theta+1) K_{\frac{\theta}{2}}(2\pi p) \right)$$

$$= \frac{\theta\pi^{-\frac{\theta}{2}}}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(4(\pi p)^{\frac{\theta+2}{2}} K_{\frac{\theta+2}{2}}(2\pi p) - (\theta+1)(\pi p)^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi p) \right)$$

$$= \frac{\theta \pi^{-\frac{\theta}{2}}}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(4 \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy - (\theta+1) \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right) \quad (\theta \in \mathbb{C}).$$

Here, the relation of equation (19-74) is used.

Therefore

$$(8-37) \quad \frac{\theta(\theta+1)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) = \frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) + \frac{\theta \pi^{-\frac{\theta}{2}}}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(4 \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy - (\theta+1) \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right) \quad (\theta \in \mathbb{C}).$$

Both sides of the equation (8-37) are multiplied by

$$\frac{2\pi}{\theta \Gamma\left(1 + \frac{1+\theta}{2}\right)} \pi^{\frac{1+\theta}{2}}$$

to obtain the function $\zeta(1+\theta)$.

$$(8-38) \quad \zeta(1+\theta) = \frac{(\theta-1) \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \sqrt{\pi}}{2 \Gamma\left(1 + \frac{1+\theta}{2}\right)} + \frac{2 \sqrt{\pi}}{\Gamma\left(1 + \frac{1+\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(4 \int_0^{\infty} y^{\theta+1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy - (\theta+1) \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right) \quad (\theta \in \mathbb{C} \setminus \{0\}).$$

For equation (8-38), -1 is substituted for θ .

$$(8-39) \quad \zeta(0) = \frac{-2 \Gamma\left(-\frac{1}{2}\right) \zeta(-1) \sqrt{\pi}}{2 \Gamma(1)} + \frac{2 \sqrt{\pi}}{\Gamma(1)} \sum_{p=1}^{\infty} \sigma_1(p) \left(4 \int_0^{\infty} y^0 e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy - 0 \cdot \int_0^{\infty} y^{-2} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right) \\ = 2\pi \zeta(-1) + 2 \sqrt{\pi} \sum_{p=1}^{\infty} \sigma_1(p) \cdot 4 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi p} \\ = 2\pi \zeta(-1) + 4\pi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m e^{-2\pi m n} = 2\pi \zeta(-1) + \pi \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n)}.$$

Therefore

$$(8-40) \quad \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n)} = \frac{1}{\pi} (\zeta(0) - 2\pi \zeta(-1)) = \frac{1}{\pi} \left(-\frac{1}{2} - 2\pi \left(-\frac{1}{12}\right)\right) = \frac{1}{6} - \frac{1}{2\pi}.$$

This result is known as “[1].”

For equation (8-38), 1 is substituted for θ .

$$(8-41) \zeta(2) =$$

$$\begin{aligned} & \frac{\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(2)} + \frac{2\sqrt{\pi}}{\Gamma(2)} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(4 \int_0^{\infty} y^2 e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy - 2 \int_0^{\infty} y^0 e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right) \\ &= \frac{\pi}{2} + 2\sqrt{\pi} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(4 \cdot \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} - 2 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi p} \right) \\ &= \frac{\pi}{2} + 2\pi \sum_{p=1}^{\infty} \sigma_{-1}(p) (2\pi p) e^{-2\pi p} = \frac{\pi}{2} + 4\pi^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n} (m n) e^{-2\pi m n}. \end{aligned}$$

Thus, the double series representation is obtained.

$$(8-42) \zeta(2) = \frac{\pi}{2} + 4\pi^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m e^{-2\pi m n}.$$

The hyperbolic sine is used to obtain the series representation.

$$(8-43) \zeta(2) = \frac{\pi}{2} + \pi^2 \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n)}.$$

The exact value of $\zeta(2)$ is finally reached:

$$(8-44) \zeta(2) = \frac{\pi}{2} + \pi^2 \left(\frac{1}{6} - \frac{1}{2\pi} \right) = \frac{\pi^2}{6}.$$

9. Hadamard product representation for the zeta function and derivation of a new Chi function

The Riemann Xi function's Hadamard product is shown as follows:

$$(9-1) \xi(\theta) = \xi(0) \prod_{m=1}^{\infty} \left(1 - \frac{\theta}{\frac{1}{2} + i\rho_m} \right) \left(1 - \frac{\theta}{\frac{1}{2} - i\rho_m} \right) = \frac{1}{2} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (2\theta - 1)^2}{(2\rho_m)^2 + 1} \quad (\theta \in \mathbb{C}).$$

The definition of the Riemann Xi function is again presented.

$$(9-2) \xi(\theta) = \frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore, the zeta function's Hadamard product representation is as follows:

$$(9-3) \zeta(\theta) = \frac{\pi^{\frac{\theta}{2}}}{\theta(\theta-1)\Gamma\left(\frac{\theta}{2}\right)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (2\theta - 1)^2}{(2\rho_m)^2 + 1} \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

For equation (9-3), 0 is substituted for θ .

$$(9-4) \zeta(0) = \frac{\pi^{\frac{0}{2}}}{2(0-1)\Gamma\left(1 + \frac{0}{2}\right)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (2\theta - 1)^2}{(2\rho_m)^2 + 1} \Big|_{\theta=0} = -\frac{1}{2} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 1}{(2\rho_m)^2 + 1} = -\frac{1}{2}.$$

Any odd number of -1 or less is substituted for θ in equation (9-3).

$$(9-5) \zeta(1-2k) = \frac{\pi^{\frac{1-2k}{2}}}{(1-2k)(1-2k-1)\Gamma\left(\frac{1-2k}{2}\right)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (2(1-2k) - 1)^2}{(2\rho_m)^2 + 1}$$

$$\begin{aligned}
 &= \frac{\sqrt{\pi}}{2k(2k-1)(-1)^k \frac{2^k \sqrt{\pi}}{(2k-1)!!} \pi^k} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1} \\
 &= -\frac{1}{2k} \cdot \frac{(-1)^{k+1} (2k-3)!!}{2^k \pi^k} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1} \quad (k \in \mathbb{N}).
 \end{aligned}$$

Here, $(-1)!! = 1$ is agreed.

Conversely, the exact value of the zeta function for any odd number of -1 or less is well known as the formula.

$$(9-6) \quad \zeta(1-2k) = -\frac{B_{2k}}{2k} \quad (k \in \mathbb{N}).$$

By combining the equations (9-5) and (9-6),

$$(9-7) \quad \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1} = \frac{(-1)^{k+1} 2^k \pi^k}{(2k-3)!!} B_{2k} \quad (k \in \mathbb{N}).$$

Some examples of calculations are shown.

$$(9-8) \quad \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 3^2}{(2\rho_m)^2 + 1} = \frac{(-1)^2 2^2 \pi}{(-1)!!} B_2 = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3}.$$

$$(9-9) \quad \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 7^2}{(2\rho_m)^2 + 1} = \frac{(-1)^3 2^2 \pi^2}{1!!} B_4 = -4\pi^2 \cdot \left(-\frac{1}{30}\right) = \frac{2\pi^2}{15}.$$

$$(9-10) \quad \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 11^2}{(2\rho_m)^2 + 1} = \frac{(-1)^4 2^3 \pi^3}{3!!} B_6 = \frac{8\pi^3}{3} \cdot \frac{1}{42} = \frac{4\pi^3}{63}.$$

$$(9-11) \quad \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 15^2}{(2\rho_m)^2 + 1} = \frac{(-1)^5 2^4 \pi^4}{5!!} B_8 = -\frac{16\pi^4}{15} \cdot \left(-\frac{1}{30}\right) = \frac{8\pi^4}{225}.$$

For equation (9-3), any even number of 2 or more is substituted for θ .

$$\begin{aligned}
 (9-12) \quad \zeta(2k) &= \frac{\pi^k}{2k(2k-1)\Gamma(k)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1} \\
 &= \frac{\pi^k}{2(2k-1)k!} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k-1)^2}{(2\rho_m)^2 + 1} \quad (k \in \mathbb{N}).
 \end{aligned}$$

Any odd number of 3 or more is substituted for θ in equation (9-3).

$$\begin{aligned}
 (9-13) \quad \zeta(2k+1) &= \frac{\pi \frac{2k+1}{2}}{(2k+1)(2k)\Gamma\left(\frac{2k+1}{2}\right)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k+1)^2}{(2\rho_m)^2 + 1} \\
 &= \frac{2^{k-1} \pi^k}{k(2k+1)!!} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (4k+1)^2}{(2\rho_m)^2 + 1} \quad (k \in \mathbb{N}).
 \end{aligned}$$

The following are some calculation examples:

$$(9-14) \quad \zeta(2) = \frac{\pi}{2} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 3^2}{(2\rho_m)^2 + 1} = \frac{\pi}{2} \cdot \frac{\pi}{3} = \frac{\pi^2}{6}.$$

$$(9-15) \quad \zeta(3) = \frac{\pi}{3} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 5^2}{(2\rho_m)^2 + 1}.$$

$$(9-16) \quad \zeta(4) = \frac{\pi^2}{12} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 7^2}{(2\rho_m)^2 + 1} = \frac{\pi^2}{12} \cdot \frac{2\pi^2}{15} = \frac{\pi^4}{90}.$$

$$(9-17) \quad \zeta(5) = \frac{\pi^2}{15} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 9^2}{(2\rho_m)^2 + 1}.$$

$$(9-18) \quad \zeta(6) = \frac{\pi^3}{60} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 11^2}{(2\rho_m)^2 + 1} = \frac{\pi^3}{60} \cdot \frac{4\pi^3}{63} = \frac{\pi^6}{945}.$$

$$(9-19) \quad \zeta(7) = \frac{4\pi^3}{315} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 13^2}{(2\rho_m)^2 + 1}.$$

$$(9-20) \quad \zeta(8) = \frac{\pi^4}{336} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 15^2}{(2\rho_m)^2 + 1} = \frac{\pi^4}{336} \cdot \frac{8\pi^4}{225} = \frac{\pi^8}{9450}.$$

$$(9-21) \quad \zeta(9) = \frac{2\pi^4}{945} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + 17^2}{(2\rho_m)^2 + 1}.$$

9-2. Derivation of a new Chi function

By the way, the origin symmetric functional equation with correction terms is similar to the functional equation that the Riemann Xi function follows.

The new function, called the Chi function, is defined as follows:

$$(9-22) \quad \chi(\theta) := 2\pi \left(A_{[2]}(\theta) + \hat{F}_{[2]}(\theta) \right) \quad (\theta \in \mathbb{C}).$$

To generate the representation for the Chi function, equations (5-82) and (5-84) are substituted on the right side of equation (9-22).

$$(9-23) \quad \chi(\theta) = \frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \\ + 2(\theta-1) \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

The Chi function has clear origin symmetry based on functional equation (5-81).

$$(9-24) \quad \chi(\theta) = \chi(-\theta) \quad (\theta \in \mathbb{C}).$$

For equation (9-23), 1 is substituted for θ .

$$(9-25) \quad \chi(1) = \frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \Big|_{\theta=1} + 2 \cdot 0 \cdot \sum_{p=1}^{\infty} \sigma_{-\theta}(p) (0 \cdot K_0(2\pi p) + (2\pi p) K_1(2\pi p)) \\ = -\theta \pi^{-\frac{1-\theta}{2}} \Gamma\left(1 + \frac{1-\theta}{2}\right) \zeta(1-\theta) \Big|_{\theta=1} = 1/2.$$

Therefore

$$(9-26) \quad \chi(\pm 1) = 1/2.$$

For equation (9-23), 0 is substituted for θ .

$$(9-27) \quad \chi(0) = \frac{1}{2} - 2 \sum_{p=1}^{\infty} \sigma_0(p) (2\pi p) K_1(2\pi p) \approx 0.4875331150002445.$$

On the real axis, the Chi function has a minimum value at the point $\theta = 0$.

$$(9-28) \quad \chi(x) \geq \chi(0) \quad (x \in \mathbb{R}).$$

An attempt is made to find 10 non-trivial zeros of the Chi function close to the real axis in order by interpolation.

The result indicates that the real parts of 10 non-trivial zeros are all zero.

Thus, the assumption that all non-trivial zeros of the Chi function lie on the imaginary axis is proved in the narrow domain.

Extending the domain of consideration to the whole complex plane, I propose the generalized Riemann hypothesis for the Chi function that states that all non-trivial zeros lie on the imaginary axis. it will be proved

in Section 14.

Furthermore, I assume that there are infinitely many non-trivial zeros on the imaginary axis in the Chi function. I define that the positive imaginary part of the m-th non-trivial zero of the Chi function to which the number is allocated in order to be closer to the real axis is τ_m . Under the assumptions,

$$(9-29) \quad \chi(\pm i \tau_m) := 0 \quad (0 < \tau_1 \leq \tau_2 \leq \tau_3 \leq \dots).$$

The positive imaginary parts of 10 non-trivial zeros of the Chi function close to the real axis are shown in order in a table. For comparison, the positive imaginary parts of 10 non-trivial zeros of the zeta function close to the real axis are also demonstrated in order in the same table.

Table shows the positive imaginary parts of 10 non-trivial zeros of the Chi function and their comparisons

m	τ_m	<, or >	ρ_m
1	12.041...	<	14.134...
2	20.487...	<	21.022...
3	25.976...	>	25.010...
4	28.269...	<	30.424...
5	32.685...	<	32.935...
6	36.583...	<	37.586...
7	42.044...	>	40.918...
8	42.901...	<	43.327...
9	46.556...	<	48.005...
10	50.021...	>	49.773...

Fig. 9-1

Under the generalized Riemann hypothesis for the Chi function, assuming that there are infinitely many non-trivial zeros on the imaginary axis, The Hadamard product representation of the Chi function can alternatively be written down.

$$(9-30) \quad \chi(\theta) = \chi(0) \prod_{m=1}^{\infty} \left(1 - \frac{\theta}{i \tau_m}\right) \left(1 - \frac{\theta}{-i \tau_m}\right) = \chi(0) \prod_{m=1}^{\infty} \left(1 + \frac{\theta^2}{(\tau_m)^2}\right) \quad (\theta \in \mathbb{C}).$$

To obtain the value of the infinite product in equation (9-30), 1/2 is substituted for θ .

$$(9-31) \quad \chi\left(\frac{1}{2}\right) = \chi(0) \prod_{m=1}^{\infty} \left(1 + \frac{(1/2)^2}{(\tau_m)^2}\right).$$

Therefore,

$$(9-32) \quad \prod_{m=1}^{\infty} \left(1 + \frac{1}{(2 \tau_m)^2}\right) = \frac{\chi\left(\frac{1}{2}\right)}{\chi(0)} \approx 1.0063381050061486.$$

1/2 is substituted for θ in equation (9-3).

$$(9-33) \quad \zeta\left(\frac{1}{2}\right) = -\frac{4\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)} \prod_{m=1}^{\infty} \frac{(2\rho_m)^2}{(2\rho_m)^2 + 1} = -\frac{4\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)} \prod_{m=1}^{\infty} \frac{1}{1 + \frac{1}{(2\rho_m)^2}}.$$

Therefore,

$$(9-34) \quad \prod_{m=1}^{\infty} \left(1 + \frac{1}{(2\rho_m)^2}\right) = -\frac{4\pi^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)} \approx 1.0057917953503750.$$

From the equations (9-32) and (9-34), the following inequality is obtained:

$$(9-35) \quad \prod_{m=1}^{\infty} \left(1 + \frac{1}{(2\rho_m)^2}\right) < \prod_{m=1}^{\infty} \left(1 + \frac{1}{(2\tau_m)^2}\right).$$

According to the inequality (9-35), the distribution of the positive imaginary parts of the non-trivial zeros of the Chi function is closer to the real axis than the distribution of those of the zeta function on the whole.

10. Functional Equation Transformations, Part3

Compared to the previous sections of 4 and 5, this section will be conducted in a more straightforward manner.

10-1. Functional Equation Transformation for the third-order I_r type functional equation

The following are the FET operations for the third-order I_r type functional equation:

$$(10-1) \quad L_{[3]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-3n^2 x^{\theta+1} + 12\pi n^4 x^{\theta+3} - 4\pi^2 n^6 x^{\theta+5}\right) e^{-\pi(n^2 + \alpha^2)x^2} dx.$$

$$(10-2) \quad R_{[3]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(12n^2 x^{\theta-4} - 18\pi n^4 x^{\theta-6} + 4\pi^2 n^6 x^{\theta-8}\right) e^{-\pi\left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.$$

For the integral of $L_{[3]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{y}{\pi(n^2 + \alpha^2)}\right)^{\frac{1}{2}}.$$

$$(10-3) \quad L_{[3]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-3n^2 \left(\frac{y}{\pi(n^2 + \alpha^2)}\right)^{\frac{\theta+1}{2}} + 12\pi n^4 \left(\frac{y}{\pi(n^2 + \alpha^2)}\right)^{\frac{\theta+3}{2}} - 4\pi^2 n^6 \left(\frac{y}{\pi(n^2 + \alpha^2)}\right)^{\frac{\theta+5}{2}}\right) e^{-y} \frac{1}{2} \left(\frac{1}{\pi(n^2 + \alpha^2)}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy.$$

Using the left-sided limit of the positive real variable α ,

$$(10-4) \quad L_{[3]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3y^{\frac{\theta}{2}} + 12y^{\frac{\theta}{2}+1} - 4y^{\frac{\theta}{2}+2}\right) e^{-y} dy \\ = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3 \int_0^{\infty} y^{\frac{\theta}{2}} e^{-y} dy + 12 \int_0^{\infty} y^{\frac{\theta}{2}+1} e^{-y} dy - 4 \int_0^{\infty} y^{\frac{\theta}{2}+2} e^{-y} dy\right).$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\text{Re}(\theta) > 1$ can be added to the above result because it is a convergent function under the condition.

$$(10-5) \quad L_{[3]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3\Gamma\left(1 + \frac{\theta}{2}\right) + 12\Gamma\left(2 + \frac{\theta}{2}\right) - 4\Gamma\left(3 + \frac{\theta}{2}\right)\right) \quad (\text{Re}(\theta) > 1).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10-6) \quad L_{[3]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-3\Gamma\left(1 + \frac{\theta}{2}\right) + 12\Gamma\left(2 + \frac{\theta}{2}\right) - 4\Gamma\left(3 + \frac{\theta}{2}\right)\right) \quad (\theta \in \mathbb{C}).$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned}
 (10-7) \quad L_{[3]}(\theta) &= \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-3 \left(\frac{\theta}{2} \right) + 12 \left(1 + \frac{\theta}{2} \right) \left(\frac{\theta}{2} \right) - 4 \left(2 + \frac{\theta}{2} \right) \left(1 + \frac{\theta}{2} \right) \left(\frac{\theta}{2} \right) \right) \Gamma \left(\frac{\theta}{2} \right) \\
 &= \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-\frac{\theta}{2} \right) (\theta^2 - 1) \Gamma \left(\frac{\theta}{2} \right) \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

Therefore

$$(10-8) \quad L_{[3]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma \left(\frac{\theta}{2} \right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

The function $L_{[3]}(\theta)$ is a convergent function in the whole complex plane.

For the integral of $R_{[3]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned}
 (10-9) \quad R_{[3]}(\theta) &= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \left(12 n^2 \left(\frac{1}{y} \right)^{\theta-4} - 18 \pi n^4 \left(\frac{1}{y} \right)^{\theta-6} \right. \\
 &\quad \left. + 4 \pi^2 n^6 \left(\frac{1}{y} \right)^{\theta-8} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} (-y^{-2}) dy \\
 &= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \left(12 n^2 y^{2-\theta} - 18 \pi n^4 y^{4-\theta} + 4 \pi^2 n^6 y^{6-\theta} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} dy.
 \end{aligned}$$

For the integral of $R_{[3]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi n}},$$

one more time.

$$\begin{aligned}
 (10-10) \quad R_{[3]}(\theta) &= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \left(12 n^2 \left(\frac{x}{\sqrt{\pi n}} \right)^{2-\theta} - 18 \pi n^4 \left(\frac{x}{\sqrt{\pi n}} \right)^{4-\theta} \right. \\
 &\quad \left. + 4 \pi^2 n^6 \left(\frac{x}{\sqrt{\pi n}} \right)^{6-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} \frac{1}{\sqrt{\pi n}} dx \\
 &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(12 x^{2-\theta} - 18 x^{4-\theta} + 4 x^{6-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} dx.
 \end{aligned}$$

Using the left-sided limit of the positive real variable α ,

$$(10-11) \quad R_{[3]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(12 x^{2-\theta} - 18 x^{4-\theta} + 4 x^{6-\theta} \right) e^{-x^2} dx.$$

For the integral of $R_{[3]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$\begin{aligned}
 (10-12) \quad R_{[3]}(\theta) &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(12 y^{\frac{2-\theta}{2}} - 18 y^{\frac{4-\theta}{2}} + 4 y^{\frac{6-\theta}{2}} \right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\
 &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(6 \int_0^\infty y^{\frac{1-\theta}{2}} e^{-y} dy - 9 \int_0^\infty y^{\frac{3-\theta}{2}} e^{-y} dy + 2 \int_0^\infty y^{\frac{5-\theta}{2}} e^{-y} dy \right).
 \end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\operatorname{Re}(\theta) < 0$ can be added to the above result because it is a convergent function under the condition.

$$(10-13) \quad R_{[3]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(6 \Gamma \left(1 + \frac{1-\theta}{2} \right) - 9 \Gamma \left(2 + \frac{1-\theta}{2} \right) + 2 \Gamma \left(3 + \frac{1-\theta}{2} \right) \right) \quad (\operatorname{Re}(\theta) < 0).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10-14) \quad \mathbf{R}_{[3]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(6\Gamma\left(1+\frac{1-\theta}{2}\right) - 9\Gamma\left(2+\frac{1-\theta}{2}\right) + 2\Gamma\left(3+\frac{1-\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).$$

The difference formula for the gamma function is applied to the above result.

$$(10-15) \quad \begin{aligned} \mathbf{R}_{[3]}(\theta) &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(6\left(\frac{1-\theta}{2}\right) - 9\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \right. \\ &\quad \left. + 2\left(2+\frac{1-\theta}{2}\right)\left(1+\frac{1-\theta}{2}\right)\left(\frac{1-\theta}{2}\right) \right) \Gamma\left(\frac{1-\theta}{2}\right) \\ &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(-\frac{\theta}{4}\right) (\theta^2 - 1) \Gamma\left(\frac{1-\theta}{2}\right) \quad (\theta \in \mathbb{C}). \end{aligned}$$

Therefore

$$(10-16) \quad \mathbf{R}_{[3]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

The function $\mathbf{R}_{[3]}(\theta)$ is also a convergent function in the whole complex plane.

Because of the convergent equations (10-8) and (10-16), the following functional equation holds true:

$$(10-17) \quad \mathbf{L}_{[3]}(\theta) = \mathbf{R}_{[3]}(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(10-18) \quad -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

I decide that the functional equation (10-18) is called the third-order \mathbf{I}_c type functional equation.

Additionally, it is equivalent to the Riemann Xi function's functional equation.

10-2. Functional Equation Transformation for the third-order \mathbf{II}_r type functional equation

As you will see later, another way to prove the absolute convergence of the correction term is chosen due to a symmetry collapse in the third-order \mathbf{II}_c type functional equation.

The operations of the FET for the third-order \mathbf{II}_r type functional equation are shown as follows:

$$(10-19) \quad \mathbf{A}_{[3]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (-3n^2 x^{\theta+1} + 12\pi n^4 x^{\theta+3} - 4\pi^2 n^6 x^{\theta+5}) e^{-\pi(n^2 + \alpha^2)x^2} dx.$$

$$(10-20) \quad \mathbf{F}_{[3]}(\theta) :=$$

$$\lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty (-6n^2 x^{\theta+1} + 24\pi n^4 x^{\theta+3} - 8\pi^2 n^6 x^{\theta+5}) e^{-\pi\left((n^2 + \alpha^2)x^2 + \frac{m^2}{x^2}\right)} dx.$$

$$(10-21) \quad \mathbf{B}_{[3]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (12n^2 x^{\theta-3} - 18\pi n^4 x^{\theta-5} + 4\pi^2 n^6 x^{\theta-7}) e^{-\pi\left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.$$

$$(10-22) \quad \mathbf{G}_{[3]}(\theta) :=$$

$$\lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty (24n^2 x^{\theta-3} - 36\pi n^4 x^{\theta-5} + 8\pi^2 n^6 x^{\theta-7}) e^{-\pi\left((m^2 + \alpha^2)x^2 + \frac{n^2}{x^2}\right)} dx.$$

Immediately, the following obvious result is obtained:

$$(10-23) \quad \mathbf{A}_{[3]}(\theta) = \mathbf{L}_{[3]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

For the integral of $B_{[3]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned}
 (10-24) \quad B_{[3]}(\theta) &= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^{\infty} \left(12 n^2 \left(\frac{1}{y}\right)^{\theta-3} - 18 \pi n^4 \left(\frac{1}{y}\right)^{\theta-5} \right. \\
 &\quad \left. + 4 \pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-7} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} (-y^{-2}) dy \\
 &= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^{\infty} \left(12 n^2 y^{1-\theta} - 18 \pi n^4 y^{3-\theta} + 4 \pi^2 n^6 y^{5-\theta} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} dy.
 \end{aligned}$$

For the integral of $B_{[3]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi n}},$$

one more time.

$$\begin{aligned}
 (10-25) \quad B_{[3]}(\theta) &= \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^{\infty} \left(12 n^2 \left(\frac{x}{\sqrt{\pi n}}\right)^{1-\theta} - 18 \pi n^4 \left(\frac{x}{\sqrt{\pi n}}\right)^{3-\theta} \right. \\
 &\quad \left. + 4 \pi^2 n^6 \left(\frac{x}{\sqrt{\pi n}}\right)^{5-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi n}} dx \\
 &= \frac{\theta}{\pi} \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(12 x^{1-\theta} - 18 x^{3-\theta} + 4 x^{5-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx.
 \end{aligned}$$

Using the left-sided limit of the positive real variable α ,

$$(10-26) \quad B_{[3]}(\theta) = \frac{\theta}{\pi} \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(12 x^{1-\theta} - 18 x^{3-\theta} + 4 x^{5-\theta} \right) e^{-x^2} dx.$$

For the integral of $B_{[3]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$\begin{aligned}
 (10-27) \quad B_{[3]}(\theta) &= \frac{\theta}{\pi} \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(12 y^{\frac{1-\theta}{2}} - 18 y^{\frac{3-\theta}{2}} + 4 y^{\frac{5-\theta}{2}} \right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\
 &= \frac{\theta}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(6 \int_0^\infty y^{\frac{-\theta}{2}} e^{-y} dy - 9 \int_0^\infty y^{\frac{2-\theta}{2}} e^{-y} dy + 2 \int_0^\infty y^{\frac{4-\theta}{2}} e^{-y} dy \right).
 \end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\operatorname{Re}(\theta) < -1$ can be added to the above result because it is a convergent function under the condition.

$$(10-28) \quad B_{[3]}(\theta) = \frac{\theta}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} \left(6 \Gamma\left(1 + \frac{-\theta}{2}\right) - 9 \Gamma\left(2 + \frac{-\theta}{2}\right) + 2 \Gamma\left(3 + \frac{-\theta}{2}\right) \right) \quad (\operatorname{Re}(\theta) < -1).$$

Analytic continuation is used to extend the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10-29) \quad B_{[3]}(\theta) = \frac{\theta}{\pi} \zeta(-\theta) \left(6 \Gamma\left(1 + \frac{-\theta}{2}\right) - 9 \Gamma\left(2 + \frac{-\theta}{2}\right) + 2 \Gamma\left(3 + \frac{-\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).$$

The difference formula for the gamma function is applied to the above result.

$$(10-30) \quad B_{[3]}(\theta) = \frac{\theta}{\pi} \zeta(-\theta) \left(6 \left(\frac{-\theta}{2}\right) - 9 \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) + 2 \left(2 + \frac{-\theta}{2}\right) \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) \right) \Gamma\left(\frac{-\theta}{2}\right)$$

$$= \frac{\theta}{\pi^2} \zeta(-\theta) \left(\frac{-\theta}{4}\right) (\theta+1) (\theta+2) \Gamma\left(\frac{-\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

Combining the relation of the complete symmetric functional equation to the result,

$$(10-31) \quad \mathbf{B}_{[3]}(\theta) = -\frac{\theta(\theta+1)(\theta+2)}{4\pi} \pi^2 \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) \\ = -\frac{\theta(\theta+1)(\theta+2)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) \quad (\theta \in \mathbb{C}).$$

The function $\mathbf{B}_{[3]}(\theta)$ is also a convergent function in the whole complex plane.

The left-sided limit of the positive real variable α , is used for defining the equation (10-20),

$$(10-32) \quad \mathbf{F}_{[3]}(\theta) = \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty (-6n^2 x^{\theta+1} + 24\pi n^4 x^{\theta+3} - 8\pi^2 n^6 x^{\theta+5}) e^{-\pi\left(n^2 x^2 + \frac{m^2}{x^2}\right)} dx.$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $\mathbf{F}_{[3]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{m y}{n}\right)^{\frac{1}{2}}.$$

$$(10-33) \quad \mathbf{F}_{[3]}(\theta) = \sum_{n=1}^\infty \sum_{m=1}^\infty \int_0^\infty \left(-6n^2 \left(\frac{m y}{n}\right)^{\frac{\theta+1}{2}} + 24\pi n^4 \left(\frac{m y}{n}\right)^{\frac{\theta+3}{2}} - 8\pi^2 n^6 \left(\frac{m y}{n}\right)^{\frac{\theta+5}{2}}\right) e^{-\pi m n \left(y + \frac{1}{y}\right)} \frac{1}{2} \left(\frac{m}{n}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy \\ = \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+2}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y}\right)} dy + 12(\pi m n) \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+4}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y}\right)} dy - 4(\pi m n)^2 \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+6}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y}\right)} dy\right).$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(10-34) \quad \mathbf{F}_{[3]}(\theta) = \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) + 12(\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) - 4(\pi m n)^2 K_{\frac{\theta+6}{2}}(2\pi m n)\right).$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied three times.

$$(10-35) \quad \mathbf{F}_{[3]}(\theta) = \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) + 12(\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) - 4(\pi m n) \left(\frac{\theta+4}{2} K_{\frac{\theta+4}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta+2}{2}}(2\pi m n)\right)\right) \\ = \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-\left(3+4(\pi m n)^2\right) K_{\frac{\theta+2}{2}}(2\pi m n) - 2(\theta-2)(\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n)\right)$$

$$\begin{aligned}
 &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left((3+4(\pi m n)^2) K_{\frac{\theta+2}{2}}(2\pi m n) \right. \\
 &\quad \left. + 2(\theta-2) \left(\frac{\theta+2}{2} K_{\frac{\theta+2}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta}{2}}(2\pi m n) \right) \right) \\
 &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(2(\theta-2)(\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n) + (\theta^2-1+4(\pi m n)^2)(\pi m n) K_{\frac{\theta+2}{2}}(2\pi m n) \right) \\
 &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(2(\theta-2)(\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n) \right. \\
 &\quad \left. + (\theta^2-1+4(\pi m n)^2) \left(\frac{\theta}{2} K_{\frac{\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta-2}{2}}(2\pi m n) \right) \right) \\
 &= -\frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left((\theta(\theta^2-1) + 8(\theta-1)(\pi m n)^2) K_{\frac{\theta}{2}}(2\pi m n) \right. \\
 &\quad \left. + 2(\theta^2-1+4(\pi m n)^2)(\pi m n) K_{\frac{\theta-2}{2}}(2\pi m n) \right).
 \end{aligned}$$

From the equation (5-37) and the result of equation (10-35),

$$\begin{aligned}
 (10-36) \quad \mathbf{F}_{[3]}(\theta) &= -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1) + 2(\theta-1)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) \right. \\
 &\quad \left. + ((\theta-1)(\theta+1) + (2\pi p)^2)(2\pi p) K_{\frac{\theta-2}{2}}(2\pi p) \right).
 \end{aligned}$$

For the equation of definition (10-22), using the left-sided limit of the positive real variable α ,

$$(10-37) \quad \mathbf{G}_{[3]}(\theta) = \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (24n^2 x^{\theta-3} - 36\pi n^4 x^{\theta-5} + 8\pi^2 n^6 x^{\theta-7}) e^{-\pi \left(m^2 x^2 + \frac{n^2}{x^2} \right)} dx.$$

For the integral of $\mathbf{G}_{[3]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned}
 (10-38) \quad \mathbf{G}_{[3]}(\theta) &= \\
 &\int_{\infty}^0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(24n^2 \left(\frac{1}{y}\right)^{\theta-3} - 36\pi n^4 \left(\frac{1}{y}\right)^{\theta-5} + 8\pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-7} \right) e^{-\pi \left(\frac{m^2}{y^2} + n^2 y^2 \right)} (-y^{-2}) dy \\
 &= \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (24n^2 y^{1-\theta} - 36\pi n^4 y^{3-\theta} + 8\pi^2 n^6 y^{5-\theta}) e^{-\pi \left(n^2 y^2 + \frac{m^2}{y^2} \right)} dy.
 \end{aligned}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $\mathbf{G}_{[3]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{m x}{n} \right)^{\frac{1}{2}}.$$

$$\begin{aligned}
 (10-39) \quad \mathbf{G}_{[3]}(\theta) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{\infty} \left(24n^2 \left(\frac{m x}{n}\right)^{\frac{1-\theta}{2}} \right. \\
 &\quad \left. - 36\pi n^4 \left(\frac{m x}{n}\right)^{\frac{3-\theta}{2}} + 8\pi^2 n^6 \left(\frac{m x}{n}\right)^{\frac{5-\theta}{2}} \right) e^{-\pi m n \left(x + \frac{1}{x} \right)} \frac{1}{2} \left(\frac{m}{n}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx \\
 &= \frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(6 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right. \\
 &\quad \left. - 9(\pi m n) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right. \\
 &\quad \left. + 2(\pi m n)^2 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{6-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right).
 \end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(10-40) \quad \mathbf{G}_{[3]}(\theta) = \frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(6 \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) - 9 (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) + 2 (\pi m n)^2 \mathbf{K}_{\frac{6-\theta}{2}}(2\pi m n)\right).$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied twice.

$$\begin{aligned} (10-41) \quad \mathbf{G}_{[3]}(\theta) &= \frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(6 \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) - 9 (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) + 2 (\pi m n) \left(\frac{4-\theta}{2} \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n)\right)\right) \\ &= -\frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(- (6 + 2 (\pi m n)^2) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (\theta + 5) (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n)\right) \\ &= -\frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(- (6 + 2 (\pi m n)^2) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (\theta + 5) \left(\frac{2-\theta}{2} \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{\theta}{2}}(2\pi m n)\right)\right) \\ &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(2 (\theta + 5) (\pi m n) \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) - ((\theta + 1) (\theta + 2) + 4 (\pi m n)^2) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n)\right) \\ &= -\frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left((\theta + 5) (2\pi m n)^2 \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) - ((\theta + 1) (\theta + 2) + (2\pi m n)^2) (2\pi m n) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n)\right). \end{aligned}$$

From the equation (5-38) and the result of equation (10-41),

$$(10-42) \quad \mathbf{G}_{[3]}(\theta) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta + 5) (2\pi p)^2 \mathbf{K}_{\frac{\theta}{2}}(2\pi p) - ((\theta + 1) (\theta + 2) + (2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{\theta-2}{2}}(2\pi p)\right).$$

I define the function $\mathbf{H}_{[3]}(\theta)$ as follows:

$$(10-43) \quad \mathbf{H}_{[3]}(\theta) := \mathbf{G}_{[3]}(\theta) - \mathbf{F}_{[3]}(\theta).$$

From (the right side of equation (10-42)) – (the right side of equation (10-36)),

$$(10-44) \quad \mathbf{H}_{[3]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1) + (\theta-7)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) + ((2\theta+1)(\theta+1) + 2(2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{\theta-2}{2}}(2\pi p)\right).$$

The modified Bessel functions of the second kind are written using the integral respectively.

$$(10-45) \quad \mathbf{H}_{[3]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1) + (\theta-7)(2\pi p)^2) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{\theta}{2}-1} e^{-\frac{2\pi p}{2} \left(x + \frac{1}{x}\right)} dx + ((2\theta+1)(\theta+1) + 2(2\pi p)^2) (2\pi p) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{\theta-2}{2}-1} e^{-\frac{2\pi p}{2} \left(x + \frac{1}{x}\right)} dx \right).$$

For the integrals of $H_{[3]}(\theta)$, I perform the variable transformation

$$x = \frac{y^2}{\pi p}.$$

$$(10-46) \quad H_{[3]}(\theta) =$$

$$\begin{aligned} & \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1) + (\theta-7)(2\pi p)^2) \cdot \frac{1}{2} \int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta-2}{2}} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \right. \\ & \quad \left. + ((2\theta+1)(\theta+1) + 2(2\pi p)^2)(2\pi p) \cdot \frac{1}{2} \int_0^{\infty} \left(\frac{y^2}{\pi p}\right)^{\frac{\theta-4}{2}} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} \frac{2y}{\pi p} dy \right) \\ & = \frac{\pi - \frac{\theta}{2}}{2\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(2(\theta(\theta-1)(\theta+1) + (\theta-7)(2\pi p)^2) \int_0^{\infty} y^{\theta-1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right. \\ & \quad \left. + ((2\theta+1)(\theta+1) + 2(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} y^{\theta-3} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \right). \end{aligned}$$

For convenience, the above result's absolutely convergent integrals in the whole complex plane are defined as follows:

$$(10-47) \quad I_p(\theta) := \int_0^{\infty} y^{-1+\operatorname{Re}(\theta)} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (p \in \mathbb{N}, \theta \in \mathbb{C}).$$

$$(10-48) \quad J_p(\theta) := \int_0^{\infty} y^{-3+\operatorname{Re}(\theta)} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (p \in \mathbb{N}, \theta \in \mathbb{C}).$$

Using the expressions $I_p(\theta)$ and $J_p(\theta)$,

$$(10-49) \quad |H_{[3]}(\theta)| \leq \frac{\pi - \frac{\operatorname{Re}(\theta)}{2}}{2\pi} \sum_{p=1}^{\infty} \sigma_{-\operatorname{Re}(\theta)}(p)$$

$$\left(2|\theta(\theta-1)(\theta+1)| I_p(\theta) + 2|\theta-7|(2\pi p)^2 I_p(\theta) + |(2\theta+1)(\theta+1)|(2\pi p)^2 J_p(\theta) + 2(2\pi p)^4 J_p(\theta) \right).$$

Now, the following obvious fact is introduced into the inequality (10-49):

$$(10-50) \quad \sigma_{-\operatorname{Re}(\theta)}(p) < p^{2k} \zeta(2k + \operatorname{Re}(\theta)) \quad (k, p \in \mathbb{N}, \operatorname{Re}(\theta) > 1 - 2k).$$

Under the condition of $\operatorname{Re}(\theta) > 1 - 2k$,

$$(10-51) \quad |H_{[3]}(\theta)| <$$

$$\begin{aligned} & \frac{\pi - \frac{\operatorname{Re}(\theta)}{2}}{2\pi} \sum_{p=1}^{\infty} p^{2k} \zeta(2k + \operatorname{Re}(\theta)) \left(2|\theta(\theta-1)(\theta+1)| I_p(\theta) + 2|\theta-7|(2\pi p)^2 I_p(\theta) \right. \\ & \quad \left. + |(2\theta+1)(\theta+1)|(2\pi p)^2 J_p(\theta) + 2(2\pi p)^4 J_p(\theta) \right) \quad (k \in \mathbb{N}, \operatorname{Re}(\theta) > 1 - 2k). \end{aligned}$$

Because the variable of the zeta function is irrelevant to the variable p , the zeta function of the sum can move outside.

$$(10-52) \quad |H_{[3]}(\theta)| <$$

$$\begin{aligned} & \frac{\pi - \frac{\operatorname{Re}(\theta)}{2}}{\pi} \zeta(2k + \operatorname{Re}(\theta)) \sum_{p=1}^{\infty} \left(|\theta(\theta-1)(\theta+1)| p^{2k} I_p(\theta) + (2\pi)^2 |\theta-7| p^{2(k+1)} I_p(\theta) \right. \\ & \quad \left. + 2\pi^2 |(2\theta+1)(\theta+1)| p^{2(k+1)} J_p(\theta) + (2\pi)^4 p^{2(k+2)} J_p(\theta) \right) \quad (k \in \mathbb{N}, \operatorname{Re}(\theta) > 1 - 2k). \end{aligned}$$

To estimate the absolute convergence of the function $H_{[3]}(\theta)$, the following equations with the condition are defined:

$$(10-53) \quad S_k(x) := \sum_{p=1}^{\infty} p^{2k} \int_0^{\infty} y^{x-1} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (k \in \mathbb{N}, x > -2k).$$

$$(10-54) \quad T_k(x) := \sum_{p=1}^{\infty} p^{2k} \int_0^{\infty} y^{x-3} e^{-\left(y^2 + \frac{(\pi p)^2}{y^2}\right)} dy \quad (k \in \mathbb{N}, x > 2-2k).$$

For the function $S_k(x)$, because the integral is an absolutely convergent integral, I can change the order of relation between the sum and the integral.

$$(10-55) \quad S_k(x) = \int_0^{\infty} y^{x-1} e^{-y^2} \lim_{L \rightarrow +\infty} \sum_{p=1}^L p^{2k} e^{-\frac{(\pi p)^2}{y^2}} dy$$

$$< \int_0^{\infty} y^{x-1} e^{-y^2} \lim_{L \rightarrow +\infty} \sum_{p=1}^{L^2} p^k e^{-\frac{\pi^2 p}{y^2}} dy =$$

$$\int_0^{\infty} y^{x-1} e^{-y^2} \sum_{p=1}^{\infty} p^k e^{-\frac{\pi^2 p}{y^2}} dy \quad (k \in \mathbb{N}, x > -2k).$$

To evaluate the inside sum of the integral, I define the function $f_k(t, y)$ of double positive variables with the parameter k as follows:

$$(10-56) \quad f_k(t, y) := t^k e^{-\frac{\pi^2 t}{y^2}} \quad (t, y > 0, k \in \mathbb{N}).$$

Taking the limits yields function values on both sides of the open interval of the positive variable t .

$$(10-57) \quad \lim_{t \rightarrow +0} f_k(t, y) = 0 \quad (y > 0, k \in \mathbb{N}).$$

$$(10-58) \quad \lim_{t \rightarrow \infty} f_k(t, y) = 0 \quad (y > 0, k \in \mathbb{N}).$$

In order to look into the behavior of the function, it is partially differentiated.

$$(10-59) \quad \frac{\partial}{\partial t} f_k(t, y) = \left(k - \frac{\pi^2 t}{y^2}\right) t^{k-1} e^{-\frac{\pi^2 t}{y^2}} \quad (t, y > 0, k \in \mathbb{N}).$$

I assume that the solution of the above equation is α .

$$(10-60) \quad \frac{\partial}{\partial t} f_k(t, y) \Big|_{t=\alpha} = 0 \quad (t, y > 0, k \in \mathbb{N}).$$

The solution is easily obtained since it has a positive value depending on both the positive variable y and parameter k .

$$(10-61) \quad \alpha = \frac{ky^2}{\pi^2} > 0 \quad (y > 0, k \in \mathbb{N}).$$

Table of the increase and decrease for $f_k(t, y)$ of the positive variable t

t	+0	α	∞
$\frac{\partial}{\partial t} f_k(t, y)$		+	0	-	
$f_k(t, y)$	0	↗	maximum	↘	0

Fig. 10-1

By considering mainly the rectangle consisting of a base $(\alpha + 1)$ and a height $f_k(\alpha, y)$ in area, I obtain the upper limit of the infinite sum.

$$(10-62) \quad \sum_{p=1}^{\infty} p^k e^{-\frac{\pi^2 p}{y^2}} < (\alpha + 1) \cdot f_k(\alpha, y) + \int_0^{\infty} t^k e^{-\frac{\pi^2 t}{y^2}} dt =$$

$$\left(\frac{ky^2}{\pi^2} + 1\right) \cdot \left(\frac{ky^2}{\pi^2}\right)^k e^{-k} + \frac{y^2(k+1)}{\pi^2(k+1)} \Gamma(k+1) =$$

$$\frac{k^{k+1} e^{-k+k!}}{\pi^2(k+1)} y^{2(k+1)} + \frac{k^k e^{-k}}{\pi^2 k} y^{2k} \quad (y > 0, k \in \mathbb{N}).$$

This result is introduced into the right side of inequality (10-55).

$$(10-63) \quad S_k(x) < \int_0^{\infty} y^{x-1} e^{-y^2} \left(\frac{k^{k+1} e^{-k+k!}}{\pi^2(k+1)} y^{2(k+1)} + \frac{k^k e^{-k}}{\pi^2 k} y^{2k} \right) dy =$$

$$\frac{k^{k+1} e^{-k+k!}}{\pi^2(k+1)} \int_0^{\infty} y^{x+2k+1} e^{-y^2} dy + \frac{k^k e^{-k}}{\pi^2 k} \int_0^{\infty} y^{x+2k-1} e^{-y^2} dy \quad (k \in \mathbb{N}, x > -2k).$$

For the above integrals, I perform the variable transformation $y = u^{1/2}$.

$$(10-64) \quad S_k(x) <$$

$$\frac{k^{k+1} e^{-k+k!}}{\pi^2(k+1)} \int_0^{\infty} u^{\frac{x+2k+1}{2}} e^{-u} \frac{1}{2} u^{-\frac{1}{2}} du + \frac{k^k e^{-k}}{\pi^2 k} \int_0^{\infty} u^{\frac{x+2k-1}{2}} e^{-u} \frac{1}{2} u^{-\frac{1}{2}} du =$$

$$\frac{k^{k+1} e^{-k+k!}}{2\pi^2(k+1)} \Gamma\left(\frac{x+2k+2}{2}\right) + \frac{k^k e^{-k}}{2\pi^2 k} \Gamma\left(\frac{x+2k}{2}\right) \quad (k \in \mathbb{N}, x > -2k).$$

Therefore,

$$(10-65) \quad S_k(x) < \left(\frac{k^{k+1} e^{-k+k!}}{4\pi^2(k+1)} (x+2k) + \frac{k^k e^{-k}}{2\pi^2 k} \right) \Gamma\left(\frac{x+2k}{2}\right) \quad (k \in \mathbb{N}, x > -2k).$$

By the same method,

$$(10-66) \quad T_k(x) < \left(\frac{k^{k+1} e^{-k+k!}}{4\pi^2(k+1)} (x+2k-2) + \frac{k^k e^{-k}}{2\pi^2 k} \right) \Gamma\left(\frac{x+2k-2}{2}\right) \quad (k \in \mathbb{N}, x > 2-2k).$$

The upper limit function of the function $S_k(x)$ is defined by the following equation:

$$(10-67) \quad \hat{S}_k(x) := \left(\frac{k^{k+1} e^{-k+k!}}{4\pi^2(k+1)} (x+2k) + \frac{k^k e^{-k}}{2\pi^2 k} \right) \Gamma\left(\frac{x+2k}{2}\right) \quad (k \in \mathbb{N}, x > -2k).$$

The upper limit function of the function $T_k(x)$ is defined by the following equation:

$$(10-68) \quad \hat{T}_k(x) := \left(\frac{k^{k+1} e^{-k+k!}}{4\pi^2(k+1)} (x+2k-2) + \frac{k^k e^{-k}}{2\pi^2 k} \right) \Gamma\left(\frac{x+2k-2}{2}\right) \quad (k \in \mathbb{N}, x > 2-2k).$$

Using the upper limit functions,

$$(10-69) \quad |\mathbf{H}_{[3]}(\theta)| <$$

$$\frac{\pi - \operatorname{Re}(\theta)}{\pi} \zeta(2k + \operatorname{Re}(\theta)) \left(|\theta(\theta-1)(\theta+1)| \hat{S}_k(\operatorname{Re}(\theta)) + (2\pi)^2 |\theta-7| \hat{S}_{k+1}(\operatorname{Re}(\theta)) \right.$$

$$\left. + 2\pi^2 |(2\theta+1)(\theta+1)| \hat{T}_{k+1}(\operatorname{Re}(\theta)) + (2\pi)^4 \hat{T}_{k+2}(\operatorname{Re}(\theta)) \right) \quad (k \in \mathbb{N}, \operatorname{Re}(\theta) > 1-2k).$$

When any positive integer k is given, absolute value of the function $H_{[3]}(\theta)$ is a convergent function in the right half-plane of $\operatorname{Re}(\theta) > 1-2k$.

Because any positive integer for the variable k can be taken, the domain of definition can be expanded to the whole complex plane.

Thus, the function $H_{[3]}(\theta)$ is an absolutely convergent function in the whole complex plane.

Combining the above results yield the following functional equation, which holds true:

$$(10-70) \quad \mathbf{A}_{[3]}(\theta) = \mathbf{B}_{[3]}(\theta) + \mathbf{H}_{[3]}(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$\begin{aligned}
(10-71) \quad & -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = -\frac{\theta(\theta+1)(\theta+2)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) + \mathbf{H}_{[3]}(\theta) \\
& = -\frac{\theta(\theta+1)(\theta+2)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) + \mathbf{H}_{[3]}(\theta) \quad (\theta \in \mathbb{C}).
\end{aligned}$$

Where

$$\begin{aligned}
(10-72) \quad & \mathbf{H}_{[3]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \\
& \left((\theta(\theta-1)(\theta+1) + (\theta-7)(2\pi p)^2) \mathbf{K} \frac{\theta}{2}(2\pi p) + (2\theta+1)(\theta+1) + 2(2\pi p)^2 \right) (2\pi p) \mathbf{K} \frac{\theta-2}{2}(2\pi p).
\end{aligned}$$

I decide that the functional equation (10-71) is called the third-order Π_c type functional equation.

10-3. Functional Equation Transformation for the fourth-order I_r type functional equation

The following are the FET operations for the fourth-order I_r type functional equation:

$$\begin{aligned}
(10-73) \quad & \mathbf{L}_{[4]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (-3n^2 x^{\theta+1} + 42\pi n^4 x^{\theta+3} \\
& - 44\pi^2 n^6 x^{\theta+5} + 8\pi^3 n^8 x^{\theta+7}) e^{-\pi(n^2 + \alpha^2)x^2} dx.
\end{aligned}$$

$$\begin{aligned}
(10-74) \quad & \mathbf{R}_{[4]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (-48n^2 x^{\theta-4} + 132\pi n^4 x^{\theta-6} \\
& - 68\pi^2 n^6 x^{\theta-8} + 8\pi^3 n^8 x^{\theta-10}) e^{-\pi\left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.
\end{aligned}$$

For the integral of $\mathbf{L}_{[4]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{y}{\pi(n^2 + \alpha^2)} \right)^{\frac{1}{2}}.$$

$$\begin{aligned}
(10-75) \quad & \mathbf{L}_{[4]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-3n^2 \left(\frac{y}{\pi(n^2 + \alpha^2)} \right)^{\frac{\theta+1}{2}} + 42\pi n^4 \left(\frac{y}{\pi(n^2 + \alpha^2)} \right)^{\frac{\theta+3}{2}} \right. \\
& \left. - 44\pi^2 n^6 \left(\frac{y}{\pi(n^2 + \alpha^2)} \right)^{\frac{\theta+5}{2}} + 8\pi^3 n^8 \left(\frac{y}{\pi(n^2 + \alpha^2)} \right)^{\frac{\theta+7}{2}} \right) e^{-y} \frac{1}{2} \left(\frac{1}{\pi(n^2 + \alpha^2)} \right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy.
\end{aligned}$$

Using the left-sided limit of the positive variable α ,

$$\begin{aligned}
(10-76) \quad & \mathbf{L}_{[4]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3y^{\frac{\theta}{2}} + 42y^{\frac{\theta}{2}+1} - 44y^{\frac{\theta}{2}+2} + 8y^{\frac{\theta}{2}+3} \right) e^{-y} dy. \\
& = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3 \int_0^{\infty} y^{\frac{\theta}{2}} e^{-y} dy + 42 \int_0^{\infty} y^{\frac{\theta}{2}+1} e^{-y} dy \right. \\
& \quad \left. - 44 \int_0^{\infty} y^{\frac{\theta}{2}+2} e^{-y} dy + 8 \int_0^{\infty} y^{\frac{\theta}{2}+3} e^{-y} dy \right).
\end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\text{Re}(\theta) > 1$ can be added to the above result because it is a convergent function under the condition.

(10 – 77) $L_{[4]}(\theta) =$

$$\frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3\Gamma\left(1+\frac{\theta}{2}\right) + 42\Gamma\left(2+\frac{\theta}{2}\right) - 44\Gamma\left(3+\frac{\theta}{2}\right) + 8\Gamma\left(4+\frac{\theta}{2}\right) \right) \quad (\operatorname{Re}(\theta) > 1).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10 – 78) \quad L_{[4]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-3\Gamma\left(1+\frac{\theta}{2}\right) + 42\Gamma\left(2+\frac{\theta}{2}\right) - 44\Gamma\left(3+\frac{\theta}{2}\right) + 8\Gamma\left(4+\frac{\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).$$

The difference formula for the gamma function is applied to the above result.

(10 – 79) $L_{[4]}(\theta) =$

$$\begin{aligned} & \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-3\left(\frac{\theta}{2}\right) + 42\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) - 44\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) + 8\left(3+\frac{\theta}{2}\right)\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) \right) \Gamma\left(\frac{\theta}{2}\right) \\ & = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(\frac{\theta}{2}\right) (\theta-1)(\theta+1)^2 \Gamma\left(\frac{\theta}{2}\right) \quad (\theta \in \mathbb{C}). \end{aligned}$$

Therefore

$$(10 – 80) \quad L_{[4]}(\theta) = \frac{\theta(\theta-1)(\theta+1)^2}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

The function $L_{[4]}(\theta)$ is a convergent function in the whole complex plane.

For the integral of $R_{[4]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned} (10 – 81) \quad R_{[4]}(\theta) &= \lim_{\alpha \rightarrow +0} \int_{\infty}^0 \sum_{n=1}^{\infty} \left(-48n^2 \left(\frac{1}{y}\right)^{\theta-4} + 132\pi n^4 \left(\frac{1}{y}\right)^{\theta-6} \right. \\ & \quad \left. - 68\pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-8} + 8\pi^3 n^8 \left(\frac{1}{y}\right)^{\theta-10} \right) e^{-\pi\left(n^2 y^2 + \frac{\alpha^2}{y^2}\right)} (-y^{-2}) dy \\ &= \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-48n^2 y^{2-\theta} + 132\pi n^4 y^{4-\theta} - 68\pi^2 n^6 y^{6-\theta} + 8\pi^3 n^8 y^{8-\theta} \right) e^{-\pi\left(n^2 y^2 + \frac{\alpha^2}{y^2}\right)} dy. \end{aligned}$$

For the integral of $R_{[4]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi n}},$$

one more time.

$$\begin{aligned} (10 – 82) \quad R_{[4]}(\theta) &= \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-48n^2 \left(\frac{x}{\sqrt{\pi n}}\right)^{2-\theta} + 132\pi n^4 \left(\frac{x}{\sqrt{\pi n}}\right)^{4-\theta} \right. \\ & \quad \left. - 68\pi^2 n^6 \left(\frac{x}{\sqrt{\pi n}}\right)^{6-\theta} + 8\pi^3 n^8 \left(\frac{x}{\sqrt{\pi n}}\right)^{8-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi n}} dx \\ &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(-48x^{2-\theta} + 132x^{4-\theta} - 68x^{6-\theta} + 8x^{8-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx. \end{aligned}$$

Using the left-sided limit of the positive real variable α ,

$$(10 – 83) \quad R_{[4]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(-48x^{2-\theta} + 132x^{4-\theta} - 68x^{6-\theta} + 8x^{8-\theta} \right) e^{-x^2} dx.$$

For the integral of $R_{[4]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$\begin{aligned}
 (10-84) \quad R_{[4]}(\theta) &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(-48 y^{\frac{2-\theta}{2}} + 132 y^{\frac{4-\theta}{2}} - 68 y^{\frac{6-\theta}{2}} + 8 y^{\frac{8-\theta}{2}} \right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\
 &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(-24 \int_0^\infty y^{\frac{1-\theta}{2}} e^{-y} dy + 66 \int_0^\infty y^{\frac{3-\theta}{2}} e^{-y} dy \right. \\
 &\quad \left. - 34 \int_0^\infty y^{\frac{5-\theta}{2}} e^{-y} dy + 4 \int_0^\infty y^{\frac{7-\theta}{2}} e^{-y} dy \right).
 \end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\text{Re}(\theta) < 0$ can be added to the above result because it is a convergent function under the condition.

$$\begin{aligned}
 (10-85) \quad R_{[4]}(\theta) &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \sum_{n=1}^\infty \frac{1}{n^{1-\theta}} \left(-24 \Gamma\left(1 + \frac{1-\theta}{2}\right) + 66 \Gamma\left(2 + \frac{1-\theta}{2}\right) \right. \\
 &\quad \left. - 34 \Gamma\left(3 + \frac{1-\theta}{2}\right) + 4 \Gamma\left(4 + \frac{1-\theta}{2}\right) \right) \quad (\text{Re}(\theta) < 0).
 \end{aligned}$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$\begin{aligned}
 (10-86) \quad R_{[4]}(\theta) &= \\
 &\frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(-24 \Gamma\left(1 + \frac{1-\theta}{2}\right) + 66 \Gamma\left(2 + \frac{1-\theta}{2}\right) - 34 \Gamma\left(3 + \frac{1-\theta}{2}\right) + 4 \Gamma\left(4 + \frac{1-\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned}
 (10-87) \quad R_{[4]}(\theta) &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(-24 \left(\frac{1-\theta}{2}\right) + 66 \left(1 + \frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2}\right) \right. \\
 &\quad \left. - 34 \left(2 + \frac{1-\theta}{2}\right) \left(1 + \frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2}\right) + 4 \left(3 + \frac{1-\theta}{2}\right) \left(2 + \frac{1-\theta}{2}\right) \left(1 + \frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2}\right) \right) \Gamma\left(\frac{1-\theta}{2}\right) \\
 &= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) (\theta-1) \left(\frac{\theta}{4}\right) (\theta+1)^2 \Gamma\left(\frac{1-\theta}{2}\right) \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

Therefore

$$(10-88) \quad R_{[4]}(\theta) = \frac{\theta(\theta-1)(\theta+1)^2}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

The function $R_{[4]}(\theta)$ is also a convergent function in the whole complex plane.

Because of the convergent equations (10-80) and (10-88), the following functional equation holds true:

$$(10-89) \quad L_{[4]}(\theta) = R_{[4]}(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(10-90) \quad \frac{\theta(\theta-1)(\theta+1)^2}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\theta(\theta-1)(\theta+1)^2}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

I decide that the functional equation (10-90) is called the fourth-order I_c type functional equation.

Additionally, it is equivalent to the Riemann Xi function's functional equation.

10-4. Functional Equation Transformation for the fourth-order II_r type functional equation

The operations of the FET for the fourth-order II_r type functional equation are shown as follows:

$$(10-91) \mathbf{A}_{[4]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (-3n^2 x^{\theta+1} + 42\pi n^4 x^{\theta+3} - 44\pi^2 n^6 x^{\theta+5} + 8\pi^3 n^8 x^{\theta+7}) e^{-\pi(n^2 + \alpha^2)x^2} dx.$$

$$(10-92) \mathbf{F}_{[4]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-6n^2 x^{\theta+1} + 84\pi n^4 x^{\theta+3} - 88\pi^2 n^6 x^{\theta+5} + 16\pi^3 n^8 x^{\theta+7}) e^{-\pi\left((n^2 + \alpha^2)x^2 + \frac{m^2}{x^2}\right)} dx.$$

$$(10-93) \mathbf{B}_{[4]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (-48n^2 x^{\theta-3} + 132\pi n^4 x^{\theta-5} - 68\pi^2 n^6 x^{\theta-7} + 8\pi^3 n^8 x^{\theta-9}) e^{-\pi\left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.$$

$$(10-94) \mathbf{G}_{[4]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-96n^2 x^{\theta-3} + 264\pi n^4 x^{\theta-5} - 136\pi^2 n^6 x^{\theta-7} + 16\pi^3 n^8 x^{\theta-9}) e^{-\pi\left((m^2 + \alpha^2)x^2 + \frac{n^2}{x^2}\right)} dx.$$

I immediately obtain the following obvious result:

$$(10-95) \mathbf{A}_{[4]}(\theta) = \mathbf{L}_{[4]}(\theta) = \frac{\theta(\theta-1)(\theta+1)^2}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

For the integral of $\mathbf{B}_{[4]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$(10-96) \mathbf{B}_{[4]}(\theta) = \lim_{\alpha \rightarrow +0} \int_{\infty}^0 \sum_{n=1}^{\infty} \left(-48n^2 \left(\frac{1}{y}\right)^{\theta-3} + 132\pi n^4 \left(\frac{1}{y}\right)^{\theta-5} - 68\pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-7} + 8\pi^3 n^8 \left(\frac{1}{y}\right)^{\theta-9}\right) e^{-\pi\left(n^2 y^2 + \frac{\alpha^2}{y^2}\right)} (-y^{-2}) dy$$

$$= \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-48n^2 y^{1-\theta} + 132\pi n^4 y^{3-\theta} - 68\pi^2 n^6 y^{5-\theta} + 8\pi^3 n^8 y^{7-\theta}\right) e^{-\pi\left(n^2 y^2 + \frac{\alpha^2}{y^2}\right)} dy.$$

For the integral of $\mathbf{B}_{[4]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi n}},$$

one more time.

$$(10-97) \mathbf{B}_{[4]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-48n^2 \left(\frac{x}{\sqrt{\pi n}}\right)^{1-\theta} + 132\pi n^4 \left(\frac{x}{\sqrt{\pi n}}\right)^{3-\theta} - 68\pi^2 n^6 \left(\frac{x}{\sqrt{\pi n}}\right)^{5-\theta} + 8\pi^3 n^8 \left(\frac{x}{\sqrt{\pi n}}\right)^{7-\theta}\right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi n}} dx$$

$$= \frac{\theta}{\pi} \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} (-48x^{1-\theta} + 132x^{3-\theta} - 68x^{5-\theta} + 8x^{7-\theta}) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx.$$

Using the left-sided limit of the positive real variable α ,

$$(10-98) \mathbf{B}_{[4]}(\theta) = \frac{\theta}{\pi} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{-\theta}} (-48x^{1-\theta} + 132x^{3-\theta} - 68x^{5-\theta} + 8x^{7-\theta}) e^{-x^2} dx.$$

For the integral of $\mathbf{B}_{[4]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$\begin{aligned}
(10-99) \quad \mathbf{B}_{[4]}(\theta) &= \frac{\theta}{\pi} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(-48 y^{\frac{1-\theta}{2}} + 132 y^{\frac{3-\theta}{2}} - 68 y^{\frac{5-\theta}{2}} + 8 y^{\frac{7-\theta}{2}} \right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\
&= \frac{\theta}{\pi} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(-24 \int_0^\infty y^{\frac{-\theta}{2}} e^{-y} dy + 66 \int_0^\infty y^{\frac{2-\theta}{2}} e^{-y} dy \right. \\
&\quad \left. - 34 \int_0^\infty y^{\frac{4-\theta}{2}} e^{-y} dy + 4 \int_0^\infty y^{\frac{6-\theta}{2}} e^{-y} dy \right).
\end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\operatorname{Re}(\theta) < -1$ can be added to the above result because it is a convergent function under the condition.

$$\begin{aligned}
(10-100) \quad \mathbf{B}_{[4]}(\theta) &= \frac{\theta}{\pi} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(-24 \Gamma\left(1 + \frac{-\theta}{2}\right) + 66 \Gamma\left(2 + \frac{-\theta}{2}\right) \right. \\
&\quad \left. - 34 \Gamma\left(3 + \frac{-\theta}{2}\right) + 4 \Gamma\left(4 + \frac{-\theta}{2}\right) \right) \quad (\operatorname{Re}(\theta) < -1).
\end{aligned}$$

Analytic continuation is used to extend the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$\begin{aligned}
(10-101) \quad \mathbf{B}_{[4]}(\theta) &= \\
&\frac{\theta}{\pi} \zeta(-\theta) \left(-24 \Gamma\left(1 + \frac{-\theta}{2}\right) + 66 \Gamma\left(2 + \frac{-\theta}{2}\right) - 34 \Gamma\left(3 + \frac{-\theta}{2}\right) + 4 \Gamma\left(4 + \frac{-\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).
\end{aligned}$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned}
(10-102) \quad \mathbf{B}_{[4]}(\theta) &= \frac{\theta}{\pi} \zeta(-\theta) \left(-24 \left(\frac{-\theta}{2}\right) + 66 \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) \right. \\
&\quad \left. - 34 \left(2 + \frac{-\theta}{2}\right) \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) + 4 \left(3 + \frac{-\theta}{2}\right) \left(2 + \frac{-\theta}{2}\right) \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) \right) \Gamma\left(\frac{-\theta}{2}\right) \\
&= \frac{\theta}{\pi} \zeta(-\theta) \left(\frac{\theta}{4}\right) (\theta+1) (\theta+2)^2 \Gamma\left(\frac{-\theta}{2}\right) \quad (\theta \in \mathbb{C}).
\end{aligned}$$

Combining the relation of the complete symmetric functional equation to the result,

$$\begin{aligned}
(10-103) \quad \mathbf{B}_{[4]}(\theta) &= \frac{\theta(\theta+1)(\theta+2)^2}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) \\
&= \frac{\theta(\theta+1)(\theta+2)^2}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) \quad (\theta \in \mathbb{C}).
\end{aligned}$$

The function $\mathbf{B}_{[4]}(\theta)$ is also a convergent function in the whole complex plane.

The left-sided limit of the positive real variable α , is used for defining the equation (10-92),

$$\begin{aligned}
(10-104) \quad \mathbf{F}_{[4]}(\theta) &= \\
&\int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty \left(-6 n^2 x^{\theta+1} + 84 \pi n^4 x^{\theta+3} - 88 \pi^2 n^6 x^{\theta+5} + 16 \pi^3 n^8 x^{\theta+7} \right) e^{-\pi \left(n^2 x^2 + \frac{m^2}{x^2} \right)} dx.
\end{aligned}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $\mathbf{F}_{[4]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{m y}{n} \right)^{\frac{1}{2}}.$$

$$\begin{aligned}
 (10-105) \quad F_{[4]}(\theta) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{\infty} \left(-6n^2 \left(\frac{m}{n}\right)^{\frac{\theta+1}{2}} + 84\pi n^4 \left(\frac{m}{n}\right)^{\frac{\theta+3}{2}} \right. \\
 &\quad \left. - 88\pi^2 n^6 \left(\frac{m}{n}\right)^{\frac{\theta+5}{2}} + 16\pi^3 n^8 \left(\frac{m}{n}\right)^{\frac{\theta+7}{2}} \right) e^{-\pi m n \left(y + \frac{1}{y}\right)} \frac{1}{2} \left(\frac{m}{n}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \\
 &\quad \left(-3 \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+2}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y}\right)} dy + 42 (\pi m n) \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+4}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y}\right)} dy \right. \\
 &\quad \left. - 44 (\pi m n)^2 \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+6}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y}\right)} dy + 8 (\pi m n)^3 \cdot \frac{1}{2} \int_0^{\infty} y^{\frac{\theta+8}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y}\right)} dy \right).
 \end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$\begin{aligned}
 (10-106) \quad F_{[4]}(\theta) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) + 42 (\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
 &\quad \left. - 44 (\pi m n)^2 K_{\frac{\theta+6}{2}}(2\pi m n) + 8 (\pi m n)^3 K_{\frac{\theta+8}{2}}(2\pi m n) \right).
 \end{aligned}$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied four times.

$$\begin{aligned}
 (10-107) \quad F_{[4]}(\theta) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) + 42 (\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
 &\quad \left. - 44 (\pi m n)^2 K_{\frac{\theta+6}{2}}(2\pi m n) + 8 (\pi m n)^2 \left(\frac{\theta+6}{2} K_{\frac{\theta+6}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) \right) \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) + 2(21 + 4(\pi m n)^2) (\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
 &\quad \left. + 4(\theta - 5) (\pi m n)^2 K_{\frac{\theta+6}{2}}(2\pi m n) \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 K_{\frac{\theta+2}{2}}(2\pi m n) + 2(21 + 4(\pi m n)^2) (\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
 &\quad \left. + 4(\theta - 5) (\pi m n) \left(\frac{\theta+4}{2} K_{\frac{\theta+4}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta+2}{2}}(2\pi m n) \right) \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left((-3 + 4(\theta - 5) (\pi m n)^2) K_{\frac{\theta+2}{2}}(2\pi m n) \right. \\
 &\quad \left. + 2(\theta^2 - \theta + 1 + 4(\pi m n)^2) (\pi m n) K_{\frac{\theta+4}{2}}(2\pi m n) \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left((-3 + 4(\theta - 5) (\pi m n)^2) K_{\frac{\theta+2}{2}}(2\pi m n) \right. \\
 &\quad \left. + 2(\theta^2 - \theta + 1 + 4(\pi m n)^2) \left(\frac{\theta+2}{2} K_{\frac{\theta+2}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta}{2}}(2\pi m n) \right) \right) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(2(\theta^2 - \theta + 1 + 4(\pi m n)^2) (\pi m n) K_{\frac{\theta}{2}}(2\pi m n) \right. \\
 &\quad \left. + ((\theta - 1)(\theta + 1)^2 + 4(2\theta - 3) (\pi m n)^2) K_{\frac{\theta+2}{2}}(2\pi m n) \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(2(\theta^2 - \theta + 1 + 4(\pi m n)^2)(\pi m n)^2 \mathbb{K} \frac{\theta}{2}(2\pi m n)\right. \\
&\quad \left.+ ((\theta - 1)(\theta + 1)^2 + 4(2\theta - 3)(\pi m n)^2) \left(\frac{\theta}{2} \mathbb{K} \frac{\theta}{2}(2\pi m n) + (\pi m n) \mathbb{K} \frac{\theta-2}{2}(2\pi m n)\right)\right) \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(4(\theta(\theta - 1)(\theta + 1)^2 + ((\theta - 1)(3\theta - 1) + 4(\pi m n)^2)(\pi m n)^2) \mathbb{K} \frac{\theta}{2}(2\pi m n)\right. \\
&\quad \left.+ 2((\theta - 1)(\theta + 1)^2 + 4(2\theta - 3)(\pi m n)^2)(\pi m n) \mathbb{K} \frac{\theta-2}{2}(2\pi m n)\right).
\end{aligned}$$

From the equation (5-37) and the result of the equation (10-107),

$$\begin{aligned}
(10 - 108) \quad \mathbf{F}_{[4]}(\theta) &= \\
&\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta - 1)(\theta + 1)^2 + ((\theta - 1)(3\theta - 1) + (2\pi p)^2)(2\pi p)^2) \mathbb{K} \frac{\theta}{2}(2\pi p)\right. \\
&\quad \left.+ ((\theta - 1)(\theta + 1)^2 + (2\theta - 3)(2\pi p)^2)(2\pi p) \mathbb{K} \frac{\theta-2}{2}(2\pi p)\right).
\end{aligned}$$

The left-sided limit of the positive real variable α , is used for defining the equation (10-94),

$$\begin{aligned}
(10 - 109) \quad \mathbf{G}_{[4]}(\theta) &= \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-96 n^2 x^{\theta-3} + 264 \pi n^4 x^{\theta-5}\right. \\
&\quad \left.- 136 \pi^2 n^6 x^{\theta-7} + 16 \pi^3 n^8 x^{\theta-9}\right) e^{-\pi\left(m^2 x^2 + \frac{n^2}{x^2}\right)} dx.
\end{aligned}$$

For the integrals of $\mathbf{G}_{[4]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$\begin{aligned}
(10 - 110) \quad \mathbf{G}_{[4]}(\theta) &= \int_{\infty}^0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-96 n^2 \left(\frac{1}{y}\right)^{\theta-3} + 264 \pi n^4 \left(\frac{1}{y}\right)^{\theta-5}\right. \\
&\quad \left.- 136 \pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-7} + 16 \pi^3 n^8 \left(\frac{1}{y}\right)^{\theta-9}\right) e^{-\pi\left(\frac{m^2}{y^2} + n^2 y^2\right)} (-y^{-2}) dy \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(-96 n^2 y^{1-\theta} + 264 \pi n^4 y^{3-\theta}\right. \\
&\quad \left.- 136 \pi^2 n^6 y^{5-\theta} + 16 \pi^3 n^8 y^{7-\theta}\right) e^{-\pi\left(n^2 y^2 + \frac{m^2}{y^2}\right)} dy.
\end{aligned}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $\mathbf{G}_{[4]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{m x}{n}\right)^{\frac{1}{2}}.$$

$$\begin{aligned}
(10 - 111) \quad \mathbf{G}_{[4]}(\theta) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{\infty} \left(-96 n^2 \left(\frac{m x}{n}\right)^{\frac{1-\theta}{2}} + 264 \pi n^4 \left(\frac{m x}{n}\right)^{\frac{3-\theta}{2}}\right. \\
&\quad \left.- 136 \pi^2 n^6 \left(\frac{m x}{n}\right)^{\frac{5-\theta}{2}} + 16 \pi^3 n^8 \left(\frac{m x}{n}\right)^{\frac{7-\theta}{2}}\right) e^{-\pi m n \left(x + \frac{1}{x}\right)} \frac{1}{2} \left(\frac{m}{n}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx \\
&= \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \\
&\quad \left(-12 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x}\right)} dx + 33 (\pi m n) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x}\right)} dx\right. \\
&\quad \left.- 17 (\pi m n)^2 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{6-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x}\right)} dx + 2 (\pi m n)^3 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{8-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x}\right)} dx\right).
\end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(10-112) \quad \mathbf{G}_{[4]}(\theta) = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-12 \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + 33 (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) - 17 (\pi m n)^2 \mathbf{K}_{\frac{6-\theta}{2}}(2\pi m n) + 2 (\pi m n)^3 \mathbf{K}_{\frac{8-\theta}{2}}(2\pi m n)\right).$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied three times.

$$\begin{aligned} (10-113) \quad \mathbf{G}_{[4]}(\theta) &= \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-12 \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + 33 (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) - 17 (\pi m n)^2 \mathbf{K}_{\frac{6-\theta}{2}}(2\pi m n) + 2 (\pi m n)^2 \left(\frac{6-\theta}{2} \mathbf{K}_{\frac{6-\theta}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n)\right)\right) \\ &= \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-12 \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (33 + 2 (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) - (\theta + 11) (\pi m n)^2 \mathbf{K}_{\frac{6-\theta}{2}}(2\pi m n)\right) \\ &= \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-12 \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (33 + 2 (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) - (\theta + 11) (\pi m n) \left(\frac{4-\theta}{2} \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n)\right)\right) \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-2 (12 + (\theta + 11) (\pi m n)^2) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (\theta^2 + 7\theta + 22 + 4 (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{4-\theta}{2}}(2\pi m n)\right) \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-2 (12 + (\theta + 11) (\pi m n)^2) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (\theta^2 + 7\theta + 22 + 4 (\pi m n)^2) \left(\frac{2-\theta}{2} \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{-\theta}{2}}(2\pi m n)\right)\right) \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(2 (\theta^2 + 7\theta + 22 + 4 (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) - ((\theta + 1) (\theta + 2)^2 + 4 (2\theta + 9) (\pi m n)^2) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n)\right) \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left((\theta^2 + 7\theta + 22 + 2 (\pi m n)^2) (2\pi m n)^2 \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) - ((\theta + 1) (\theta + 2)^2 + 4 (2\theta + 9) (2\pi m n)^2) (2\pi m n) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi m n)\right). \end{aligned}$$

From the equation (5-38) and the result of equation (10-113),

$$(10-114) \quad \mathbf{G}_{[4]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta^2 + 7\theta + 22 + 2 (\pi p)^2) (2\pi p)^2 \mathbf{K}_{\frac{\theta}{2}}(2\pi p) - ((\theta + 1) (\theta + 2)^2 + 4 (2\theta + 9) (2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p)\right).$$

I define the function $\mathbf{H}_{[4]}(\theta)$ as follows:

$$(10-115) \quad \mathbf{H}_{[4]}(\theta) := \mathbf{G}_{[4]}(\theta) - \mathbf{F}_{[4]}(\theta).$$

From (the right side of equation (10-114)) – (the right side of equation (10-108)), I obtain the above result, which will be shown soon. Additionally, since the procedure is the same as the one used in the previous subsection, I may confirm the absolute convergence of the function $\mathbf{H}_{[4]}(\theta)$ without a proof. Combining the above results, the following functional equation holds true:

$$(10-116) \mathbf{A}_{[4]}(\theta) = \mathbf{B}_{[4]}(\theta) + \mathbf{H}_{[4]}(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(10-117) \frac{\theta(\theta-1)(\theta+1)^2}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\theta(\theta+1)(\theta+2)^2}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) + \mathbf{H}_{[4]}(\theta)$$

$$= \frac{\theta(\theta+1)(\theta+2)^2}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) + \mathbf{H}_{[4]}(\theta) \quad (\theta \in \mathbb{C}).$$

Where

$$(10-118) \mathbf{H}_{[4]}(\theta) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1)^2 + (2\theta+3)(\theta-7)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta+1)(2\theta^2+4\theta+3) + 2(2\theta+3)(2\pi p)^2)(2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right).$$

The functional equation (10-117) is called the fourth-order Π_c type functional equation.

10-5. Functional Equation Transformation for the fifth-order I_r type functional equation

The following are the FET operations for the fifth-order I_r type functional equation:

$$(10-119) \mathbf{L}_{[5]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (-3n^2 x^{\theta+1} + 72\pi n^4 x^{\theta+3} - 224\pi^2 n^6 x^{\theta+5} \\ + 128\pi^3 n^8 x^{\theta+7} - 16\pi^4 n^{10} x^{\theta+9}) e^{-\pi(n^2+\alpha^2)x^2} dx.$$

$$(10-120) \mathbf{R}_{[5]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} (312n^2 x^{\theta-4} - 1188\pi n^4 x^{\theta-6} + 952\pi^2 n^6 x^{\theta-8} \\ - 232\pi^3 n^8 x^{\theta-10} + 16\pi^4 n^{10} x^{\theta-12}) e^{-\pi\left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.$$

For the integral of $\mathbf{L}_{[5]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{y}{\pi(n^2+\alpha^2)} \right)^{\frac{1}{2}}.$$

$$(10-121) \mathbf{L}_{[5]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(-3n^2 \left(\frac{y}{\pi(n^2+\alpha^2)} \right)^{\frac{\theta+1}{2}} + 72\pi n^4 \left(\frac{y}{\pi(n^2+\alpha^2)} \right)^{\frac{\theta+3}{2}} - 224\pi^2 n^6 \left(\frac{y}{\pi(n^2+\alpha^2)} \right)^{\frac{\theta+5}{2}} \right. \\ \left. + 128\pi^3 n^8 \left(\frac{y}{\pi(n^2+\alpha^2)} \right)^{\frac{\theta+7}{2}} - 16\pi^4 n^{10} \left(\frac{y}{\pi(n^2+\alpha^2)} \right)^{\frac{\theta+9}{2}} \right) e^{-y} \frac{1}{2} \left(\frac{1}{\pi(n^2+\alpha^2)} \right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy.$$

Using the left-sided limit of the positive variable α ,

$$(10-122) \mathbf{L}_{[5]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3y^{\frac{\theta}{2}} + 72y^{\frac{\theta}{2}+1} - 224y^{\frac{\theta}{2}+2} + 128y^{\frac{\theta}{2}+3} - 16y^{\frac{\theta}{2}+4} \right) e^{-y} dy.$$

$$= \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3 \int_0^{\infty} y^{\frac{\theta}{2}} e^{-y} dy + 72 \int_0^{\infty} y^{\frac{\theta}{2}+1} e^{-y} dy - 224 \int_0^{\infty} y^{\frac{\theta}{2}+2} e^{-y} dy \right. \\ \left. + 128 \int_0^{\infty} y^{\frac{\theta}{2}+3} e^{-y} dy - 16 \int_0^{\infty} y^{\frac{\theta}{2}+4} e^{-y} dy \right).$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\text{Re}(\theta) > 1$ can be added to the above result because it is a convergent function under the

condition.

$$(10-123) \quad \mathbf{L}_{[5]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^{\theta}} \left(-3\Gamma\left(1+\frac{\theta}{2}\right) + 72\Gamma\left(2+\frac{\theta}{2}\right) - 224\Gamma\left(3+\frac{\theta}{2}\right) \right. \\ \left. + 128\Gamma\left(4+\frac{\theta}{2}\right) - 16\Gamma\left(5+\frac{\theta}{2}\right) \right) \quad (\operatorname{Re}(\theta) > 1).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10-124) \quad \mathbf{L}_{[5]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-3\Gamma\left(1+\frac{\theta}{2}\right) + 72\Gamma\left(2+\frac{\theta}{2}\right) - 224\Gamma\left(3+\frac{\theta}{2}\right) \right. \\ \left. + 128\Gamma\left(4+\frac{\theta}{2}\right) - 16\Gamma\left(5+\frac{\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).$$

The difference formula for the gamma function is applied to the above result.

$$(10-125) \quad \mathbf{L}_{[5]}(\theta) = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-3\left(\frac{\theta}{2}\right) + 72\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) - 224\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) \right. \\ \left. + 128\left(3+\frac{\theta}{2}\right)\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) - 16\left(4+\frac{\theta}{2}\right)\left(3+\frac{\theta}{2}\right)\left(2+\frac{\theta}{2}\right)\left(1+\frac{\theta}{2}\right)\left(\frac{\theta}{2}\right) \right) \Gamma\left(\frac{\theta}{2}\right) \\ = \frac{\pi^{-\frac{\theta}{2}}}{2\pi} \zeta(\theta) \left(-\frac{\theta}{2}\right) (\theta-1)(\theta+1)(\theta^2+4\theta+5) \Gamma\left(\frac{\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(10-126) \quad \mathbf{L}_{[5]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)(\theta^2+4\theta+5)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

The function $\mathbf{L}_{[5]}(\theta)$ is a convergent function in the whole complex plane.

For the integral of $\mathbf{R}_{[5]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$(10-127) \quad \mathbf{R}_{[5]}(\theta) = \lim_{\alpha \rightarrow +0} \int_{\alpha}^0 \sum_{n=1}^{\infty} \left(312 n^2 \left(\frac{1}{y}\right)^{\theta-4} - 1188 \pi n^4 \left(\frac{1}{y}\right)^{\theta-6} + 952 \pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-8} \right. \\ \left. - 232 \pi^3 n^8 \left(\frac{1}{y}\right)^{\theta-10} + 16 \pi^4 n^{10} \left(\frac{1}{y}\right)^{\theta-12} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} (-y^{-2}) dy \\ = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(312 n^2 y^{2-\theta} - 1188 \pi n^4 y^{4-\theta} + 952 \pi^2 n^6 y^{6-\theta} \right. \\ \left. - 232 \pi^3 n^8 y^{8-\theta} + 16 \pi^4 n^{10} y^{10-\theta} \right) e^{-\pi \left(n^2 y^2 + \frac{\alpha^2}{y^2} \right)} dy.$$

For the integral of $\mathbf{R}_{[5]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi} n},$$

one more time.

$$(10-128) \quad \mathbf{R}_{[5]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^{\infty} \sum_{n=1}^{\infty} \left(312 n^2 \left(\frac{x}{\sqrt{\pi} n} \right)^{2-\theta} - 1188 \pi n^4 \left(\frac{x}{\sqrt{\pi} n} \right)^{4-\theta} \right. \\ \left. + 952 \pi^2 n^6 \left(\frac{x}{\sqrt{\pi} n} \right)^{6-\theta} - 232 \pi^3 n^8 \left(\frac{x}{\sqrt{\pi} n} \right)^{8-\theta} \right. \\ \left. + 16 \pi^4 n^{10} \left(\frac{x}{\sqrt{\pi} n} \right)^{10-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2} \right)} \frac{1}{\sqrt{\pi} n} dx$$

$$= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} (312 x^{2-\theta} - 1188 x^{4-\theta} + 952 x^{6-\theta} - 232 x^{8-\theta} + 16 x^{10-\theta}) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx.$$

Using the left-sided limit of the positive real variable α ,

$$(10 - 129) \mathbf{R}_{[5]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} (312 x^{2-\theta} - 1188 x^{4-\theta} + 952 x^{6-\theta} - 232 x^{8-\theta} + 16 x^{10-\theta}) e^{-x^2} dx.$$

For the integral of $\mathbf{R}_{[5]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$(10 - 130) \mathbf{R}_{[5]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(312 y^{\frac{2-\theta}{2}} - 1188 y^{\frac{4-\theta}{2}} + 952 y^{\frac{6-\theta}{2}} - 232 y^{\frac{8-\theta}{2}} + 16 y^{\frac{10-\theta}{2}} \right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy$$

$$= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(156 \int_0^\infty y^{\frac{1-\theta}{2}} e^{-y} dy - 594 \int_0^\infty y^{\frac{3-\theta}{2}} e^{-y} dy + 476 \int_0^\infty y^{\frac{5-\theta}{2}} e^{-y} dy - 116 \int_0^\infty y^{\frac{7-\theta}{2}} e^{-y} dy + 8 \int_0^\infty y^{\frac{9-\theta}{2}} e^{-y} dy \right).$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\operatorname{Re}(\theta) < 0$ can be added to the above result because it is a convergent function under the condition.

$$(10 - 131) \mathbf{R}_{[5]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{1-\theta}} \left(156 \Gamma\left(1 + \frac{1-\theta}{2}\right) - 594 \Gamma\left(2 + \frac{1-\theta}{2}\right) + 476 \Gamma\left(3 + \frac{1-\theta}{2}\right) - 116 \Gamma\left(4 + \frac{1-\theta}{2}\right) + 8 \Gamma\left(5 + \frac{1-\theta}{2}\right) \right) \quad (\operatorname{Re}(\theta) < 0).$$

Analytic continuation extends the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10 - 132) \mathbf{R}_{[5]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(156 \Gamma\left(1 + \frac{1-\theta}{2}\right) - 594 \Gamma\left(2 + \frac{1-\theta}{2}\right) + 476 \Gamma\left(3 + \frac{1-\theta}{2}\right) - 116 \Gamma\left(4 + \frac{1-\theta}{2}\right) + 8 \Gamma\left(5 + \frac{1-\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).$$

The difference formula for the gamma function is applied to the above result.

$$(10 - 133) \mathbf{R}_{[5]}(\theta) = \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(156 \left(\frac{1-\theta}{2}\right) - 594 \left(1 + \frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2}\right) + 476 \left(2 + \frac{1-\theta}{2}\right) \left(1 + \frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2}\right) - 116 \left(3 + \frac{1-\theta}{2}\right) \left(2 + \frac{1-\theta}{2}\right) \left(1 + \frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2}\right) + 8 \left(4 + \frac{1-\theta}{2}\right) \left(3 + \frac{1-\theta}{2}\right) \left(2 + \frac{1-\theta}{2}\right) \left(1 + \frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2}\right) \right) \Gamma\left(\frac{1-\theta}{2}\right)$$

$$= \frac{\pi^{-\frac{1-\theta}{2}}}{\pi} \zeta(1-\theta) \left(-\frac{\theta}{4}\right) (\theta-1) (\theta+1) (\theta^2 + 4\theta + 5) \Gamma\left(\frac{1-\theta}{2}\right), \quad (\theta \in \mathbb{C}).$$

Therefore

$$(10 - 134) \mathbf{R}_{[5]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)(\theta^2+4\theta+5)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

The function $\mathbf{R}_{[5]}(\theta)$ is also a convergent function in the whole complex plane.

Because of the convergent equations (10-126) and (10-134), the following functional equation holds true:

$$(10-135) \quad \mathbf{L}_{[5]}(\theta) = \mathbf{R}_{[5]}(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(10-136) \quad -\frac{\theta(\theta-1)(\theta+1)(\theta^2+4\theta+5)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \\ -\frac{\theta(\theta-1)(\theta+1)(\theta^2+4\theta+5)}{4\pi} \pi^{-\frac{1-\theta}{2}} \Gamma\left(\frac{1-\theta}{2}\right) \zeta(1-\theta) \quad (\theta \in \mathbb{C}).$$

The functional equation (10-136) is called the fifth-order I_c type functional equation. Additionally, it is equivalent to the Riemann Xi function's functional equation.

10-6. Functional Equation Transformation for the fifth-order Π_r type functional equation

The operations of the FET for the fifth-order Π_r type functional equation are shown as follows:

$$(10-137) \quad \mathbf{A}_{[5]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (-3n^2 x^{\theta+1} + 72\pi n^4 x^{\theta+3} - 224\pi^2 n^6 x^{\theta+5} \\ + 128\pi^3 n^8 x^{\theta+7} - 16\pi^4 n^{10} x^{\theta+9}) e^{-\pi(n^2 + \alpha^2)x^2} dx.$$

$$(10-138) \quad \mathbf{F}_{[5]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty (-6n^2 x^{\theta+1} + 144\pi n^4 x^{\theta+3} - 448\pi^2 n^6 x^{\theta+5} \\ + 256\pi^3 n^8 x^{\theta+7} - 32\pi^4 n^{10} x^{\theta+9}) e^{-\pi\left((n^2 + \alpha^2)x^2 + \frac{m^2}{x^2}\right)} dx.$$

$$(10-139) \quad \mathbf{B}_{[5]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (312n^2 x^{\theta-3} - 1188\pi n^4 x^{\theta-5} + 952\pi^2 n^6 x^{\theta-7} \\ - 232\pi^3 n^8 x^{\theta-9} + 16\pi^4 n^{10} x^{\theta-11}) e^{-\pi\left(\frac{n^2}{x^2} + \alpha^2 x^2\right)} dx.$$

$$(10-140) \quad \mathbf{G}_{[5]}(\theta) := \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty (624n^2 x^{\theta-3} - 2376\pi n^4 x^{\theta-5} + 1904\pi^2 n^6 x^{\theta-7} \\ - 464\pi^3 n^8 x^{\theta-9} + 32\pi^4 n^{10} x^{\theta-11}) e^{-\pi\left((m^2 + \alpha^2)x^2 + \frac{n^2}{x^2}\right)} dx.$$

I immediately obtain the following obvious result:

$$(10-141) \quad \mathbf{A}_{[5]}(\theta) = \mathbf{L}_{[5]}(\theta) = -\frac{\theta(\theta-1)(\theta+1)(\theta^2+4\theta+5)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \quad (\theta \in \mathbb{C}).$$

For the integral of $\mathbf{B}_{[5]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$(10-142) \quad \mathbf{B}_{[5]}(\theta) = \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \left(312n^2 \left(\frac{1}{y}\right)^{\theta-3} - 1188\pi n^4 \left(\frac{1}{y}\right)^{\theta-5} + 952\pi^2 n^6 \left(\frac{1}{y}\right)^{\theta-7} \right. \\ \left. - 232\pi^3 n^8 \left(\frac{1}{y}\right)^{\theta-9} + 16\pi^4 n^{10} \left(\frac{1}{y}\right)^{\theta-11}\right) e^{-\pi\left(n^2 y^2 + \frac{\alpha^2}{y^2}\right)} (-y^{-2}) dy \\ = \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty (312n^2 y^{1-\theta} - 1188\pi n^4 y^{3-\theta} + 952\pi^2 n^6 y^{5-\theta} \\ - 232\pi^3 n^8 y^{7-\theta} + 16\pi^4 n^{10} y^{9-\theta}) e^{-\pi\left(n^2 y^2 + \frac{\alpha^2}{y^2}\right)} dy.$$

For the integral of $\mathbf{B}_{[5]}(\theta)$, I perform the variable transformation

$$y = \frac{x}{\sqrt{\pi n}},$$

one more time.

$$\begin{aligned} (10-143) \quad \mathbf{B}_{[5]}(\theta) &= \\ \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty &\left(312 n^2 \left(\frac{x}{\sqrt{\pi n}} \right)^{1-\theta} - 1188 \pi n^4 \left(\frac{x}{\sqrt{\pi n}} \right)^{3-\theta} + 952 \pi^2 n^6 \left(\frac{x}{\sqrt{\pi n}} \right)^{5-\theta} \right. \\ &\left. - 232 \pi^3 n^8 \left(\frac{x}{\sqrt{\pi n}} \right)^{7-\theta} + 16 \pi^4 n^{10} \left(\frac{x}{\sqrt{\pi n}} \right)^{9-\theta} \right) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} \frac{1}{\sqrt{\pi n}} dx \\ &= \frac{\theta}{\pi} \lim_{\alpha \rightarrow +0} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{-\theta}} (312 x^{1-\theta} - 1188 x^{3-\theta} + 952 x^{5-\theta} \\ &\quad - 232 x^{7-\theta} + 16 x^{9-\theta}) e^{-\left(x^2 + \frac{(\pi n \alpha)^2}{x^2}\right)} dx. \end{aligned}$$

Using the left-sided limit of the positive real variable α ,

$$(10-144) \quad \mathbf{B}_{[5]}(\theta) = \frac{\theta}{\pi} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{-\theta}} (312 x^{1-\theta} - 1188 x^{3-\theta} + 952 x^{5-\theta} - 232 x^{7-\theta} + 16 x^{9-\theta}) e^{-x^2} dx.$$

For the integral of $\mathbf{B}_{[5]}(\theta)$, I perform the variable transformation $x = y^{1/2}$.

$$\begin{aligned} (10-145) \quad \mathbf{B}_{[5]}(\theta) &= \frac{\theta}{\pi} \int_0^\infty \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(312 y^{\frac{1-\theta}{2}} - 1188 y^{\frac{3-\theta}{2}} + 952 y^{\frac{5-\theta}{2}} \right. \\ &\quad \left. - 232 y^{\frac{7-\theta}{2}} + 16 y^{\frac{9-\theta}{2}} \right) e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy \\ &= \frac{\theta}{\pi} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(156 \int_0^\infty y^{\frac{-\theta}{2}} e^{-y} dy - 594 \int_0^\infty y^{\frac{2-\theta}{2}} e^{-y} dy + 476 \int_0^\infty y^{\frac{4-\theta}{2}} e^{-y} dy \right. \\ &\quad \left. - 116 \int_0^\infty y^{\frac{6-\theta}{2}} e^{-y} dy + 8 \int_0^\infty y^{\frac{8-\theta}{2}} e^{-y} dy \right). \end{aligned}$$

Because the sum is irrelevant to the integrals, the sum is shifted outside.

Now, the condition $\operatorname{Re}(\theta) < -1$ can be added to the above result because it is a convergent function under the condition.

$$(10-146) \quad \mathbf{B}_{[5]}(\theta) = \frac{\theta}{\pi} \sum_{n=1}^\infty \frac{1}{n^{-\theta}} \left(156 \Gamma\left(1 + \frac{-\theta}{2}\right) - 594 \Gamma\left(2 + \frac{-\theta}{2}\right) + 476 \Gamma\left(3 + \frac{-\theta}{2}\right) \right. \\ \left. - 116 \Gamma\left(4 + \frac{-\theta}{2}\right) + 8 \Gamma\left(5 + \frac{-\theta}{2}\right) \right) \quad (\operatorname{Re}(\theta) < -1).$$

Analytic continuation is used to extend the domain of definition to the whole complex plane for both the zeta and gamma functions.

$$(10-147) \quad \mathbf{B}_{[5]}(\theta) = \frac{\theta}{\pi} \zeta(-\theta) \left(156 \Gamma\left(1 + \frac{-\theta}{2}\right) - 594 \Gamma\left(2 + \frac{-\theta}{2}\right) + 476 \Gamma\left(3 + \frac{-\theta}{2}\right) \right. \\ \left. - 116 \Gamma\left(4 + \frac{-\theta}{2}\right) + 8 \Gamma\left(5 + \frac{-\theta}{2}\right) \right) \quad (\theta \in \mathbb{C}).$$

The difference formula for the gamma function is applied to the above result.

$$\begin{aligned}
 (10-148) \quad \mathbf{B}_{[5]}(\theta) &= \frac{\theta}{\pi^2} \zeta(-\theta) \left(156 \left(\frac{-\theta}{2}\right) - 594 \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) + 476 \left(2 + \frac{-\theta}{2}\right) \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) \right. \\
 &\quad \left. - 116 \left(3 + \frac{-\theta}{2}\right) \left(2 + \frac{-\theta}{2}\right) \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) + 8 \left(4 + \frac{-\theta}{2}\right) \left(3 + \frac{-\theta}{2}\right) \left(2 + \frac{-\theta}{2}\right) \left(1 + \frac{-\theta}{2}\right) \left(\frac{-\theta}{2}\right) \right) \Gamma\left(\frac{-\theta}{2}\right) \\
 &= \frac{\theta}{\pi^2} \zeta(-\theta) \left(\frac{-\theta}{4}\right) (\theta+1) (\theta+2) (\theta^2 + 6\theta + 10) \Gamma\left(\frac{-\theta}{2}\right) \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

Combining the relation of the complete symmetric functional equation to the result,

$$\begin{aligned}
 (10-149) \quad \mathbf{B}_{[5]}(\theta) &= -\frac{\theta(\theta+1)(\theta+2)(\theta^2+6\theta+10)}{4\pi} \pi^2 \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) \\
 &= -\frac{\theta(\theta+1)(\theta+2)(\theta^2+6\theta+10)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) \quad (\theta \in \mathbb{C}).
 \end{aligned}$$

The function $\mathbf{B}_{[5]}(\theta)$ is also a convergent function in the whole complex plane.

The left-sided limit of the positive real variable α , is used for defining the equation (10-138),

$$\begin{aligned}
 (10-150) \quad \mathbf{F}_{[5]}(\theta) &= \int_0^\infty \sum_{n=1}^\infty \sum_{m=1}^\infty \left(-6n^2 x^{\theta+1} + 144\pi n^4 x^{\theta+3} - 448\pi^2 n^6 x^{\theta+5} \right. \\
 &\quad \left. + 256\pi^3 n^8 x^{\theta+7} - 32\pi^4 n^{10} x^{\theta+9} \right) e^{-\pi \left(n^2 x^2 + \frac{m^2}{x^2} \right)} dx.
 \end{aligned}$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $\mathbf{F}_{[5]}(\theta)$, I perform the variable transformation

$$x = \left(\frac{m y}{n}\right)^{\frac{1}{2}}.$$

$$\begin{aligned}
 (10-151) \quad \mathbf{F}_{[5]}(\theta) &= \sum_{n=1}^\infty \sum_{m=1}^\infty \int_0^\infty \left(-6n^2 \left(\frac{m y}{n}\right)^{\frac{\theta+1}{2}} + 144\pi n^4 \left(\frac{m y}{n}\right)^{\frac{\theta+3}{2}} - 448\pi^2 n^6 \left(\frac{m y}{n}\right)^{\frac{\theta+5}{2}} \right. \\
 &\quad \left. + 256\pi^3 n^8 \left(\frac{m y}{n}\right)^{\frac{\theta+7}{2}} - 32\pi^4 n^{10} \left(\frac{m y}{n}\right)^{\frac{\theta+9}{2}} \right) e^{-\pi m n \left(y + \frac{1}{y} \right)} \frac{1}{2} \left(\frac{m}{n}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} dy \\
 &= \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+2}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y} \right)} dy \right. \\
 &\quad + 72 (\pi m n) \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+4}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y} \right)} dy \\
 &\quad - 224 (\pi m n)^2 \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+6}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y} \right)} dy \\
 &\quad + 128 (\pi m n)^3 \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+8}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y} \right)} dy \\
 &\quad \left. - 16 (\pi m n)^4 \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta+10}{2}-1} e^{-\frac{2\pi m n}{2} \left(y + \frac{1}{y} \right)} dy \right).
 \end{aligned}$$

The integrals can be written using the modified Bessel functions of the second kind.

$$\begin{aligned}
 (10-152) \quad \mathbf{F}_{[5]}(\theta) &= \frac{2}{\pi} \sum_{n=1}^\infty \sum_{m=1}^\infty (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) + 72 (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
 &\quad \left. - 224 (\pi m n)^2 \mathbf{K}_{\frac{\theta+6}{2}}(2\pi m n) + 128 (\pi m n)^3 \mathbf{K}_{\frac{\theta+8}{2}}(2\pi m n) - 16 (\pi m n)^4 \mathbf{K}_{\frac{\theta+10}{2}}(2\pi m n) \right).
 \end{aligned}$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied five times.

$$\begin{aligned}
(10-153) \quad \mathbf{F}_{[5]}(\theta) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) + 72 (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
&\quad - 224 (\pi m n)^2 \mathbf{K}_{\frac{\theta+6}{2}}(2\pi m n) + 128 (\pi m n)^3 \mathbf{K}_{\frac{\theta+8}{2}}(2\pi m n) \\
&\quad \left. - 16 (\pi m n)^3 \left(\frac{\theta+8}{2} \mathbf{K}_{\frac{\theta+8}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{\theta+6}{2}}(2\pi m n)\right)\right) \\
&= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) + 72 (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
&\quad \left. - 16 (14 + (\pi m n)^2) (\pi m n)^2 \mathbf{K}_{\frac{\theta+6}{2}}(2\pi m n) - 8 (\theta-8) (\pi m n)^3 \mathbf{K}_{\frac{\theta+8}{2}}(2\pi m n)\right) \\
&= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) + 72 (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
&\quad - 16 (14 + (\pi m n)^2) (\pi m n)^2 \mathbf{K}_{\frac{\theta+6}{2}}(2\pi m n) \\
&\quad \left. - 8 (\theta-8) (\pi m n)^2 \left(\frac{\theta+6}{2} \mathbf{K}_{\frac{\theta+6}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n)\right)\right) \\
&= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) - 8 (-9 + (\theta-8) (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
&\quad \left. - 4 (\theta^2 - 2\theta + 8 + 4 (\pi m n)^2) (\pi m n)^2 \mathbf{K}_{\frac{\theta+6}{2}}(2\pi m n)\right) \\
&= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(-3 \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) - 8 (-9 + (\theta-8) (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n) \right. \\
&\quad \left. - 4 (\theta^2 - 2\theta + 8 + 4 (\pi m n)^2) (\pi m n) \left(\frac{\theta+4}{2} \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n)\right)\right) \\
&= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left((3 + 4 (\theta^2 - 2\theta + 8 + 4 (\pi m n)^2) (\pi m n)^2) \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) \right. \\
&\quad \left. + 2 (\theta^3 + 2\theta^2 - 4 + 8 (\theta-2) (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{\theta+4}{2}}(2\pi m n)\right) \\
&= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left((3 + 4 (\theta^2 - 2\theta + 8 + 4 (\pi m n)^2) (\pi m n)^2) \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) \right. \\
&\quad \left. + 2 (\theta^3 + 2\theta^2 - 4 + 8 (\theta-2) (\pi m n)^2) \left(\frac{\theta+2}{2} \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{\theta}{2}}(2\pi m n)\right)\right) \\
&= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(2 (\theta^3 + 2\theta^2 - 4 + 8 (\theta-2) (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) \right. \\
&\quad \left. + (\theta^4 + 4\theta^3 + 4\theta^2 - 4\theta - 5 + 4 (3\theta^2 - 2\theta + 4 (\pi m n)^2) (\pi m n)^2) \mathbf{K}_{\frac{\theta+2}{2}}(2\pi m n)\right) \\
&= -\frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \left(4 (\theta^3 + 2\theta^2 - 4 + 8 (\theta-2) (\pi m n)^2) (\pi m n)^2 \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) \right. \\
&\quad \left. + 2 (\theta^4 + 4\theta^3 + 4\theta^2 - 4\theta - 5 + 4 (3\theta^2 - 2\theta + 4 (\pi m n)^2) (\pi m n)^2) \right. \\
&\quad \left. \left(\frac{\theta}{2} \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) + (\pi m n) \mathbf{K}_{\frac{\theta-2}{2}}(2\pi m n)\right)\right) \\
&= -\frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{\theta}{2}} \\
&\quad \left((\theta^5 + 4\theta^4 + 4\theta^3 - 4\theta^2 - 5\theta + 4 (4\theta^3 - 4 + 4 (3\theta - 4) (\pi m n)^2) (\pi m n)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi m n) \right. \\
&\quad \left. + 2 (\theta^4 + 4\theta^3 + 4\theta^2 - 4\theta - 5 + 4 (3\theta^2 - 2\theta + 4 (\pi m n)^2) (\pi m n)^2) (\pi m n) \mathbf{K}_{\frac{\theta-2}{2}}(2\pi m n)\right).
\end{aligned}$$

From the equation (5-37) and the result of equation (10-153),

$$(10-154) \quad \mathbf{F}_{[5]}(\theta) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}}$$

$$\left((\theta^5 + 4\theta^4 + 4\theta^3 - 4\theta^2 - 5\theta + (4\theta^3 - 4 + (3\theta - 4)(2\pi p)^2)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + (\theta^4 + 4\theta^3 + 4\theta^2 - 4\theta - 5 + (3\theta^2 - 2\theta + (2\pi p)^2)(2\pi p)^2)(2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right).$$

The left-sided limit of the positive real variable α , is used for defining the equation (10-140),

$$(10-155) \quad \mathbf{G}_{[5]}(\theta) = \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (624 n^2 x^{\theta-3} - 2376 \pi n^4 x^{\theta-5} + 1904 \pi^2 n^6 x^{\theta-7} \\ - 464 \pi^3 n^8 x^{\theta-9} + 32 \pi^4 n^{10} x^{\theta-11}) e^{-\pi \left(m^2 x^2 + \frac{n^2}{x^2} \right)} dx.$$

For the integral of $\mathbf{G}_{[5]}(\theta)$, I perform the variable transformation $x = y^{-1}$.

$$(10-156) \quad \mathbf{G}_{[5]}(\theta) = \int_{\infty}^0 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(624 n^2 \left(\frac{1}{y} \right)^{\theta-3} - 2376 \pi n^4 \left(\frac{1}{y} \right)^{\theta-5} + 1904 \pi^2 n^6 \left(\frac{1}{y} \right)^{\theta-7} \right. \\ \left. - 464 \pi^3 n^8 \left(\frac{1}{y} \right)^{\theta-9} + 32 \pi^4 n^{10} \left(\frac{1}{y} \right)^{\theta-11} \right) e^{-\pi \left(\frac{m^2}{y^2} + n^2 y^2 \right)} (-y^{-2}) dy \\ = \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (624 n^2 y^{1-\theta} - 2376 \pi n^4 y^{3-\theta} + 1904 \pi^2 n^6 y^{5-\theta} \\ - 464 \pi^3 n^8 y^{7-\theta} + 32 \pi^4 n^{10} y^{9-\theta}) e^{-\pi \left(n^2 y^2 + \frac{m^2}{y^2} \right)} dy.$$

Assuming that the integral and the double sum can be interchanged, for the integrals of $\mathbf{G}_{[5]}(\theta)$, I perform the variable transformation

$$y = \left(\frac{m x}{n} \right)^{\frac{1}{2}}.$$

$$(10-157) \quad \mathbf{G}_{[5]}(\theta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^{\infty} \left(624 n^2 \left(\frac{m x}{n} \right)^{\frac{1-\theta}{2}} - 2376 \pi n^4 \left(\frac{m x}{n} \right)^{\frac{3-\theta}{2}} + 1904 \pi^2 n^6 \left(\frac{m x}{n} \right)^{\frac{5-\theta}{2}} \right. \\ \left. - 464 \pi^3 n^8 \left(\frac{m x}{n} \right)^{\frac{7-\theta}{2}} + 32 \pi^4 n^{10} \left(\frac{m x}{n} \right)^{\frac{9-\theta}{2}} \right) e^{-\pi m n \left(x + \frac{1}{x} \right)} \frac{1}{2} \left(\frac{m}{n} \right)^{\frac{1}{2}} x^{-\frac{1}{2}} dx \\ = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n} \right)^{-\frac{\theta}{2}} \left(78 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{2-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right. \\ - 297 (\pi m n) \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{4-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \\ + 238 (\pi m n)^2 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{6-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \\ - 58 (\pi m n)^3 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{8-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \\ \left. + 4 (\pi m n)^4 \cdot \frac{1}{2} \int_0^{\infty} x^{\frac{10-\theta}{2}-1} e^{-\frac{2\pi m n}{2} \left(x + \frac{1}{x} \right)} dx \right).$$

The integrals can be written using the modified Bessel functions of the second kind.

$$(10-158) \quad G_{[5]}(\theta) = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(78 K_{\frac{2-\theta}{2}}(2\pi m n) - 297 (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n)\right) \\ + 238 (\pi m n)^2 K_{\frac{6-\theta}{2}}(2\pi m n) - 58 (\pi m n)^3 K_{\frac{8-\theta}{2}}(2\pi m n) + 4 (\pi m n)^4 K_{\frac{10-\theta}{2}}(2\pi m n).$$

Because any modified Bessel function of the second kind of the double sum converges absolutely, the assumed exchange is justified. The recurrence formula for the modified Bessel function of the second kind is applied four times.

$$(10-159) \quad G_{[5]}(\theta) = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(78 K_{\frac{2-\theta}{2}}(2\pi m n) - 297 (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n)\right) \\ + 238 (\pi m n)^2 K_{\frac{6-\theta}{2}}(2\pi m n) - 58 (\pi m n)^3 K_{\frac{8-\theta}{2}}(2\pi m n) \\ + 4 (\pi m n)^3 \left(\frac{8-\theta}{2} K_{\frac{8-\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{6-\theta}{2}}(2\pi m n)\right) \\ = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(78 K_{\frac{2-\theta}{2}}(2\pi m n) - 297 (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n)\right) \\ + (238 + 4 (\pi m n)^2) (\pi m n)^2 K_{\frac{6-\theta}{2}}(2\pi m n) - 2 (\theta + 21) (\pi m n)^3 K_{\frac{8-\theta}{2}}(2\pi m n) \\ = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(78 K_{\frac{2-\theta}{2}}(2\pi m n) - 297 (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n)\right) \\ + (238 + 4 (\pi m n)^2) (\pi m n)^2 K_{\frac{6-\theta}{2}}(2\pi m n) \\ - 2 (\theta + 21) (\pi m n)^2 \left(\frac{6-\theta}{2} K_{\frac{6-\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n)\right) \\ = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(78 K_{\frac{2-\theta}{2}}(2\pi m n)\right) \\ - (297 + 2 (\theta + 21) (\pi m n)^2) (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n) \\ + (\theta^2 + 15\theta + 112 + 4 (\pi m n)^2) (\pi m n)^2 K_{\frac{6-\theta}{2}}(2\pi m n) \\ = \frac{8}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(78 K_{\frac{2-\theta}{2}}(2\pi m n)\right) \\ - (297 + 2 (\theta + 21) (\pi m n)^2) (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n) \\ + (\theta^2 + 15\theta + 112 + 4 (\pi m n)^2) (\pi m n) \left(\frac{4-\theta}{2} K_{\frac{4-\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{2-\theta}{2}}(2\pi m n)\right) \\ = \frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left((156 + 2 (\theta^2 + 15\theta + 112 + 4 (\pi m n)^2) (\pi m n)^2) K_{\frac{2-\theta}{2}}(2\pi m n)\right) \\ - (\theta^3 + 11\theta^2 + 52\theta + 146 + 4 (2\theta + 17) (\pi m n)^2) (\pi m n) K_{\frac{4-\theta}{2}}(2\pi m n) \\ = \frac{4}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left((156 + 2 (\theta^2 + 15\theta + 112 + 4 (\pi m n)^2) (\pi m n)^2) K_{\frac{2-\theta}{2}}(2\pi m n)\right) \\ - (\theta^3 + 11\theta^2 + 52\theta + 146 + 4 (2\theta + 17) (\pi m n)^2) \left(\frac{2-\theta}{2} K_{\frac{2-\theta}{2}}(2\pi m n) + (\pi m n) K_{\frac{\theta}{2}}(2\pi m n)\right) \\ = \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\pi m n) \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(-2 (\theta^3 + 11\theta^2 + 52\theta + 146 + 4 (2\theta + 17) (\pi m n)^2) (\pi m n) K_{\frac{\theta}{2}}(2\pi m n)\right) \\ + (\theta^4 + 9\theta^3 + 30\theta^2 + 42\theta + 20 + 4 (3\theta^2 + 28\theta + 78 + 4 (\pi m n)^2) (\pi m n)^2) K_{\frac{2-\theta}{2}}(2\pi m n) \\ = -\frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{m}{n}\right)^{-\frac{\theta}{2}} \left(4 (\theta^3 + 11\theta^2 + 52\theta + 146 + 4 (2\theta + 17) (\pi m n)^2) (\pi m n)^2 K_{\frac{\theta}{2}}(2\pi m n)\right) \\ - 2 (\theta^4 + 9\theta^3 + 30\theta^2 + 42\theta + 20 + 4 (3\theta^2 + 28\theta + 78 + 4 (\pi m n)^2) (\pi m n)^2) (\pi m n) K_{\frac{2-\theta}{2}}(2\pi m n).$$

From the equation (5-38) and the result of equation (10-159),

$$(10-160) \quad \mathbf{G}_{[5]}(\theta) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}}$$

$$\left((\theta^3 + 11\theta^2 + 52\theta + 146 + (2\theta + 17)(2\pi p)^2)(2\pi p)^2 \mathbf{K} \frac{\theta}{2}(2\pi p) \right. \\ \left. - (\theta^4 + 9\theta^3 + 30\theta^2 + 42\theta + 20 + (3\theta^2 + 28\theta + 78 + (2\pi p)^2)(2\pi p)^2)(2\pi p) \mathbf{K} \frac{2-\theta}{2}(2\pi p) \right).$$

The function $\mathbf{H}_{[5]}(\theta)$ is defined as follows:

$$(10-161) \quad \mathbf{H}_{[5]}(\theta) := \mathbf{G}_{[5]}(\theta) - \mathbf{F}_{[5]}(\theta).$$

From (the right side of equation (10-160)) – (the right side of equation (10-154)), I obtain the above result, which will be shown soon. And I may also omit a proof for the absolute convergence of the function $\mathbf{H}_{[5]}(\theta)$.

Combining the above results, the following functional equation holds true:

$$(10-162) \quad \mathbf{A}_{[5]}(\theta) = \mathbf{B}_{[5]}(\theta) + \mathbf{H}_{[5]}(\theta) \quad (\theta \in \mathbb{C}).$$

Therefore

$$(10-163) \quad -\frac{\theta(\theta-1)(\theta+1)(\theta^2+4\theta+5)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \\ -\frac{\theta(\theta+1)(\theta+2)(\theta^2+6\theta+10)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) + \mathbf{H}_{[5]}(\theta) \\ = -\frac{\theta(\theta+1)(\theta+2)(\theta^2+6\theta+10)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) + \mathbf{H}_{[5]}(\theta) \quad (\theta \in \mathbb{C}).$$

Where

$$(10-164) \quad \mathbf{H}_{[5]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}}$$

$$\left((\theta^5 + 4\theta^4 + 4\theta^3 - 4\theta^2 - 5\theta + (3\theta^3 - 11\theta^2 - 52\theta - 150 + (\theta - 21)(2\pi p)^2)(2\pi p)^2) \mathbf{K} \frac{\theta}{2}(2\pi p) \right. \\ \left. + (2\theta^4 + 13\theta^3 + 34\theta^2 + 38\theta + 15 + 2(3\theta^2 + 13\theta + 39 + (2\pi p)^2)(2\pi p)^2)(2\pi p) \mathbf{K} \frac{2-\theta}{2}(2\pi p) \right).$$

The functional equation (10-163) is called the fifth-order Π_c type functional equation.

11. Derivation of a new explicit formula for the zeta function and the second proof of the Riemann hypothesis

11-1. Derivation of a new explicit formula for the zeta function

The fourth-order Π_c type functional equation is shown again.

$$(11-1) \quad \frac{\theta(\theta-1)(\theta+1)^2}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{\theta(\theta+1)(\theta+2)^2}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) + \mathbf{H}_{[4]}(\theta) \\ = \frac{\theta(\theta+1)(\theta+2)^2}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) + \mathbf{H}_{[4]}(\theta), \quad (\theta \in \mathbb{C}).$$

The third-order Π_c type functional equation is also presented.

$$(11-2) \quad -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = -\frac{\theta(\theta+1)(\theta+2)}{4\pi} \pi^{\frac{\theta}{2}} \Gamma\left(-\frac{\theta}{2}\right) \zeta(-\theta) + \mathbf{H}_{[3]}(\theta) \\ = -\frac{\theta(\theta+1)(\theta+2)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) + \mathbf{H}_{[3]}(\theta) \quad (\theta \in \mathbb{C}).$$

From (fourth-order Π_c type functional equation)+(third-order Π_c type functional equation) $\times(\theta+2)$,

$$(11-3) \quad -\frac{\theta(\theta-1)(\theta+1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = (\theta+2) \mathbf{H}_{[3]}(\theta) + \mathbf{H}_{[4]}(\theta) \quad (\theta \in \mathbb{C}).$$

Where

$$(11-4) \quad \mathbf{H}_{[3]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1) + (\theta-7)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((2\theta+1)(\theta+1) + 2(2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right).$$

$$(11-5) \quad \mathbf{H}_{[4]}(\theta) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1)^2 + (2\theta+3)(\theta-7)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta+1)(2\theta^2+4\theta+3) + 2(2\theta+3)(2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right).$$

The right side of equation (11-3) is calculated to obtain the following equation:

$$(11-6) \quad (\theta+2) \mathbf{H}_{[3]}(\theta) + \mathbf{H}_{[4]}(\theta) = \frac{\theta+1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

The common factor $(\theta+1)$ is simplified to obtain the following equation:

$$(11-7) \quad -\frac{\theta(\theta-1)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

Both sides of the above equation are multiplied by

$$\frac{2\pi^{\frac{\theta}{2}+1}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \quad (\theta \in \mathbb{C} \setminus \{1\})$$

to obtain the new explicit formula for the zeta function.

$$(11-8) \quad \zeta(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2) (2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

11-2. The second proof of the Riemann hypothesis

Both x and y are assumed to be real numbers, and $(x + iy)$ is assumed to be a non-trivial zeta function zero. From the point symmetry of the Riemann Xi function, $(1 - x - iy)$ is also non-trivial zero of the zeta function. The real axis must be excluded from the target region because there are trivial zeros of the zeta function on it. I consider that the non-trivial zeros are determined as the solutions of the simultaneous equations comprising the following equations:

$$(11-9) \quad \begin{cases} \zeta(x+iy) = 0 & (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}), \\ \zeta(1-x-iy) = 0 & (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}). \end{cases}$$

For the explicit formula for the zeta function, $(x + iy)$ is substituted for θ .

$$(11-10) \quad \zeta(x+iy) = \frac{2\pi^{\frac{x+iy}{2}}}{(1-x-iy)\Gamma\left(1+\frac{x+iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-x-iy}(p) p^{\frac{x+iy}{2}}$$

$$\left(((x+iy)(x+iy-1) + (7-x-iy)(2\pi p)^2) K \frac{x+iy}{2}(2\pi p) \right. \\ \left. + ((x+iy-1) - 2(2\pi p)^2)(2\pi p) K \frac{2-x-iy}{2}(2\pi p) \right) = 0 \quad (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}).$$

Similarly, for the explicit formula for the zeta function, $(1-x-iy)$ is substituted for θ .

$$(11-11) \quad \zeta(1-x-iy) = \frac{2\pi^{\frac{1-x-iy}{2}}}{(x+iy)\Gamma\left(1+\frac{1-x-iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{x+iy-1}(p) p^{\frac{1-x-iy}{2}}$$

$$\left(((1-x-iy)(-x-iy) + (6+x+iy)(2\pi p)^2) K \frac{1-x-iy}{2}(2\pi p) \right. \\ \left. + (-x-iy - 2(2\pi p)^2)(2\pi p) K \frac{1+x+iy}{2}(2\pi p) \right) = 0 \quad (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}).$$

The non-trivial zeros are determined as the solutions of the simultaneous equations comprising the equations (11-10) and (11-11).

Because of the symmetric pair of the equations,

$$(11-12) \quad x = 1/2.$$

is immediately determined.

For equation (11-10), $1/2$ is substituted for x .

$$(11-13) \quad \zeta(1/2+iy) = \frac{2\pi^{\frac{1+iy}{4}}}{\left(\frac{1}{2}-iy\right)\Gamma\left(1+\frac{1+iy}{4}\right)} \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+iy}{4}}$$

$$\left(\left(-\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right) + \left(\frac{13}{2}-iy\right)(2\pi p)^2\right) K \frac{1+iy}{4}(2\pi p) \right. \\ \left. + \left(-\left(\frac{1}{2}-iy\right) - 2(2\pi p)^2\right)(2\pi p) K \frac{3-iy}{4}(2\pi p) \right) = 0 \quad (y \in \mathbb{R} \setminus \{0\}).$$

For equation (11-11), $1/2$ is substituted for x .

$$(11-14) \quad \zeta(1/2-iy) = \frac{2\pi^{\frac{1-iy}{4}}}{\left(\frac{1}{2}+iy\right)\Gamma\left(1+\frac{1-iy}{4}\right)} \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}+iy}(p) p^{\frac{1-iy}{4}}$$

$$\left(\left(-\left(\frac{1}{2}-iy\right)\left(\frac{1}{2}+iy\right) + \left(\frac{13}{2}+iy\right)(2\pi p)^2\right) K \frac{1-iy}{4}(2\pi p) \right. \\ \left. + \left(-\left(\frac{1}{2}+iy\right) - 2(2\pi p)^2\right)(2\pi p) K \frac{3+iy}{4}(2\pi p) \right) = 0 \quad (y \in \mathbb{R} \setminus \{0\}).$$

The equations (11-13) and (11-14) are complex conjugates of one another. And I can choose either equation (11-13) or (11-14) as the determining equation of the real variable y . On the critical line, the gamma factor of the zeta function does not take the value zero. Furthermore, the condition of $y = 0$ can be excepted because there is no non-trivial zero of the zeta function on the real axis.

Therefore, the following equation gives the determining equation of the real variable y :

$$(11-15) \quad \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+iy}{4}} \left((-y^2 - \frac{1}{4} + \left(\frac{13}{2}-iy\right)(2\pi p)^2) K \frac{1+iy}{4}(2\pi p) \right. \\ \left. + \left(-\frac{1}{2}+iy - 2(2\pi p)^2\right)(2\pi p) K \frac{3-iy}{4}(2\pi p) \right) = 0 \quad (y \in \mathbb{R}).$$

The explicit formula for the Riemann Xi function is also calculated.

$$(11-16) \quad \xi(\theta) = \frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta)$$

$$= -2 \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}}$$

$$\left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

The function value of the Riemann Xi function on the critical line is

$$(11-17) \quad \xi\left(\frac{1}{2} + iy\right) = -2 \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+i2y}{4}} \left((-y^2 - \frac{1}{4} + (\frac{13}{2} - iy)(2\pi p)^2) K_{\frac{1+i2y}{4}}(2\pi p) \right. \\ \left. + \left(-\frac{1}{2} + iy - 2(2\pi p)^2\right)(2\pi p) K_{\frac{3-i2y}{4}}(2\pi p) \right) \quad (y \in \mathbb{R}).$$

This result shows that the Riemann Xi function on the critical line is equivalent to the left side of the determining equation with difference constant multiplication.

Applying the recurrence formula and the origin symmetry with respect to the index of the modified Bessel function of the second kind,

$$(11-18) \quad \xi\left(\frac{1}{2} + iy\right) = -2 \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+i2y}{4}} \left((-y^2 - \frac{1}{4} + (\frac{13}{2} - iy)(2\pi p)^2) K_{\frac{1+i2y}{4}}(2\pi p) \right. \\ \left. + \left(-\frac{1}{2} + iy - 2(2\pi p)^2\right)(2\pi p) K_{\frac{-3+i2y}{4}}(2\pi p) \right) \\ = -2 \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+i2y}{4}} \left((-y^2 - \frac{1}{4} + (\frac{13}{2} - iy)(2\pi p)^2) K_{\frac{1+i2y}{4}}(2\pi p) \right. \\ \left. + \left(\frac{1}{2} - iy + 2(2\pi p)^2\right) \left(\frac{1+i2y}{2} K_{\frac{1+i2y}{4}}(2\pi p) - (2\pi p) K_{\frac{5+i2y}{4}}(2\pi p) \right) \right) \\ = -2 \sum_{p=1}^{\infty} (2\pi p) \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+i2y}{4}} \left(\left(\frac{15}{2} + iy\right)(2\pi p) K_{\frac{1+i2y}{4}}(2\pi p) \right. \\ \left. - \left(\frac{1}{2} - iy + 2(2\pi p)^2\right) K_{\frac{5+i2y}{4}}(2\pi p) \right) \quad (y \in \mathbb{R}).$$

Using the complex conjugate on both sides of the above equation,

$$(11-19) \quad \xi^*\left(\frac{1}{2} + iy\right) = -2 \sum_{p=1}^{\infty} (2\pi p) \sigma_{-\frac{1}{2}+iy}(p) p^{\frac{1-i2y}{4}} \left(\left(\frac{15}{2} - iy\right)(2\pi p) K_{\frac{1-i2y}{4}}(2\pi p) \right. \\ \left. - \left(\frac{1}{2} + iy + 2(2\pi p)^2\right) K_{\frac{5-i2y}{4}}(2\pi p) \right) \quad (y \in \mathbb{R}).$$

Therefore, it is shown that the Riemann Xi function takes real value on the critical line as follows:

$$(11-20) \quad \text{Im}\left(\xi\left(\frac{1}{2} + iy\right)\right) = \frac{1}{2i} \left(\xi\left(\frac{1}{2} + iy\right) - \xi^*\left(\frac{1}{2} + iy\right) \right) = 0 \quad (y \in \mathbb{R}).$$

The Hadamard product representation for the Riemann Xi function on the critical line is

$$(11-21) \quad \xi\left(\frac{1}{2} + iy\right) = \frac{1}{2} \prod_{m=1}^{\infty} \left(1 - \frac{4y^2 + 1}{(2\rho_m)^2 + 1} \right) \quad (y \in \mathbb{R}).$$

From the equations (11-17) and (11-21),

$$(11-22) \quad \sum_{p=1}^{\infty} \sigma_{-\frac{1}{2}-iy}(p) p^{\frac{1+i2y}{4}} \left((-y^2 - \frac{1}{4} + (\frac{13}{2} - iy)(2\pi p)^2) K_{\frac{1+i2y}{4}}(2\pi p) \right. \\ \left. + \left(-\frac{1}{2} + iy - 2(2\pi p)^2\right)(2\pi p) K_{\frac{3-i2y}{4}}(2\pi p) \right) = -\frac{1}{4} \prod_{m=1}^{\infty} \left(1 - \frac{4y^2 + 1}{(2\rho_m)^2 + 1} \right) \in \mathbb{R} \quad (y \in \mathbb{R}).$$

The equation (11-22) insists on two points. One is that the left side of the determining equation has a real value on the critical line, and the other is that there are an unlimited number of non-trivial zeros. It is possible to assert that there is no non-trivial zero off the critical line.

The second proof of the Riemann hypothesis for the zeta function is thus completed.

It may appear strange because the left side of the determining equation takes a real value on the critical line, although the imaginary unit i is described as explicit in equation (11-15).

Then see the following analogy as a reference:

Consider the cubic equations with three different real solutions. Additionally, the imaginary unit i appeared when the formula for solving cubic equations was applied. When you actually proceed with the calculation according to the formula, the imaginary unit has disappeared and the real solutions have been obtained.

It is possible to design a zeta function variant that takes real value on the critical line.

The method is to take the absolute value for the gamma factor in the explicit formula for the zeta function.

I decide to write the zeta function variation as zeta-tilde by adding the symbol \sim (tilde) just above the Greek letter ζ .

$$(11-23) \quad \tilde{\zeta}(\theta) := \left| \frac{2\pi^2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \right| \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

From now, I consider the behavior of the number-theoretic function $\sigma_{-\theta}(n) n^{\theta/2}$ on the critical line by presenting specific examples.

As its preparation, I begin with the classification of the whole positive integers into four subsets.

Assuming k as a positive integer, I consider the first subset: $\{1\}$, the second subset of the $(2k-1)$ power of prime numbers: $\{p^{2k-1}\}$, the third subset of the $2k$ power of prime numbers: $\{p^{2k}\}$, and the fourth subset of the composite numbers not belonging to any one of the aforementioned three subsets.

In the following discussions, a , b , and c are assumed to be three different arbitrary prime numbers.

The magnitude correlation of a , b , and c does not matter.

When $n = 1$,

$$(11-24) \quad n^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}(n) \Big|_{n=1} = 1.$$

This is a direct current component when y is considered as the time variable and applied in the analogy of an electric circuit's current waveform.

When n is the prime number a ,

$$(11-25) \quad a^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}(a) = \left(1 + a^{-\frac{1+i2y}{2}} \right) a^{\frac{1+i2y}{4}} = 2 \cosh\left(\frac{1+i2y}{4} \log(a)\right) \\ = 2 \cosh\left(\frac{\log(a)}{4}\right) \cos\left(\frac{y \log(a)}{2}\right) + i 2 \sinh\left(\frac{\log(a)}{4}\right) \sin\left(\frac{y \log(a)}{2}\right).$$

This corresponds to an alternate current waveform with a single spectrum.

When n is the cubic of the prime number a ,

$$(11-26) \quad (a^3)^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}(a^3) = \left(1 + a^{-\frac{1+i2y}{2}} + (a^2)^{-\frac{1+i2y}{2}} + (a^3)^{-\frac{1+i2y}{2}} \right) (a^3)^{\frac{1+i2y}{4}} \\ = 2 \cosh\left(\frac{1+i2y}{4} \log(a)\right) + 2 \cosh\left(\frac{3+i6y}{4} \log(a)\right).$$

When n is the $(2k-1)$ power of the prime number a ,

$$(11-27) \quad (a^{2k-1})^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}(a^{2k-1}) = \sum_{m=1}^k 2 \cosh\left(\frac{2m-1+i2(2m-1)y}{4} \log(a)\right) \quad (k \in \mathbb{N}).$$

This corresponds to an alternate current waveform with multiple spectra.

When n is the square of the prime number a ,

$$(11-28) \left(a^2\right)^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}\left(a^2\right) = \left(1+a^{-\frac{1+i2y}{2}}+(a^2)^{-\frac{1+i2y}{2}}\right)\left(a^2\right)^{\frac{1+i2y}{4}}$$

$$= 1+2 \cosh\left(\frac{1+i2y}{2} \log (a)\right).$$

When n is the fourth power of the prime number a ,

$$(11-29) \left(a^4\right)^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}\left(a^4\right) =$$

$$\left(1+a^{-\frac{1+i2y}{2}}+(a^2)^{-\frac{1+i2y}{2}}+(a^3)^{-\frac{1+i2y}{2}}+(a^4)^{-\frac{1+i2y}{2}}\right)\left(a^4\right)^{\frac{1+i2y}{4}}$$

$$= 1+2 \cosh\left(\frac{1+i2y}{2} \log (a)\right)+2 \operatorname{Cosh}\left((1+i2y) \log (a)\right).$$

When n is the $2k$ power of the prime number a ,

$$(11-30) \left(a^{2k}\right)^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}\left(a^{2k}\right) = 1+\sum_{m=1}^k 2 \cosh\left(\frac{m+i2my}{2} \log (a)\right) \quad (k \in \mathbb{N}).$$

This corresponds to an alternate current waveform with multiple spectra and superimposed direct current components. When n is the power of each prime number, the functional type can be determined additively.

When n is the composite number ab ,

$$(11-31) (a \cdot b)^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}(a \cdot b) = \left(1+a^{-\frac{1+i2y}{2}}+b^{-\frac{1+i2y}{2}}+(a \cdot b)^{-\frac{1+i2y}{2}}\right)(a \cdot b)^{\frac{1+i2y}{4}}$$

$$= 4 \cosh\left(\frac{1+i2y}{4} \log (a)\right) \cosh\left(\frac{1+i2y}{4} \log (b)\right).$$

When n is the composite number abc ,

$$(11-32) (a \cdot b \cdot c)^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}(a \cdot b \cdot c) = \left(1+a^{-\frac{1+i2y}{2}}+b^{-\frac{1+i2y}{2}}+c^{-\frac{1+i2y}{2}}\right.$$

$$\left.+ (a \cdot b)^{-\frac{1+i2y}{2}}+(b \cdot c)^{-\frac{1+i2y}{2}}+(c \cdot a)^{-\frac{1+i2y}{2}}+(a \cdot b \cdot c)^{-\frac{1+i2y}{2}}\right)(a \cdot b \cdot c)^{\frac{1+i2y}{4}}$$

$$= 8 \cosh\left(\frac{1+i2y}{4} \log (a)\right) \cosh\left(\frac{1+i2y}{4} \log (b)\right) \cosh\left(\frac{1+i2y}{4} \log (c)\right).$$

When n is the composite number ab^2 ,

$$(11-33) (a \cdot b^2)^{\frac{1+i2y}{4}} \sigma_{-\frac{1+i2y}{2}}\left(a \cdot b^2\right) =$$

$$\left(1+a^{-\frac{1+i2y}{2}}+b^{-\frac{1+i2y}{2}}+(b^2)^{-\frac{1+i2y}{2}}+(a \cdot b)^{-\frac{1+i2y}{2}}+(a \cdot b^2)^{-\frac{1+i2y}{2}}\right)\left(a \cdot b^2\right)^{\frac{1+i2y}{4}}$$

$$= 2 \cosh\left(\frac{1+i2y}{4} \log (a)\right)\left(1+2 \cosh\left(\frac{1+i2y}{2} \log (b)\right)\right).$$

When n is a composite number other than the powers of each prime number, the functional type can be mainly determined using multiplicative analysis.

The number-theoretic function $\sigma_{-\theta}(n) n^{\theta/2}$ is described by the complex hyperbolic cosine in the explicit formula for the zeta function on the critical line.

12. Some representations for the zeta function for any odd number except 1

I presented some representations for the zeta function for any odd number except 1 and relational mentions in the first half of this section. In the second half, I will show another two types of general representations for the zeta function for any odd number of either 3 or 7, or more.

12-1. Some representations for the zeta function for any odd number except 1 and relational mentions.

The explicit formula for the zeta function is shown again.

$$(12-1) \quad \zeta(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

For rewriting the modified Bessel function of the second kind into the quadrable integral representation, the following equation is prepared in subsection 19-5:

$$(12-2) \quad \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \alpha^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\alpha) \quad (\alpha > 0, \theta \in \mathbb{C}).$$

Here, putting as

$$\alpha = \pi p \quad (p \in \mathbb{N})$$

the following two relations are obtained:

$$(12-3) \quad (\pi p)^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi p) = \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \quad (\theta \in \mathbb{C}, p \in \mathbb{N}).$$

$$(12-4) \quad (\pi p)^{\frac{\theta-2}{2}} K_{\frac{\theta-2}{2}}(2\pi p) = (\pi p)^{\frac{\theta-2}{2}} K_{\frac{2-\theta}{2}}(2\pi p) \\ = \int_0^{\infty} x^{\theta-3} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \quad (\theta \in \mathbb{C}, p \in \mathbb{N}).$$

To apply those transformation formulae, the right side of equation (12-1) is modified as follows:

$$(12-5) \quad \zeta(\theta) = \frac{1}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(2(\theta(1-\theta) + (\theta-7)(2\pi p)^2) (\pi p)^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((1-\theta) + 2(2\pi p)^2)(2\pi p)^2 (\pi p)^{\frac{\theta-2}{2}} K_{\frac{\theta-2}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

Therefore,

$$(12-6) \quad \zeta(\theta) = \\ \frac{1}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(2(\theta(1-\theta) + (\theta-7)(2\pi p)^2) \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ \left. + ((1-\theta) + 2(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} x^{\theta-3} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

The equation (12-6) can be used to calculate each zeta function representation for any odd number of 3 or more. Conversely, for calculating each zeta function representation for any odd number of -1 or less, for

equation (12-1), $-\theta$ is substituted for θ to obtain the following equation:

$$(12-7) \quad \zeta(-\theta) = \frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{\theta}(p) p^{-\frac{\theta}{2}} \left(2(\theta(\theta+1) + (7+\theta)(2\pi p)^2) K_{-\frac{\theta}{2}}(2\pi p) - 4((\theta+1) + 2(2\pi p)^2)(\pi p) K_{\frac{2+\theta}{2}}(2\pi p)\right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

The relations of equations (3-30) and (5-39) are applied to the equation (12-7).

$$(12-8) \quad \zeta(-\theta) = \frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(2(\theta(\theta+1) + (7+\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) - 4((\theta+1) + 2(2\pi p)^2)(\pi p) K_{\frac{2+\theta}{2}}(2\pi p)\right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

The recurrence formula for the modified Bessel function of the second kind is applied.

$$(12-9) \quad \begin{aligned} \zeta(-\theta) &= \frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(2(\theta(\theta+1) + (7+\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) - 4((\theta+1) + 2(2\pi p)^2) \left(\frac{\theta}{2} K_{\frac{\theta}{2}}(2\pi p) + (\pi p) K_{\frac{\theta-2}{2}}(2\pi p)\right)\right) \\ &= \frac{\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(2(7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) - 2((\theta+1) + 2(2\pi p)^2)(2\pi p) K_{\frac{\theta-2}{2}}(2\pi p)\right) \\ &= \frac{\pi^{-\theta}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(2(7-\theta)(2\pi p)^2 (\pi p)^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi p) - ((\theta+1) + 2(2\pi p)^2)(2\pi p)^2 (\pi p)^{\frac{\theta-2}{2}} K_{\frac{\theta-2}{2}}(2\pi p)\right) \quad (\theta \in \mathbb{C} \setminus \{-1\}). \end{aligned}$$

Therefore

$$(12-10) \quad \zeta(-\theta) = \frac{\pi^{-\theta}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left(2(7-\theta)(2\pi p)^2 \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx - ((\theta+1) + 2(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} x^{\theta-3} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx\right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

Thus, the equation (12-10) can be used to calculate each zeta function representation for any odd number of -1 or less.

Some calculation examples and the results will be shown soon.

Moreover, because the values are well known, I will post exact value for each of the zeta function for any odd number of -1 or less together.

For equation (12-10), 1 is substituted for θ ,

$$\begin{aligned}
 (12-11) \quad \zeta(-1) &= \frac{\pi^{-1}}{2\Gamma\left(1-\frac{1}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(12(2\pi p)^2 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (2+2(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} x^{-2} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= \frac{1}{2\pi\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(12(2\pi p)^2 \frac{\sqrt{\pi}}{2} e^{-2\pi p} - (2+2(2\pi p)^2)(2\pi p)^2 \frac{\sqrt{\pi}}{2\pi p} e^{-2\pi p} \right).
 \end{aligned}$$

Therefore,

$$(12-12) \quad \zeta(-1) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-1}(p) ((2\pi p)^2 - 3(2\pi p) + 1)(2\pi p) e^{-2\pi p} = -\frac{1}{12}.$$

For equation (12-10), 3 is substituted for θ ,

$$\begin{aligned}
 (12-13) \quad \zeta(-3) &= \frac{\pi^{-3}}{4\Gamma\left(1-\frac{3}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(8(2\pi p)^2 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (4+2(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= -\frac{1}{8\pi^3\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(8(2\pi p)^2 \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} - (4+2(2\pi p)^2)(2\pi p)^2 \frac{\sqrt{\pi}}{2} e^{-2\pi p} \right).
 \end{aligned}$$

Therefore,

$$(12-14) \quad \zeta(-3) = \frac{1}{8\pi^3} \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p) - 2)(2\pi p)^3 e^{-2\pi p} = \frac{1}{120}.$$

For equation (12-10), 5 is substituted for θ ,

$$\begin{aligned}
 (12-15) \quad \zeta(-5) &= \frac{\pi^{-5}}{6\Gamma\left(1-\frac{5}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(4(2\pi p)^2 \int_0^{\infty} x^4 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (6+2(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= \frac{1}{8\pi^5\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(4(2\pi p)^2 \frac{(2\pi p)^2+3(2\pi p)+3}{8} \sqrt{\pi} e^{-2\pi p} \right. \\
 &\quad \left. - (6+2(2\pi p)^2)(2\pi p)^2 \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} \right).
 \end{aligned}$$

Therefore,

$$(12-16) \quad \zeta(-5) = -\frac{1}{16\pi^5} \sum_{p=1}^{\infty} \sigma_{-5}(p) (2\pi p)^5 e^{-2\pi p} = -\frac{1}{252}.$$

For equation (12-10), 7 and 9 are substituted for θ , only the results are shown.

$$\begin{aligned}
 (12-17) \quad \zeta(-7) &= \\
 &= \frac{15}{256\pi^7} \sum_{p=1}^{\infty} \sigma_{-7}(p) ((2\pi p)^4 + 3(2\pi p)^3 + 7(2\pi p)^2 + 12(2\pi p) + 12)(2\pi p)^2 e^{-2\pi p} = \frac{1}{240}.
 \end{aligned}$$

$$(12-18) \quad \zeta(-9) = -\frac{21}{256\pi^9} \sum_{p=1}^{\infty} \sigma_{-9}(p) \left((2\pi p)^5 + 7(2\pi p)^4 + 30(2\pi p)^3 + 90(2\pi p)^2 + 180(2\pi p) + 180 \right) (2\pi p)^2 e^{-2\pi p} = -\frac{1}{132}.$$

For equation (12-6), 3 is substituted for θ ,

$$(12-19) \quad \zeta(3) = \frac{1}{2\Gamma\left(1+\frac{3}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(2(-6-4(2\pi p)^2) \int_0^{\infty} x^2 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx + (-2+2(2\pi p)^2) (2\pi p)^2 \int_0^{\infty} x^0 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \right) \\ = \frac{2}{3\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(2(-6-4(2\pi p)^2) \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} + (-2+2(2\pi p)^2) (2\pi p)^2 \frac{\sqrt{\pi}}{2} e^{-2\pi p} \right).$$

Therefore,

$$(12-20) \quad \zeta(3) = \frac{2}{3} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^4 - 2(2\pi p)^3 - 3(2\pi p)^2 - 3(2\pi p) - 3 \right) e^{-2\pi p}.$$

The right side of equation (12-20) can be expanded as follows:

$$(12-21) \quad \zeta(3) = \frac{2}{3} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^3 - 2(2\pi p)^2 - 3(2\pi p) - 3 \right) (2\pi p) e^{-2\pi p} - \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} + \zeta(3).$$

Thus, I can obtain the representation for the series

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3},$$

and also its exact value is known.

$$(12-22) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} = \frac{2}{3} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^3 - 2(2\pi p)^2 - 3(2\pi p) - 3 \right) (2\pi p) e^{-2\pi p} = \frac{7\pi^3}{180}.$$

Further, the right side of equation (12-20) can be also expanded as follows:

$$(12-23) \quad \zeta(3) = \frac{2}{3} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p) - 2 \right) (2\pi p)^3 e^{-2\pi p} - 2 \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^2 + (2\pi p) + 1 \right) e^{-2\pi p}.$$

Because the first term on the right side of equation (12-23) has already been determined, I obtain the representation that gives the transcendental number a fixed value.

$$(12-24) \quad \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p) - 2 \right) (2\pi p)^3 e^{-2\pi p} = \frac{\pi^3}{15}.$$

For equation (12-6), 5 is substituted for θ ,

$$\begin{aligned}
 (12-25) \quad \zeta(5) &= \frac{1}{4\Gamma\left(1+\frac{5}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(2(-20-2(2\pi p)^2) \int_0^{\infty} x^4 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. + (-4+2(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} x^2 e^{-\left(x^2+\frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= \frac{2}{15\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(2(-20-2(2\pi p)^2) \frac{(2\pi p)^2+3(2\pi p)+3}{8} \sqrt{\pi} e^{-2\pi p} \right. \\
 &\quad \left. + (-4+2(2\pi p)^2)(2\pi p)^2 \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} \right).
 \end{aligned}$$

Therefore,

$$(12-26) \quad \zeta(5) = \frac{1}{15} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left((2\pi p)^5 - 5(2\pi p)^3 - 15(2\pi p)^2 - 30(2\pi p) - 30 \right) e^{-2\pi p}.$$

The right side of equation (12-26) can be expanded as follows:

$$\begin{aligned}
 (12-27) \quad \zeta(5) &= \frac{1}{15} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left((2\pi p)^4 - 5(2\pi p)^2 - 15(2\pi p) - 30 \right) (2\pi p) e^{-2\pi p} \\
 &- \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^5} + \zeta(5).
 \end{aligned}$$

Thus, I can obtain the representation for the series

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^5},$$

but its exact value is unknown.

$$(12-28) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^5} = \frac{1}{15} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left((2\pi p)^4 - 5(2\pi p)^2 - 15(2\pi p) - 30 \right) (2\pi p) e^{-2\pi p}.$$

Furthermore, the right side of equation (12-26) can also be expanded as follows:

$$\begin{aligned}
 (12-29) \quad \zeta(5) &= \\
 &\frac{1}{15} \sum_{p=1}^{\infty} \sigma_{-5}(p) (2\pi p)^5 e^{-2\pi p} - \frac{1}{3} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left((2\pi p)^3 + 3(2\pi p)^2 + 6(2\pi p) + 6 \right) e^{-2\pi p}.
 \end{aligned}$$

Because the first term on the right side of equation (12-29) has already been determined, I obtain the representation that gives the transcendental number a fixed value.

$$(12-30) \quad \sum_{p=1}^{\infty} \sigma_{-5}(p) (2\pi p)^5 e^{-2\pi p} = \frac{4\pi^5}{63}.$$

For equation (12-6), 7 is substituted for θ , and the principal results are shown.

$$\begin{aligned}
 (12-31) \quad \zeta(7) &= \\
 &\frac{2}{315} \sum_{p=1}^{\infty} \sigma_{-7}(p) \left((2\pi p)^6 + 3(2\pi p)^5 - 30(2\pi p)^3 - 135(2\pi p)^2 - 315(2\pi p) - 315 \right) e^{-2\pi p}.
 \end{aligned}$$

$$\begin{aligned}
 (12-32) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} &= \frac{2}{315} \sum_{p=1}^{\infty} \sigma_{-7}(p) \left((2\pi p)^5 + 3(2\pi p)^4 \right. \\
 &\quad \left. - 30(2\pi p)^2 - 135(2\pi p) - 315 \right) (2\pi p) e^{-2\pi p} = \frac{19\pi^7}{56700}.
 \end{aligned}$$

$$(12-33) \quad \sum_{p=1}^{\infty} \sigma_{-7}(p) \left((2\pi p)^4 + 3(2\pi p)^3 + 7(2\pi p)^2 + 12(2\pi p) + 12 \right) (2\pi p)^2 e^{-2\pi p} = \frac{16\pi^7}{225}.$$

For equation (12-6), 9 is substituted for θ , and the principal results are shown.

$$(12-34) \quad \zeta(9) = \frac{1}{1890} \sum_{p=1}^{\infty} \sigma_{-9}(p) \left((2\pi p)^7 + 7(2\pi p)^6 + 21(2\pi p)^5 \right. \\ \left. - 315(2\pi p)^3 - 1575(2\pi p)^2 - 3780(2\pi p) - 3780 \right) e^{-2\pi p}.$$

$$(12-35) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^9} = \frac{1}{1890} \sum_{p=1}^{\infty} \sigma_{-9}(p) \left((2\pi p)^6 + 7(2\pi p)^5 + 21(2\pi p)^4 \right. \\ \left. - 315(2\pi p)^2 - 1575(2\pi p) - 3780 \right) (2\pi p) e^{-2\pi p}.$$

$$(12-36) \quad \sum_{p=1}^{\infty} \sigma_{-9}(p) \left((2\pi p)^5 + 7(2\pi p)^4 + 30(2\pi p)^3 \right. \\ \left. + 90(2\pi p)^2 + 180(2\pi p) + 180 \right) (2\pi p)^2 e^{-2\pi p} = \frac{64\pi^9}{693}.$$

For equation (12-6), 11 is substituted for θ , and the principal results are shown.

$$(12-37) \quad \zeta(11) = \frac{2}{51975} \sum_{p=1}^{\infty} \sigma_{-11}(p) \left((2\pi p)^8 + 12(2\pi p)^7 + 70(2\pi p)^6 + 210(2\pi p)^5 \right. \\ \left. - 105(2\pi p)^4 - 4410(2\pi p)^3 - 21735(2\pi p)^2 - 51975(2\pi p) - 51975 \right) e^{-2\pi p}.$$

$$(12-38) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{11}} = \frac{2}{51975} \sum_{p=1}^{\infty} \sigma_{-11}(p) \left((2\pi p)^7 + 12(2\pi p)^6 + 70(2\pi p)^5 + 210(2\pi p)^4 \right. \\ \left. - 105(2\pi p)^3 - 4410(2\pi p)^2 - 21735(2\pi p) - 51975 \right) (2\pi p) e^{-2\pi p} = \frac{1453\pi^{11}}{425675250}.$$

$$(12-39) \quad \sum_{p=1}^{\infty} \sigma_{-11}(p) \left((2\pi p)^6 + 12(2\pi p)^5 + 81(2\pi p)^4 + 375(2\pi p)^3 \right. \\ \left. + 1215(2\pi p)^2 + 2520(2\pi p) + 2520 \right) (2\pi p)^2 e^{-2\pi p} = \frac{176896\pi^{11}}{1289925}.$$

For equation (12-6), 13 is substituted for θ , and the principal results are shown.

$$(12-40) \quad \zeta(13) = \frac{1}{405405} \sum_{p=1}^{\infty} \sigma_{-13}(p) \left((2\pi p)^9 + 18(2\pi p)^8 + 162(2\pi p)^7 + 882(2\pi p)^6 \right. \\ \left. + 2457(2\pi p)^5 - 3780(2\pi p)^4 - 72765(2\pi p)^3 - 343035(2\pi p)^2 - 810810(2\pi p) - 810810 \right) e^{-2\pi p}.$$

$$(12-41) \quad \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{13}} = \frac{1}{405405} \sum_{p=1}^{\infty} \sigma_{-13}(p) \left((2\pi p)^8 + 18(2\pi p)^7 + 162(2\pi p)^6 + 882(2\pi p)^5 \right. \\ \left. + 2457(2\pi p)^4 - 3780(2\pi p)^3 - 72765(2\pi p)^2 - 343035(2\pi p) - 810810 \right) (2\pi p) e^{-2\pi p}.$$

$$(12-42) \quad \sum_{p=1}^{\infty} \sigma_{-13}(p) \left((2\pi p)^7 + 18(2\pi p)^6 + 175(2\pi p)^5 + 1155(2\pi p)^4 \right. \\ \left. + 5460(2\pi p)^3 + 18060(2\pi p)^2 + 37800(2\pi p) + 37800 \right) (2\pi p)^2 e^{-2\pi p} = \frac{1024\pi^{13}}{4455}.$$

For equation (12-6), 15 is substituted for θ , and the principal results are shown.

$$(12-43) \quad \zeta(15) = \frac{2}{14189175} \sum_{p=1}^{\infty} \sigma_{-15}(p) \left((2\pi p)^{10} + 25(2\pi p)^9 + 315(2\pi p)^8 \right. \\ \left. + 2520(2\pi p)^7 + 12915(2\pi p)^6 + 31185(2\pi p)^5 - 103950(2\pi p)^4 - 1351350(2\pi p)^3 \right. \\ \left. - 6081075(2\pi p)^2 - 14189175(2\pi p) - 14189175 \right) e^{-2\pi p}.$$

$$(12-44) \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{15}} = \frac{2}{14\,189\,175} \sum_{p=1}^{\infty} \sigma_{-15}(p) \left((2\pi p)^9 + 25(2\pi p)^8 + 315(2\pi p)^7 \right. \\ \left. + 2520(2\pi p)^6 + 12\,915(2\pi p)^5 + 31\,185(2\pi p)^4 - 103\,950(2\pi p)^3 - 1\,351\,350(2\pi p)^2 \right. \\ \left. - 6\,081\,075(2\pi p) - 14\,189\,175 \right) (2\pi p) e^{-2\pi p} = \frac{13\,687\pi^{15}}{390\,769\,879\,500}.$$

$$(12-45) \sum_{p=1}^{\infty} \sigma_{-15}(p) \left((2\pi p)^8 + 25(2\pi p)^7 + 330(2\pi p)^6 + 2940(2\pi p)^5 + 19\,005(2\pi p)^4 \right. \\ \left. + 89\,775(2\pi p)^3 + 297\,675(2\pi p)^2 + 623\,700(2\pi p) + 623\,700 \right) (2\pi p)^2 e^{-2\pi p} = \frac{14\,815\,232\pi^{15}}{34\,459\,425}.$$

For equation (12-6), 17 is substituted for θ , and the principal results are shown.

$$(12-46) \zeta(17) = \frac{1}{137\,837\,700} \sum_{p=1}^{\infty} \sigma_{-17}(p) \left((2\pi p)^{11} + 33(2\pi p)^{10} + 550(2\pi p)^9 \right. \\ \left. + 5940(2\pi p)^8 + 44\,055(2\pi p)^7 + 211\,365(2\pi p)^6 + 405\,405(2\pi p)^5 - 2\,702\,700(2\pi p)^4 \right. \\ \left. - 27\,702\,675(2\pi p)^3 - 119\,594\,475(2\pi p)^2 - 275\,675\,400(2\pi p) - 275\,675\,400 \right) e^{-2\pi p}.$$

$$(12-47) \sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^{17}} = \frac{1}{137\,837\,700} \sum_{p=1}^{\infty} \sigma_{-17}(p) \left((2\pi p)^{10} + 33(2\pi p)^9 + 550(2\pi p)^8 \right. \\ \left. + 5940(2\pi p)^7 + 44\,055(2\pi p)^6 + 211\,365(2\pi p)^5 + 405\,405(2\pi p)^4 - 2\,702\,700(2\pi p)^3 \right. \\ \left. - 27\,702\,675(2\pi p)^2 - 119\,594\,475(2\pi p) - 275\,675\,400 \right) (2\pi p) e^{-2\pi p}.$$

$$(12-48) \sum_{p=1}^{\infty} \sigma_{-17}(p) \left((2\pi p)^9 + 33(2\pi p)^8 + 567(2\pi p)^7 + 6552(2\pi p)^6 + 55\,377(2\pi p)^5 \right. \\ \left. + 350\,595(2\pi p)^4 + 1\,642\,410(2\pi p)^3 + 5\,426\,190(2\pi p)^2 \right. \\ \left. + 11\,351\,340(2\pi p) + 11\,351\,340 \right) (2\pi p)^2 e^{-2\pi p} = \frac{718\,716\,928\pi^{17}}{808\,782\,975}.$$

12-2. Another two types of general representations for the zeta function for any odd number of either 3 or 7, or more

For equation (12-6), any odd number of 3 or more is substituted for θ .

$$(12-49) \zeta(2k+1) = \frac{1}{2k\Gamma\left(1 + \frac{2k+1}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \\ \left(2\left((2k+1)(-2k) + (2k-6)(2\pi p)^2\right) \int_0^{\infty} x^{2k} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ \left. + (-2k+2)(2\pi p)^2 (2\pi p)^2 \int_0^{\infty} x^{2(k-1)} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\ = \frac{2^{k+1}}{k(2k+1)!!\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left((-2k(2k+1) + 2(k-3)(2\pi p)^2) \int_0^{\infty} x^{2k} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ \left. + (-k + (2\pi p)^2) (2\pi p)^2 \int_0^{\infty} x^{2(k-1)} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \quad (k \in \mathbb{N}).$$

The general representation for the two integrals of the result of equation (12-49) is given by the description in

the subsection 19-4 i.e.,

$$(12-50) \int_0^\infty x^{2k} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^\mu \sqrt{\pi} e^{-2\pi p} \quad (p, k \in \mathbb{N}).$$

Where

$$(12-51) a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \quad (k \in \mathbb{N}, \mu = 0, 1, \dots, k).$$

And the coefficients deserving special mention are as follows:

$$(12-52) a_{k,k} = 1 \quad (k \in \mathbb{N}).$$

$$(12-53) a_{k,1} = a_{k,0} = (2k-1)!! \quad (k \in \mathbb{N}).$$

For equation (12-51), 0 is substituted directly for k and μ .

$$(12-54) a_{0,0} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \Big|_{k=\mu=0} = 1.$$

This result indicates that equation (12-50) holds even if $k = 0$. The above general representation for the two integrals is introduced into the result of equation (12-49).

$$\begin{aligned} (12-55) \zeta(2k+1) &= \frac{2^{k+1}}{k(2k+1)!! \sqrt{\pi}} \sum_{p=1}^\infty \sigma_{-(2k+1)}(p) \\ &\left((-2k(2k+1) + 2(k-3)(2\pi p)^2) \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^\mu \sqrt{\pi} e^{-2\pi p} \right. \\ &\quad \left. + (-k + (2\pi p)^2) (2\pi p)^2 \frac{1}{2^k} \sum_{\mu=0}^{k-1} a_{k-1,\mu} (2\pi p)^\mu \sqrt{\pi} e^{-2\pi p} \right) \\ &= \frac{2}{k(2k+1)!!} \sum_{p=1}^\infty \sigma_{-(2k+1)}(p) \left((-k(2k+1) + (k-3)(2\pi p)^2) \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^\mu \right. \\ &\quad \left. + (-k + (2\pi p)^2) (2\pi p)^2 \sum_{\mu=0}^{k-1} a_{k-1,\mu} (2\pi p)^\mu \right) e^{-2\pi p} \\ &= \frac{2}{k(2k+1)!!} \sum_{p=1}^\infty \sigma_{-(2k+1)}(p) \left(a_{k-1,k-1} (2\pi p)^{k+3} + (a_{k-1,k-2} + (k-3)a_{k,k}) (2\pi p)^{k+2} \right. \\ &\quad \left. + \sum_{\mu=4}^{k+1} a_{k-1,\mu-4} (2\pi p)^\mu + (k-3) \sum_{\mu=2}^{k+1} a_{k,\mu-2} (2\pi p)^\mu - k \sum_{\mu=2}^{k+1} a_{k-1,\mu-2} (2\pi p)^\mu \right. \\ &\quad \left. - k(2k+1) \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^\mu \right) e^{-2\pi p} \quad (k \in \mathbb{N}). \end{aligned}$$

From the equations (12-52) and (12-54), the coefficient of $(2\pi p)^{k+3}$ is immediately determined as follows:

$$(12-56) a_{k-1,k-1} = 1 \quad (k \in \mathbb{N}).$$

The coefficient of $(2\pi p)^{k+2}$ is determined under the condition of $k > 1$ as follows:

$$\begin{aligned} (12-57) a_{k-1,k-2} + (k-3)a_{k,k} &= \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \Big|_{k=k-1, \mu=k-2} + (k-3) \cdot 1 \\ &= \frac{k!}{2^1 (k-2)!!} + (k-3) = \frac{(k-2)(k+3)}{2} \quad (k \in \mathbb{N} \setminus \{1\}). \end{aligned}$$

For the result of equation (12-57), 1 is substituted directly for k.

$$(12-58) \quad \frac{(k-2)(k+3)}{2} \Big|_{k=1} = -2.$$

This result is the same as the coefficient of $(2\pi p)^3$ of the representation for $\zeta(3)$. Thus, the right side of equation (12-57) holds even for the case when $k = 1$.

The above results are introduced into the result of equation (12-55).

$$(12-59) \quad \zeta(2k+1) = \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left((2\pi p)^{k+3} + \frac{(k-2)(k+3)}{2} (2\pi p)^{k+2} \right. \\ \left. + \sum_{\mu=4}^{k+1} a_{k-1,\mu-4} (2\pi p)^{\mu} + (k-3) \sum_{\mu=2}^{k+1} a_{k,\mu-2} (2\pi p)^{\mu} \right. \\ \left. - k \sum_{\mu=2}^{k+1} a_{k-1,\mu-2} (2\pi p)^{\mu} - k(2k+1) \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^{\mu} \right) \quad (k \in \mathbb{N}).$$

Here, new coefficients $c_{k,\mu}$ are used for the general representation for the zeta function for any odd number of 3 or more.

The coefficients $c_{k,\mu}$ are determined by the following equation of definition:

$$(12-60) \quad \zeta(2k+1) := \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left((2\pi p)^{k+3} + \frac{(k-2)(k+3)}{2} (2\pi p)^{k+2} \right. \\ \left. + \sum_{\mu=0}^{k+1} c_{k,\mu} (2\pi p)^{\mu} \right) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

The coefficients $c_{k,k+3}$ and $c_{k,k+2}$ are immediately determined as follows:

$$(12-61) \quad c_{k,k+3} = 1 \quad (k \in \mathbb{N}).$$

$$(12-62) \quad c_{k,k+2} = \frac{(k-2)(k+3)}{2} \quad (k \in \mathbb{N}).$$

And the coefficients $c_{k,\mu}$ are determined with the condition as follows:

$$(12-63) \quad c_{k,\mu} =$$

$$a_{k-1,\mu-4} + (k-3) a_{k,\mu-2} - k a_{k-1,\mu-2} - k(2k+1) a_{k,\mu} \quad (k \in \mathbb{N} \setminus \{1, 2\}, \mu = 4, 5, \dots, k+1).$$

The relation of equation (12-51) is introduced into the above equation.

$$(12-64) \quad c_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \Big|_{k=k-1, \mu=\mu-4} + (k-3) \cdot \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \Big|_{k=k, \mu=\mu-2} \\ - k \cdot \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \Big|_{k=k-1, \mu=\mu-2} - k(2k+1) \cdot \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \\ (k \in \mathbb{N} \setminus \{1, 2\}, \mu = 4, 5, \dots, k+1).$$

I demonstrate the calculation steps..

$$(12-65) \quad c_{k,\mu} = \frac{(2k+2-\mu)!}{2^{k+3-\mu} (\mu-4)! (k+3-\mu)!} + (k-3) \frac{(2k+2-\mu)!}{2^{k+2-\mu} (\mu-2)! (k+2-\mu)!} \\ - k \frac{(2k-\mu)!}{2^{k+1-\mu} (\mu-2)! (k+1-\mu)!} - k(2k+1) \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \\ = \left(\frac{(2k+2-\mu)(2k+1-\mu)}{2^{k+3-\mu} (\mu-4)! (k+3-\mu)!} + \frac{(k-3)(2k+2-\mu)(2k+1-\mu)}{2^{k+2-\mu} (\mu-2)! (k+2-\mu)!} \right. \\ \left. - \frac{k}{2^{k+1-\mu} (\mu-2)! (k+1-\mu)!} - \frac{k(2k+1)}{2^{k-\mu} \mu! (k-\mu)!} \right) (2k-\mu)!$$

$$= \left(\frac{\mu(\mu-1)(\mu-2)(\mu-3)(2k+2-\mu)(2k+1-\mu)}{2^{k+3-\mu}\mu!(k+3-\mu)!} + \frac{2\mu(\mu-1)(k-3)(k+3-\mu)(2k+2-\mu)(2k+1-\mu)}{2^{k+3-\mu}\mu!(k+3-\mu)!} \right. \\ \left. - \frac{4\mu(\mu-1)k(k+3-\mu)(k+2-\mu)}{2^{k+3-\mu}\mu!(k+3-\mu)!} - \frac{8k(2k+1)(k+3-\mu)(k+2-\mu)(k+1-\mu)}{2^{k+3-\mu}\mu!(k+3-\mu)!} \right) (2k-\mu)!$$

The result of the above calculation is shown.

$$= \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!(k+3-\mu)!} (\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13) \\ + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) \\ + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24))) \quad (k \in \mathbb{N} \setminus \{1, 2\}, \mu = 4, 5, \dots, k+1).$$

For equation (12-65), 0 is substituted for μ .

$$(12-66) \mathbf{c}_{k,0} = -\frac{2k(8k^4 + 52k^3 + 112k^2 + 92k + 24)}{2^{k+3}(k+3)!} (2k)! \\ = -\frac{8k(2k+1)(k+1)(k+2)(k+3)}{2^{k+3}(k+3)!} (2k)! = -\frac{k}{2^k k!} (2k+1)! = -k(2k+1)!! \quad (k \in \mathbb{N} \setminus \{1, 2\}).$$

For equation (12-65), 1 is substituted for μ .

$$(12-67) \mathbf{c}_{k,1} = -\frac{2k(8k^4 + 28k^3 + 28k^2 + 8k)}{2^{k+2}(k+2)!} (2k-1)! \\ = -\frac{8k^2(2k+1)(k+1)(k+2)}{2^{k+2}(k+2)!} (2k-1)! = -\frac{k}{2^k k!} (2k+1)! = -k(2k+1)!! \quad (k \in \mathbb{N} \setminus \{1, 2\}).$$

For equation (12-65), 2 is substituted for μ .

$$(12-68) \mathbf{c}_{k,2} = -\frac{2k(8k^4 - 4k^3 + 16k^2 + 16k - 12)}{2^{k+1} \cdot 2!(k+1)!} (2k-2)! = -\frac{8k(2k-1)(k+1)(k^2 - k + 3)}{2^{k+1} \cdot 2!(k+1)!} (2k-2)! \\ = -\frac{k^2 - k + 3}{2^{k-1}(k-1)!} (2k-1)! = -(k^2 - k + 3)(2k-1)!! \quad (k \in \mathbb{N} \setminus \{1, 2\}).$$

For equation (12-65), 3 is substituted for μ .

$$(12-69) \mathbf{c}_{k,3} = -\frac{2k(8k^4 - 44k^3 + 124k^2 - 124k + 36)}{2^k \cdot 3! k!} (2k-3)! \\ = -\frac{8k(2k-1)(k-1)(k^2 - 4k + 9)}{2^k \cdot 3! k!} (2k-3)! = -\frac{k^2 - 4k + 9}{3 \cdot 2^{k-1}(k-1)!} (2k-1)! \\ = -\frac{k^2 - 4k + 9}{3} (2k-1)!! \quad (k \in \mathbb{N} \setminus \{1, 2\}).$$

For equation (12-65), 1 is substituted for k , and 0, 1, and 2 are substituted for μ .

$$(12-70) \mathbf{c}_{1,\mu} = \frac{(2k-\mu)!}{2^{k+3-\mu}\mu!(k+3-\mu)!} (\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13) \\ + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) \\ + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24))) \Big|_{k=1} \quad (\mu = 0, 1, 2).$$

The results of the above calculation are shown.

$$(12-71) \{\mathbf{c}_{1,2}, \mathbf{c}_{1,1}, \mathbf{c}_{1,0}\} = \{-3, -3, -3\}.$$

Because the coefficients $\mathbf{c}_{1,4}$ and $\mathbf{c}_{1,3}$ have already determined,

$$(12-72) \{\mathbf{c}_{1,4}, \mathbf{c}_{1,3}, \mathbf{c}_{1,2}, \mathbf{c}_{1,1}, \mathbf{c}_{1,0}\} = \{1, -2, -3, -3, -3\}.$$

This result is the same to the coefficient elements of the representation for $\zeta(3)$.

For equation (12-65), 2 is substituted for k , and 0,1,2,3, and 4 are substituted for μ .

$$(12-73) \quad c_{2,\mu} = \frac{(2k-\mu)!}{2^{k+3-\mu} \mu! (k+3-\mu)!} \left(\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13)) \right. \\ \left. + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) \right. \\ \left. + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24) \right) \Big|_{k=2} \quad (\mu = 0, 1, 2, 3, 4).$$

The results of the above calculation are shown.

$$(12-74) \quad \{c_{2,4}, c_{2,3}, c_{2,2}, c_{2,1}, c_{2,0}\} = \{0, -5, -15, -30, -30\}.$$

Because the coefficient $c_{2,5}$ has already determined,

$$(12-75) \quad \{c_{2,5}, c_{2,4}, c_{2,3}, c_{2,2}, c_{2,1}, c_{2,0}\} = \{1, 0, -5, -15, -30, -30\}.$$

This result is the same to the coefficient elements of the representation for $\zeta(5)$.

The above results are combined with the results of equations (12-66), (12-67), (12-68), and (12-69). Therefore,

$$(12-76) \quad c_{k,1} = c_{k,0} = -k(2k+1)!! \quad (k \in \mathbb{N}).$$

$$(12-77) \quad c_{k,2} = -(k^2 - k + 3)(2k-1)!! \quad (k \in \mathbb{N}).$$

$$(12-78) \quad c_{k,3} = -\frac{k^2 - 4k + 9}{3}(2k-1)!! \quad (k \in \mathbb{N}).$$

Combining the above results, I arrive at one of the conclusions in this subsection.

The following is the second general representation for the zeta function for any odd number of 3 or more:

$$(12-79) \quad \zeta(2k+1) = \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left((2\pi p)^{k+3} + \frac{(k-2)(k+3)}{2} (2\pi p)^{k+2} \right. \\ \left. + \sum_{\mu=0}^{k+1} c_{k,\mu} (2\pi p)^{\mu} \right) e^{-2\pi p} \quad (k \in \mathbb{N}).$$

Where

$$(12-80) \quad c_{k,\mu} = \frac{(2k-\mu)!}{2^{k+3-\mu} \mu! (k+3-\mu)!} \left(\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13)) \right. \\ \left. + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) \right. \\ \left. + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24) \right) \quad (k \in \mathbb{N}, \mu = 0, 1, \dots, k+1).$$

Hereafter, I consider the third general representation differently.

For equation (12-80), $k+3$ is substituted directly for μ under the condition of $k > 2$.

$$(12-81) \quad c_{k,k+3} = \frac{(2k-\mu)!}{2^{k+3-\mu} \mu! (k+3-\mu)!} \left(\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13)) \right. \\ \left. + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) \right. \\ \left. + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24) \right) \Big|_{k=k+3} \quad (k \in \mathbb{N} \setminus \{1, 2\}).$$

I show the final step of the calculation.

$$(12-82) \quad c_{k,k+3} = \frac{(k+3)(k+2)(k+1)k(k-1)(k-2)}{(k+3)!} (k-3)! = 1 \quad (k \in \mathbb{N} \setminus \{1, 2\}).$$

For equation (12-80), $k+2$ is substituted directly for μ under the condition of $k > 1$.

$$(12-83) \ C_{k,k+2} = \frac{(2k-\mu)!}{2^{k+3-\mu} \mu! (k+3-\mu)!} (\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13) + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24))) \Big|_{k=k+2} \quad (k \in \mathbb{N} \setminus \{1\}).$$

I show the final step of the calculation.

$$(12-84) \ C_{k,k+2} = \frac{(k+3)(k+2)(k+1)k(k-1)(k-2)}{2(k+2)!} (k-2)! = \frac{(k-2)(k+3)}{2} \quad (k \in \mathbb{N} \setminus \{1\}).$$

The above results are the same as the coefficients of both $(2\pi p)^{k+3}$ and $(2\pi p)^{k+2}$ of equation (12-79) under the condition of $k > 2$. And I arrive at another conclusion in this subsection.

The following is the third general representation for the zeta function for any odd number of 7 or more:

$$(12-85) \ \zeta(2k+1) = \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(\sum_{\mu=0}^{k+3} C_{k,\mu} (2\pi p)^\mu \right) e^{-2\pi p} \quad (k \in \mathbb{N} \setminus \{1, 2\}).$$

Where

$$(12-86) \ C_{k,\mu} = \frac{(2k-\mu)!}{2^{k+3-\mu} \mu! (k+3-\mu)!} (\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13) + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24))) \quad (k \in \mathbb{N} \setminus \{1, 2\}, \mu = 0, 1, \dots, k+3).$$

And the coefficients deserving special mention are as follows:

$$(12-87) \ C_{k,k+3} = 1 \quad (k \in \mathbb{N}).$$

$$(12-88) \ C_{k,k+2} = \frac{(k-2)(k+3)}{2} \quad (k \in \mathbb{N}).$$

$$(12-89) \ C_{k,3} = -\frac{k^2 - 4k + 9}{3} (2k-1)!! \quad (k \in \mathbb{N}).$$

$$(12-90) \ C_{k,2} = -(k^2 - k + 3) (2k-1)!! \quad (k \in \mathbb{N}).$$

$$(12-91) \ C_{k,1} = C_{k,0} = -k(2k+1)!! \quad (k \in \mathbb{N}).$$

Table of the coefficients of the infinite series

k	1	2	3	4	5	6	7	8
$\frac{2}{k(2k+1)!!}$	$\frac{2}{3}$	$\frac{1}{15}$	$\frac{2}{315}$	$\frac{1}{1890}$	$\frac{2}{51975}$	$\frac{1}{405405}$	$\frac{2}{14189175}$	$\frac{1}{137837700}$

Fig. 12-1

Table of the coefficients $C_{k,\mu}$

k	$C_{k,11}$	$C_{k,10}$	$C_{k,9}$	$C_{k,8}$	$C_{k,7}$	$C_{k,6}$	$C_{k,5}$	$C_{k,4}$	$C_{k,3}$	$C_{k,2}$	$C_{k,1}$	$C_{k,0}$
1								1	-2	-3	-3	-3
2							1	0	-5	-15	-30	-30
3						1	3	0	-30	-135	-315	-315
4					1	7	21	0	-315	-1575	-3780	-3780
5				1	12	70	210	-105	-4410	-21735	-51975	-51975
6			1	18	162	882	2457	-3780	-72765	-343035	-810810	-810810
7		1	25	315	2520	12915	31185	-103950	-1351350	-6081075	-14189175	-14189175
8	1	33	550	5940	44055	211365	405405	-2702700	-27702675	-119594475	-275675400	-275675400

Fig. 12-2

When calculating the values of the function $(\zeta)_\lambda(2k+1)$, the Mathematica inputs are shown as follows.

$$c[\{k_ , \mu_ \}] := \frac{(2k-\mu)!}{2^{k+3-\mu} \mu! (k+3-\mu)!} (\mu(\mu-1)^2(\mu-2)(\mu-3)(\mu+4) - 2k(8k^4 - 4k^3(\mu^2 + 5\mu - 13) + 4k^2(2\mu^3 + 3\mu^2 - 26\mu + 28) - k(7\mu^4 - 2\mu^3 - 89\mu^2 + 168\mu - 92) + (\mu-1)(3\mu^4 - 2\mu^3 - 39\mu^2 + 68\mu - 24));$$

$$zeta[\{k_ , \lambda_ \}] := \frac{2}{k(2k+1)!!} \text{Sum}[\text{DivisorSigma}[-(2k+1), p] \left((2\pi p)^{k+3} + \frac{(k-2)(k+3)}{2} (2\pi p)^{k+2} + \text{Sum}[c[\{k, \mu\}] (2\pi p)^\mu, \{\mu, 0, k+1\}] \right) e^{-2\pi p}, \{p, 1, \lambda\}];$$

$$N[\{zeta[\{1, 1\}], zeta[\{1, 2\}], zeta[\{1, 3\}], zeta[\{1, 4\}], zeta[\{1, 5\}], Zeta[3]\}, 10]$$

By increasing the degree of approximation, it asymptotes the true value of the zeta function for any odd number of 3 or more.

$$(12-92) \{ \zeta_1(3), \zeta_2(3), \zeta_3(3), \zeta_4(3), \zeta_5(3), \zeta(3) \} \\ \simeq \{1.148054268, 1.201550493, 1.202053511, 1.202056889, 1.202056903, 1.202056903\}.$$

$$(12-93) \{ \zeta_1(5), \zeta_2(5), \zeta_3(5), \zeta_4(5), \zeta_5(5), \zeta(5) \} \\ \simeq \{0.9638129674, 1.035899189, 1.036919392, 1.036927709, 1.036927755, 1.036927755\}.$$

$$(12-94) \{ \zeta_1(7), \zeta_2(7), \zeta_3(7), \zeta_4(7), \zeta_5(7), \zeta(7) \} \\ \simeq \{0.8992359468, 1.006186929, 1.008327215, 1.008349125, 1.008349277, 1.008349277\}.$$

$$(12-95) \{ \zeta_1(9), \zeta_2(9), \zeta_3(9), \zeta_4(9), \zeta_5(9), \zeta_6(9), \zeta(9) \} \simeq \{0.8448946784, \\ 0.9977962524, 1.001954409, 1.002007939, 1.002008390, 1.002008393, 1.002008393\}.$$

By choosing the approximation appropriately, for example, a value can be obtained with 30 decimal places of precision.

$$(12-96) \{ \zeta_{13}(3), \zeta(3) \} \\ \simeq \{1.202056903159594285399738161511, 1.202056903159594285399738161511\}.$$

$$(12-97) \{ \zeta_{14}(5), \zeta(5) \} \\ \simeq \{1.036927755143369926331365486457, 1.036927755143369926331365486457\}.$$

$$(12-98) \{ \zeta_{14}(7), \zeta(7) \} \\ \simeq \{1.008349277381922826839797549850, 1.008349277381922826839797549850\}.$$

$$(12-99) \{ \zeta_{14}(9), \zeta(9) \} \\ \simeq \{1.002008392826082214417852769232, 1.002008392826082214417852769232\}.$$

12-3. Two special series with fixed values

The left side of equation (12-30) can be written as follows:

$$(12-100) \sum_{p=1}^{\infty} \sigma_{-5}(p) (2\pi p)^5 e^{-2\pi p} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^5} (2\pi m n)^5 e^{-2\pi m n} \\ = (2\pi)^5 \sum_{n=1}^{\infty} \frac{15}{8} \left(\frac{2}{15} + \frac{1}{\sinh^2(\pi n)} + \frac{1}{\sinh^4(\pi n)} \right) \frac{1}{\sinh^2(\pi n)} \\ = 60\pi^5 \sum_{n=1}^{\infty} \left(\frac{2}{15} + \frac{1}{\sinh^2(\pi n)} + \frac{1}{\sinh^4(\pi n)} \right) \frac{1}{\sinh^2(\pi n)}.$$

Therefore,

$$(12-101) \quad \sum_{n=1}^{\infty} \left(\frac{2}{15} + \frac{1}{\sinh^2(\pi n)} + \frac{1}{\sinh^4(\pi n)} \right) \frac{1}{\sinh^2(\pi n)} = \frac{1}{945}.$$

The following formulae are known “[1]”:

$$(12-102) \quad \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n)} = \frac{1}{6} - \frac{1}{2\pi}.$$

$$(12-103) \quad \sum_{n=1}^{\infty} \frac{1}{\sinh^4(\pi n)} = -\frac{11}{90} + \frac{1}{3\pi} + \frac{1}{1920\pi^6} \Gamma^8\left(\frac{1}{4}\right).$$

The above formulae are introduced into the equation (12-101) to obtain the following series with fixed value:

$$(12-104) \quad \sum_{n=1}^{\infty} \frac{1}{\sinh^6(\pi n)} = \frac{191}{1890} - \frac{4}{15\pi} - \frac{1}{1920\pi^6} \Gamma^8\left(\frac{1}{4}\right).$$

13. Function that takes the constant 0 anywhere in the whole complex plane, and some series that give the algebraic number 0

The fifth-order Π_c type functional equation is shown again.

$$(13-1) \quad -\frac{\theta(\theta-1)(\theta+1)(\theta^2+4\theta+5)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \\ -\frac{\theta(\theta+1)(\theta+2)(\theta^2+6\theta+10)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) + \mathbf{H}_{[5]}(\theta) \quad (\theta \in \mathbb{C}).$$

Where

$$(13-2) \quad \mathbf{H}_{[5]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \\ \left((\theta^5 + 4\theta^4 + 4\theta^3 - 4\theta^2 - 5\theta + (3\theta^3 - 11\theta^2 - 52\theta - 150 + (\theta-21)(2\pi p)^2)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + (2\theta^4 + 13\theta^3 + 34\theta^2 + 38\theta + 15 + 2(3\theta^2 + 13\theta + 39 + (2\pi p)^2)(2\pi p)^2)(2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right).$$

From (the third-order Π_c type functional equation) $\times (\theta^2 + 6\theta + 10)$,

$$(13-3) \quad -\frac{\theta(\theta-1)(\theta+1)(\theta^2+6\theta+10)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \\ -\frac{\theta(\theta+1)(\theta+2)(\theta^2+6\theta+10)}{4\pi} \pi^{-\frac{1+\theta}{2}} \Gamma\left(\frac{1+\theta}{2}\right) \zeta(1+\theta) + (\theta^2 + 6\theta + 10) \mathbf{H}_{[3]}(\theta) \quad (\theta \in \mathbb{C}).$$

Where

$$(13-4) \quad \mathbf{H}_{[3]}(\theta) = \frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1)(\theta+1) + (\theta-7)(2\pi p)^2) \mathbf{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((2\theta+1)(\theta+1) + 2(2\pi p)^2)(2\pi p) \mathbf{K}_{\frac{2-\theta}{2}}(2\pi p) \right).$$

From (equation (13-1)) - (equation(13-3)),

$$(13-5) \quad \frac{\theta(\theta-1)(\theta+1)(2\theta+5)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \mathbf{H}_{[5]}(\theta) - (\theta^2 + 6\theta + 10) \mathbf{H}_{[3]}(\theta) \quad (\theta \in \mathbb{C}).$$

In addition, from (equation (11-5)) $\times (\theta + 1)(2\theta + 5)$,

$$(13-6) \quad -\frac{\theta(\theta-1)(\theta+1)(2\theta+5)}{4\pi} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) = \frac{(\theta+1)(2\theta+5)}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}}$$

$$\left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

And also, from (equation (13-5)) + (equation (13-6)),

$$(13-7) \quad \mathbf{H}_{[5]}(\theta) - (\theta^2 + 6\theta + 10) \mathbf{H}_{[3]}(\theta) + \frac{(\theta+1)(2\theta+5)}{\pi} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}}$$

$$\left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) = 0 \quad (\theta \in \mathbb{C}).$$

Substituting the results of $\mathbf{H}_{[3]}(\theta)$ and $\mathbf{H}_{[5]}(\theta)$ for the calculation, the following equation is obtained:

$$(13-8) \quad \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right.$$

$$\left. + 2(24 + (2\pi p)^2)(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) = 0 \quad (\theta \in \mathbb{C}).$$

Let define the left side of equation (13-8) as the function $\omega(\theta)$.

$$(13-9) \quad \omega(\theta) := \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right.$$

$$\left. + 2(24 + (2\pi p)^2)(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

For above equation, $-\theta$ is substituted for θ to obtain the following equation:

$$(13-10) \quad \omega(-\theta) = \sum_{p=1}^{\infty} \sigma_{\theta}(p) p^{-\frac{\theta}{2}} \left((-3(-\theta-3)(-\theta-5) + (-\theta-21)(2\pi p)^2)(2\pi p)^2 K_{\frac{-\theta}{2}}(2\pi p) \right.$$

$$\left. + 2(24 + (2\pi p)^2)(2\pi p)^3 K_{\frac{2+\theta}{2}}(2\pi p) \right)$$

$$= \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((-3(\theta+3)(\theta+5) - (\theta+21)(2\pi p)^2)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right.$$

$$\left. + 4(24 + (2\pi p)^2)(2\pi p)^2 \left(\frac{\theta}{2} K_{\frac{\theta}{2}}(2\pi p) + (\pi p) K_{\frac{\theta-2}{2}}(2\pi p) \right) \right)$$

$$= \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right.$$

$$\left. + 2(24 + (2\pi p)^2)(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

As a result, I obtained the function that takes the constant 0 anywhere in the whole complex plane.

$$(13-11) \quad \omega(\theta) = \omega(-\theta) = \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right.$$

$$\left. + 2(24 + (2\pi p)^2)(2\pi p)^3 K_{\frac{\theta-2}{2}}(2\pi p) \right) = 0 \quad (\theta \in \mathbb{C}).$$

In order to obtain a representation that has quadrable integrals, I modify the right side of equation (13-11) as follows:

$$(13-12) \quad \omega(\theta) = \pi^{-\frac{\theta}{2}} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left((-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2)(2\pi p)^2 (\pi p)^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi p) \right.$$

$$\left. + (24 + (2\pi p)^2)(2\pi p)^4 (\pi p)^{\frac{\theta-2}{2}} K_{\frac{\theta-2}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C}).$$

From the equations (12-3) and (12-4),

$$(13-13) \quad \omega(\theta) = \pi^{-\frac{\theta}{2}} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left((-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2) (2\pi p)^2 \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ \left. + (24 + (2\pi p)^2) (2\pi p)^4 \int_0^{\infty} x^{\theta-3} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \quad (\theta \in \mathbb{C}).$$

Two integrals of the right side of equation (13-13) are quadrable when θ is any odd number of 1 or more. Some calculating examples will be shown. When $\theta = 1$,

$$(13-14) \quad \omega(1) = \pi^{-\frac{1}{2}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left((-24 - 20(2\pi p)^2) (2\pi p)^2 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ \left. + (24 + (2\pi p)^2) (2\pi p)^4 \int_0^{\infty} x^{-2} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\ = \frac{1}{\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left((-24 - 20(2\pi p)^2) (2\pi p)^2 \frac{\sqrt{\pi}}{2} e^{-2\pi p} + (24 + (2\pi p)^2) (2\pi p)^4 \frac{\sqrt{\pi}}{2\pi p} e^{-2\pi p} \right) \\ = \sum_{p=1}^{\infty} \sigma_{-1}(p) \left((2\pi p)^3 - 10(2\pi p)^2 + 24(2\pi p) - 12 \right) (2\pi p)^2 e^{-2\pi p}.$$

Therefore

$$(13-15) \quad \sum_{p=1}^{\infty} \sigma_{-1}(p) \left((2\pi p)^3 - 10(2\pi p)^2 + 24(2\pi p) - 12 \right) (2\pi p)^2 e^{-2\pi p} = \omega(\pm 1) = 0.$$

When $\theta = 3$,

$$(13-16) \quad \omega(3) = \pi^{-\frac{3}{2}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((-18(2\pi p)^2) (2\pi p)^2 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ \left. + (24 + (2\pi p)^2) (2\pi p)^4 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\ = \frac{1}{\pi \sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(-18(2\pi p)^4 \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} + (24 + (2\pi p)^2) (2\pi p)^4 \frac{\sqrt{\pi}}{2} e^{-2\pi p} \right) \\ = \frac{1}{2\pi} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^2 - 9(2\pi p) + 15 \right) (2\pi p)^4 e^{-2\pi p}.$$

Therefore

$$(13-17) \quad \sum_{p=1}^{\infty} \sigma_{-3}(p) \left((2\pi p)^2 - 9(2\pi p) + 15 \right) (2\pi p)^4 e^{-2\pi p} = 2\pi \omega(\pm 3) = 0.$$

When $\theta = 5$,

$$\begin{aligned}
 (13-18) \quad \omega(5) &= \pi^{-\frac{5}{2}} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left((-16(2\pi p)^2)(2\pi p)^2 \int_0^{\infty} x^4 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. + (24 + (2\pi p)^2)(2\pi p)^4 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= \frac{1}{\pi^2 \sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(-16(2\pi p)^4 \frac{(2\pi p)^2 + 3(2\pi p) + 3}{8} \sqrt{\pi} e^{-2\pi p} \right. \\
 &\quad \left. + (24 + (2\pi p)^2)(2\pi p)^4 \frac{(2\pi p) + 1}{4} \sqrt{\pi} e^{-2\pi p} \right) \\
 &= \frac{1}{4\pi^2} \sum_{p=1}^{\infty} \sigma_{-5}(p) ((2\pi p) - 7)(2\pi p)^6 e^{-2\pi p}.
 \end{aligned}$$

Therefore

$$(13-19) \quad \sum_{p=1}^{\infty} \sigma_{-5}(p) ((2\pi p) - 7)(2\pi p)^6 e^{-2\pi p} = 4\pi^2 \omega(\pm 5) = 0.$$

When $\theta = 7, 9, 11, 13, 15,$ and $17,$ only the results will be shown.

$$\begin{aligned}
 (13-20) \quad \sum_{p=1}^{\infty} \sigma_{-7}(p) & \left((2\pi p)^6 - 4(2\pi p)^5 - 15(2\pi p)^4 - 45(2\pi p)^3 - 105(2\pi p)^2 \right. \\
 & \left. - 180(2\pi p) - 180 \right) (2\pi p)^2 e^{-2\pi p} = 8\pi^3 \omega(\pm 7) = 0.
 \end{aligned}$$

$$\begin{aligned}
 (13-21) \quad \sum_{p=1}^{\infty} \sigma_{-9}(p) & \left((2\pi p)^7 - 21(2\pi p)^5 - 147(2\pi p)^4 - 630(2\pi p)^3 - 1890(2\pi p)^2 \right. \\
 & \left. - 3780(2\pi p) - 3780 \right) (2\pi p)^2 e^{-2\pi p} = 16\pi^4 \omega(\pm 9) = 0.
 \end{aligned}$$

$$\begin{aligned}
 (13-22) \quad \sum_{p=1}^{\infty} \sigma_{-11}(p) & \left((2\pi p)^8 + 5(2\pi p)^7 - 6(2\pi p)^6 - 252(2\pi p)^5 - 1995(2\pi p)^4 - 9765(2\pi p)^3 \right. \\
 & \left. - 32445(2\pi p)^2 - 68040(2\pi p) - 68040 \right) (2\pi p)^2 e^{-2\pi p} = 32\pi^5 \omega(\pm 11) = 0.
 \end{aligned}$$

$$\begin{aligned}
 (13-23) \quad \sum_{p=1}^{\infty} \sigma_{-13}(p) & \left((2\pi p)^9 + 11(2\pi p)^8 + 45(2\pi p)^7 - 180(2\pi p)^6 - 4095(2\pi p)^5 - 33075(2\pi p)^4 \right. \\
 & \left. - 170100(2\pi p)^3 - 585900(2\pi p)^2 - 1247400(2\pi p) \right. \\
 & \left. - 1247400 \right) (2\pi p)^2 e^{-2\pi p} = 64\pi^6 \omega(\pm 13) = 0.
 \end{aligned}$$

$$\begin{aligned}
 (13-24) \quad \sum_{p=1}^{\infty} \sigma_{-15}(p) & \left((2\pi p)^{10} + 18(2\pi p)^9 + 150(2\pi p)^8 + 450(2\pi p)^7 - 4725(2\pi p)^6 \right. \\
 & \left. - 79380(2\pi p)^5 - 630315(2\pi p)^4 - 3274425(2\pi p)^3 - 11382525(2\pi p)^2 \right. \\
 & \left. - 24324300(2\pi p) - 24324300 \right) (2\pi p)^2 e^{-2\pi p} = 128\pi^7 \omega(\pm 15) = 0.
 \end{aligned}$$

$$\begin{aligned}
 (13-25) \quad \sum_{p=1}^{\infty} \sigma_{-17}(p) & \left((2\pi p)^{11} + 26(2\pi p)^{10} + 330(2\pi p)^9 + 2310(2\pi p)^8 + 3465(2\pi p)^7 \right. \\
 & \left. - 124740(2\pi p)^6 - 1735965(2\pi p)^5 - 13357575(2\pi p)^4 - 68918850(2\pi p)^3 - 239188950(2\pi p)^2 \right. \\
 & \left. - 510810300(2\pi p) - 510810300 \right) (2\pi p)^2 e^{-2\pi p} = 256\pi^8 \omega(\pm 17) = 0.
 \end{aligned}$$

It is recognized that some series that seem to give transcendental numbers, however, give the algebraic number 0 at times. And the function $\omega(\theta)$ is the null function in the whole complex plane. Unfortunately, no one knows whether the zeta value for any odd number of 3 or more is a transcendental number at the moment.

14. Derivation of an explicit formula for a new eta function and the proof of the generalized Riemann hypothesis for the eta function

14-1. Derivation of an explicit formula for a new eta function

The following equation defines the eta function $\eta(\theta)$:

$$(14-1) \quad \eta(\theta) := \frac{\pi^2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \chi(\theta) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

According to the definition, the non-trivial zeros of the eta function are the same as those of the Chi function. Thus, I also propose the generalized Riemann hypothesis for the eta function that states that all non-trivial zeros lie on the imaginary axis. it will be proved soon.

The equation (9-24) is introduced into the equation of definition.

$$(14-2) \quad \eta(\theta) = \frac{\pi^2}{(\theta-1)\Gamma\left(1+\frac{\theta}{2}\right)} \left(\frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \zeta(\theta) \right. \\ \left. + 2(\theta-1) \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \right) \\ = \zeta(\theta) + \frac{2\pi^2}{\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left(\theta K_{\frac{\theta}{2}}(2\pi p) + (2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

The explicit formula for the zeta function is introduced into the equation (14-2).

$$(14-3) \quad \eta(\theta) = \frac{2\pi^2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2) (2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) - \frac{2\pi^2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \\ \left(\theta(\theta-1) K_{\frac{\theta}{2}}(2\pi p) + (\theta-1)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

By simplifying, I obtain the explicit formula for the eta function.

$$(14-4) \quad \eta(\theta) = \frac{2\pi^2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

For the explicit formula, $-\theta$ is substituted for θ .

$$(14-5) \quad \eta(-\theta) = \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{\theta}(p) p^{-\frac{\theta}{2}} \left((7+\theta)(2\pi p)^2 K_{\frac{-\theta}{2}}(2\pi p) \right. \\ \left. - 2(2\pi p)^3 K_{\frac{2+\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

The relation between the equations (3-30) and (5-39) is applied to the equation (14-5).

$$(14-6) \quad \eta(-\theta) = \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7+\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) - 4(2\pi p)^2 (\pi p) K_{\frac{2+\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

The recurrence formula for the modified Bessel function of the second kind is applied.

$$(14-7) \quad \eta(-\theta) = \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7+\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) - 4(2\pi p)^2 \left(\frac{\theta}{2} K_{\frac{\theta}{2}}(2\pi p) + (\pi p) K_{\frac{\theta-2}{2}}(2\pi p) \right) \right) \\ = \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

The difference between $\eta(\theta)$ and $\eta(-\theta)$ is essentially reflected in each gamma factor. And the result of equation (14-7) will be used in Section 16.

14-2. The proof of the generalized Riemann hypothesis for the eta function

Assuming that there are infinitely many non-trivial zeros on the imaginary axis in the eta function, and both x and y are assumed to be real numbers, and $(x + iy)$ is assumed to be a non-trivial zero of the eta function. Based on the origin symmetry of the Chi function, $(-x - iy)$ is also a non-trivial zero of the eta function. Because it contains trivial zeros of the eta function, the real axis must be excluded from the target region. I consider that non-trivial zeros are determined as solutions of simultaneous equations consisting of the following equations:

$$(14-8) \quad \begin{cases} \eta(x + iy) = 0 & (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}), \\ \eta(-x - iy) = 0 & (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}). \end{cases}$$

For the explicit formula for the eta function, $(x + iy)$ is substituted for θ .

$$(14-9) \quad \eta(x + iy) = \frac{2\pi^{\frac{x+iy}{2}}}{(1-x-iy)\Gamma\left(1+\frac{x+iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-x-iy}(p) p^{\frac{x+iy}{2}} \\ \left((7-x-iy)(2\pi p)^2 K_{\frac{x+iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-x-iy}{2}}(2\pi p) \right) = 0 \quad (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}).$$

Similarly, in the explicit formula for the eta function, $(-x - iy)$ is substituted for θ .

$$(14-10) \quad \eta(-x - iy) = \frac{2\pi^{\frac{-x-iy}{2}}}{(1+x+iy)\Gamma\left(1+\frac{-x-iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{x+iy}(p) p^{\frac{-x-iy}{2}} \\ \left((7+x+iy)(2\pi p)^2 K_{\frac{-x-iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2+x+iy}{2}}(2\pi p) \right) = 0 \quad (x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}).$$

The non-trivial zeros are determined as the solutions of the simultaneous equations comprising the equations (14-9) and (14-10). Owing to the symmetric pair of these equations,

$$(14-11) \quad x = 0.$$

is immediately determined.

For equation (14-9), 0 is substituted for x .

$$(14-12) \quad \eta(iy) = \frac{2\pi^{\frac{iy}{2}}}{(1-iy)\Gamma\left(1+\frac{iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}}$$

$$\left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-iy}{2}}(2\pi p)\right) = 0 \quad (y \in \mathbb{R} \setminus \{0\}).$$

For equation (14-10), 0 is substituted for x.

$$(14-13) \quad \eta(-iy) = \frac{2\pi^{-\frac{iy}{2}}}{(1+iy)\Gamma\left(1-\frac{iy}{2}\right)} \sum_{p=1}^{\infty} \sigma_{iy}(p) p^{-\frac{iy}{2}}$$

$$\left((7+iy)(2\pi p)^2 K_{-\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2+iy}{2}}(2\pi p)\right) = 0 \quad (y \in \mathbb{R} \setminus \{0\}).$$

The equations (14-12) and (14-13) are complex conjugate of each other in relation.

Hence, I can choose either of these equations as the determining equation of the real variable y.

Furthermore, the gamma factor of the eta function does not have a value of zero on the imaginary axis.

Additionally, the condition of y = 0 can be excepted, because the eta function has no non-trivial zero on the real axis.

Therefore, the following equation gives the determining equation of the real variable y.

$$(14-14) \quad \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}} \left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-iy}{2}}(2\pi p)\right) = 0 \quad (y \in \mathbb{R}).$$

The explicit formula for the Chi function is also found.

$$(14-15) \quad \chi(\theta) = (\theta-1)\pi^{-\frac{\theta}{2}}\Gamma\left(1+\frac{\theta}{2}\right)\eta(\theta) =$$

$$-2 \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p)\right) \quad (\theta \in \mathbb{C}).$$

The function value of the Chi function on the imaginary axis is

$$(14-16) \quad \chi(iy) = -2 \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}} \left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-iy}{2}}(2\pi p)\right) \quad (y \in \mathbb{R}).$$

Applying the origin symmetry with respect to the index of the modified Bessel function of the second kind and the recurrence formula,

$$(14-17) \quad \chi(iy) = -2 \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}} \left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{iy-2}{2}}(2\pi p)\right)$$

$$= -2 \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}} \left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) + 2(2\pi p)^2 \left(iy K_{\frac{iy}{2}}(2\pi p) - (2\pi p) K_{\frac{iy+2}{2}}(2\pi p)\right)\right)$$

$$= -2 \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}} \left((7+iy)(2\pi p)^2 K_{-\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2+iy}{2}}(2\pi p)\right) \quad (y \in \mathbb{R}).$$

Using the complex conjugate on both sides of the above equation,

$$(14-18) \quad \chi^*(iy) = -2 \sum_{p=1}^{\infty} \sigma_{iy}(p) p^{-\frac{iy}{2}} \left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-iy}{2}}(2\pi p)\right)$$

$$= -2 \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}} \left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-iy}{2}}(2\pi p)\right) \quad (y \in \mathbb{R}).$$

Here, the equation (5-39) is applied.

Therefore, it is shown that the Chi function takes real value on the imaginary axis as follows:

$$(14-19) \quad \mathbf{Im}(\chi(iy)) = \frac{1}{2i}(\chi(iy) - \chi^*(iy)) = 0 \quad (y \in \mathbb{R}).$$

The Hadamard product representation for the Chi function on the imaginary axis is

$$(14-20) \quad \chi(iy) = \left(\frac{1}{2} - 2 \sum_{p=1}^{\infty} \sigma_0(p) (2\pi p) K_1(2\pi p) \right) \prod_{m=1}^{\infty} \left(1 - \frac{y^2}{(\tau_m)^2} \right) \quad (y \in \mathbb{R}).$$

From the equations (14-16) and (14-20),

$$(14-21) \quad \sum_{p=1}^{\infty} \sigma_{-iy}(p) p^{\frac{iy}{2}} \left((7-iy)(2\pi p)^2 K_{\frac{iy}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-iy}{2}}(2\pi p) \right) \\ = -\frac{1}{2} \left(\frac{1}{2} - 2 \sum_{p=1}^{\infty} \sigma_0(p) (2\pi p) K_1(2\pi p) \right) \prod_{m=1}^{\infty} \left(1 - \frac{y^2}{(\tau_m)^2} \right) \in \mathbb{R} \quad (y \in \mathbb{R}).$$

On the imaginary axis, the left side of the determining equation takes a real value. In this case, the imaginary axis corresponds to the critical line of the eta function.

It is possible to assert that there is no non-trivial zero off the imaginary axis.

At this stage, the proof of the generalized Riemann hypothesis for the eta function is complete.

It should be noted that the existence of an infinite number of non-trivial zeros of the eta function remains unproven.

Furthermore, it is possible to design an eta function variant that takes real values on the imaginary axis. The method is to obtain the absolute value for the gamma factor of the eta function's explicit formula. I decide to write the eta function variant as eta-tilde ($\tilde{\eta}$) by adding the symbol \sim (tilde) just above the Greek letter η .

$$(14-22) \quad \tilde{\eta}(\theta) := \left| \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \right| \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

From now on, I consider the behavior of the number-theoretic function $\sigma_{-\theta}(n) n^{\theta/2}$ on the imaginary axis by presenting specific examples. When $n = 1$,

$$(14-23) \quad n^{\frac{iy}{2}} \sigma_{-iy}(n) \Big|_{n=1} = 1.$$

When n is the prime number a ,

$$(14-24) \quad a^{\frac{iy}{2}} \sigma_{-iy}(a) = (1 + a^{-iy}) a^{\frac{iy}{2}} = 2 \cos\left(\frac{y}{2} \log(a)\right).$$

When n is $(2k-1)$ power of the prime number a ,

$$(14-25) \quad (a^{2k-1})^{\frac{iy}{2}} \sigma_{-iy}(a^{2k-1}) = \sum_{m=1}^k 2 \cos\left(\frac{(2m-1)y}{2} \log(a)\right) \quad (k \in \mathbb{N}).$$

When n is $2k$ power of the prime number a ,

$$(14-26) \quad (a^{2k})^{\frac{iy}{2}} \sigma_{-iy}(a^{2k}) = 1 + \sum_{m=1}^k 2 \cos(m y \log(a)) \quad (k \in \mathbb{N}).$$

The number-theoretic function $\sigma_{-\theta}(n) n^{\theta/2}$ is described by a real cosine in the explicit formula for the eta function on the imaginary axis.

15. Visualization of the zeta functions and the eta functions

Let start by preparing some formulae for visualization.

The rule in subsection 6-3 is applied to the explicit formula for the zeta function.

In this case, the approximate formula for the zeta function is written as $\zeta_\lambda(\theta)$.

$$(15-1) \quad \zeta_\lambda(\theta) := \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\lambda} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

The rule is also applied for defining equation (11-23).

$$(15-2) \quad \tilde{\zeta}_\lambda(\theta) := \left| \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \right| \sum_{p=1}^{\lambda} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

The relation between $\zeta_\lambda(\theta)$ and $(\text{zeta-hat})_\lambda(\theta)$ is

$$(15-3) \quad \lim_{\lambda \rightarrow +\infty} \zeta_\lambda(\theta) = \lim_{\lambda \rightarrow +\infty} \hat{\zeta}_\lambda(\theta) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

The relation between $\zeta_\lambda(\theta)$ and $(\text{zeta-tilde})_\lambda(\theta)$ is

$$(15-4) \quad |\zeta_\lambda(\theta)| = \left| \tilde{\zeta}_\lambda(\theta) \right| \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

The rule is also applied to the explicit formula for the eta function.

$$(15-5) \quad \eta_\lambda(\theta) := \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\lambda} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

The rule is also applied for defining equation (14-22).

$$(15-6) \quad \tilde{\eta}_\lambda(\theta) := \left| \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \right| \sum_{p=1}^{\lambda} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

The relation between $\eta_\lambda(\theta)$ and $(\text{eta-tilde})_\lambda(\theta)$ is

$$(15-7) \quad |\eta_\lambda(\theta)| = \left| \tilde{\eta}_\lambda(\theta) \right| \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

The right side of the null function $\omega(\theta)$ is multiplied by the absolute value of the gamma factor of the zeta function to obtain the estimating function $(\text{omega-tilde})_\lambda(\theta)$; moreover, the function $(\text{omega-tilde})_\infty(\theta)$ is also the null function.

$$(15-8) \quad \tilde{\omega}_\lambda(\theta) := \left| \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \right| \sum_{p=1}^{\lambda} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \\ \left((-3(\theta-3)(\theta-5) + (\theta-21)(2\pi p)^2)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + 2(24 + (2\pi p)^2)(2\pi p)^3 K_{\frac{\theta-2}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

Three – dimensional graph of $Z = \text{Re}(\tilde{\omega}_8(x + iy))$ and $Z = \text{Im}(\tilde{\omega}_8(x + iy))$,
 $x \in [-35, 35]$, $y \in [0, 70]$

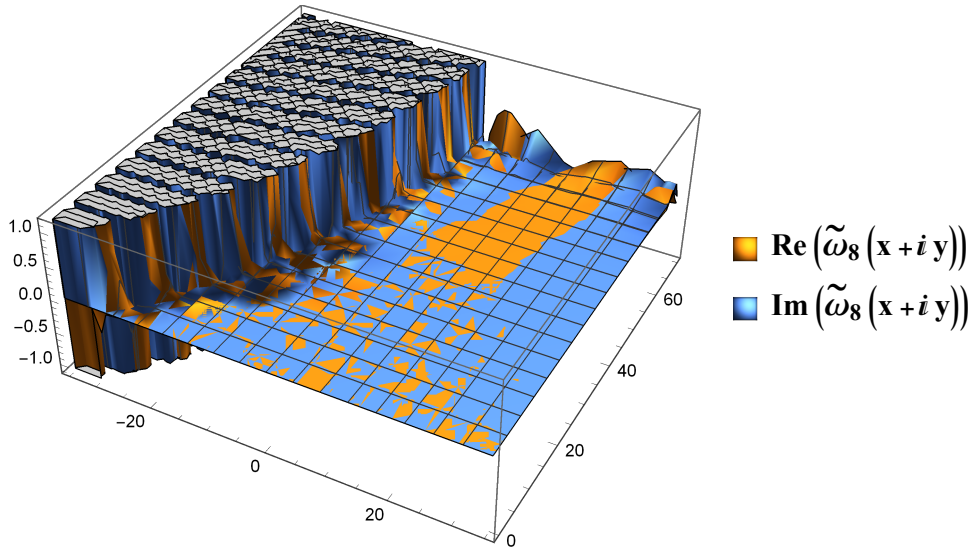


Fig. 15-1

The above three-dimensional graph shows the computational error because the null function $(\omega - \text{tilde})_{\infty}(\theta)$ takes the value zero anywhere in the whole complex plane.

Three – dimensional graph of $Z = \text{Re}(\tilde{\omega}_{11}(x + iy))$ and $Z = \text{Im}(\tilde{\omega}_{11}(x + iy))$,
 $x \in [-35, 35]$, $y \in [0, 70]$

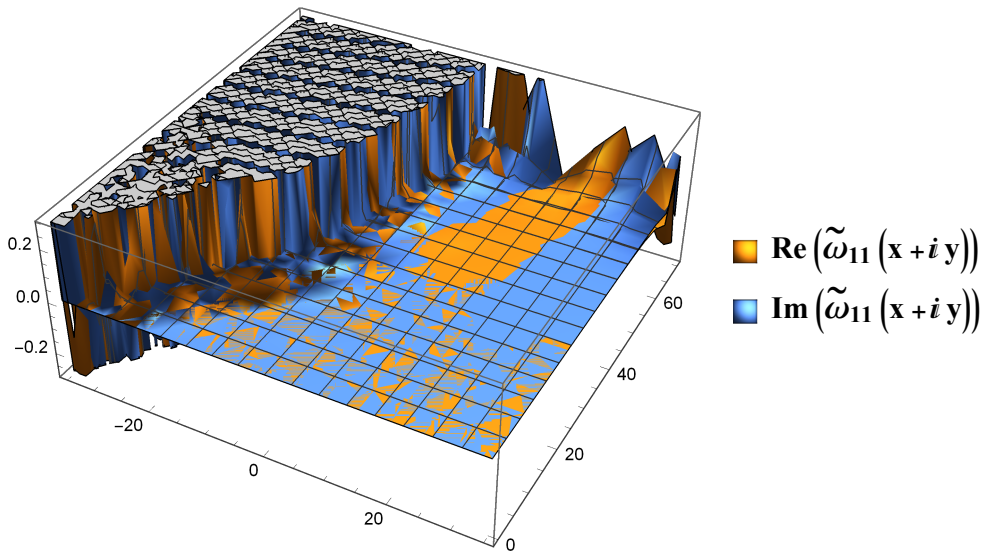


Fig. 15-2

When the following two graphs are compared, it is observed that the computational error decreases with a larger degree of approximation at each point in the observational domain. In comparison to the same domain, the above results emphasize that the degree of approximation is needed to draw another kind of graph.

Graph of $Z = \operatorname{Re} \left(\tilde{\zeta}_8 \left(\frac{1}{2} + i y \right) \right)$, $Z = \operatorname{Im} \left(\tilde{\zeta}_8 \left(\frac{1}{2} + i y \right) \right)$, and $Z = -\frac{1}{3} |\eta_8(i y)|$, $y \in [0, 52]$

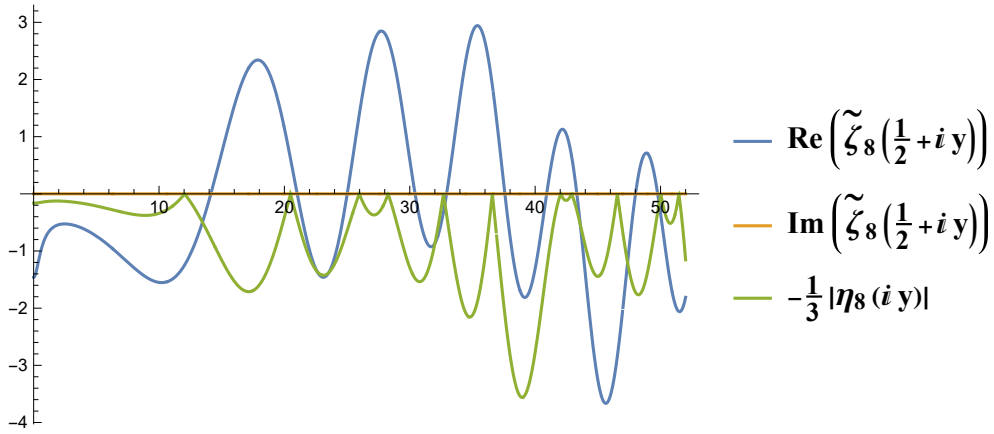


Fig. 15-3

Both the functions

$$\tilde{\zeta}_8 \left(\frac{1}{2} + i y \right) \text{ and } -\frac{1}{3} |\eta_8(i y)|$$

are projected onto the same surface. I can see a relation between the imaginary parts of the non-trivial zeros of the zeta function and those of the eta function that is neither too close nor too far away. The imaginary part of the function (zeta - tilde) (θ) is zero anywhere along the critical line because the absolute value is taken for the gamma factor of the zeta function. On the critical line, the function (zeta - tilde) (θ) takes a real value, and 10 non-trivial zeros of the zeta function lie on the interval $[1/2 + i 0, 1/2 + i 52]$.

Here are the details. (The positive imaginary parts of the 11 non-trivial zeros of the eta function will be shown later.)

$$(15-9) \left\{ \operatorname{Re} \left(\tilde{\zeta}_8 \left(\frac{1}{2} + i \left(\frac{14\,134\,725\,142}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_8 \left(\frac{1}{2} + i \left(\frac{14\,134\,725\,142}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-5.82730 \times 10^{-10}, 1.00359 \times 10^{-9}\}.$$

$$(15-10) \left\{ \operatorname{Re} \left(\tilde{\zeta}_8 \left(\frac{1}{2} + i \left(\frac{21\,022\,039\,639}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_8 \left(\frac{1}{2} + i \left(\frac{21\,022\,039\,639}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{8.77134 \times 10^{-10}, -1.39654 \times 10^{-9}\}.$$

$$(15-11) \left\{ \operatorname{Re} \left(\tilde{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{25\,010\,857\,580}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{25\,010\,857\,580}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-1.57157 \times 10^{-9}, 1.17188 \times 10^{-9}\}.$$

$$(15-12) \left\{ \operatorname{Re} \left(\tilde{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{30\,424\,876\,126}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{30\,424\,876\,126}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{1.12075 \times 10^{-9}, -1.48713 \times 10^{-9}\}.$$

$$(15-13) \left\{ \operatorname{Re} \left(\tilde{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{32\,935\,061\,588}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_9 \left(\frac{1}{2} + i \left(\frac{32\,935\,061\,588}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-1.02165 \times 10^{-9}, 1.74259 \times 10^{-9}\}.$$

$$(15-14) \left\{ \operatorname{Re} \left(\tilde{\zeta}_{10} \left(\frac{1}{2} + i \left(\frac{37\,586\,178\,159}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_{10} \left(\frac{1}{2} + i \left(\frac{37\,586\,178\,159}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{1.59892 \times 10^{-9}, -2.27410 \times 10^{-9}\}.$$

$$(15 - 15) \left\{ \operatorname{Re} \left(\tilde{\zeta}_{10} \left(\frac{1}{2} + i \left(\frac{40\,918\,719\,012}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_{10} \left(\frac{1}{2} + i \left(\frac{40\,918\,719\,012}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-1.71047 \times 10^{-9}, 1.27075 \times 10^{-9}\}.$$

$$(15 - 16) \left\{ \operatorname{Re} \left(\tilde{\zeta}_{10} \left(\frac{1}{2} + i \left(\frac{43\,327\,073\,281}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_{10} \left(\frac{1}{2} + i \left(\frac{43\,327\,073\,281}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{1.67766 \times 10^{-9}, -1.98936 \times 10^{-9}\}.$$

$$(15 - 17) \left\{ \operatorname{Re} \left(\tilde{\zeta}_{11} \left(\frac{1}{2} + i \left(\frac{48\,005\,150\,881}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_{11} \left(\frac{1}{2} + i \left(\frac{48\,005\,150\,881}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-1.83014 \times 10^{-9}, 1.30592 \times 10^{-9}\}.$$

$$(15 - 18) \left\{ \operatorname{Re} \left(\tilde{\zeta}_{12} \left(\frac{1}{2} + i \left(\frac{49\,773\,832\,478}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\zeta}_{12} \left(\frac{1}{2} + i \left(\frac{49\,773\,832\,478}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{9.53951 \times 10^{-10}, -1.88391 \times 10^{-9}\}.$$

Three - dimensional graph of $z = \operatorname{Re} \left(\tilde{\zeta}_5(x + iy) \right)$ and $z = 0$, $x \in \left[-\frac{1}{2}, \frac{3}{2} \right]$, $y \in [0, 35]$

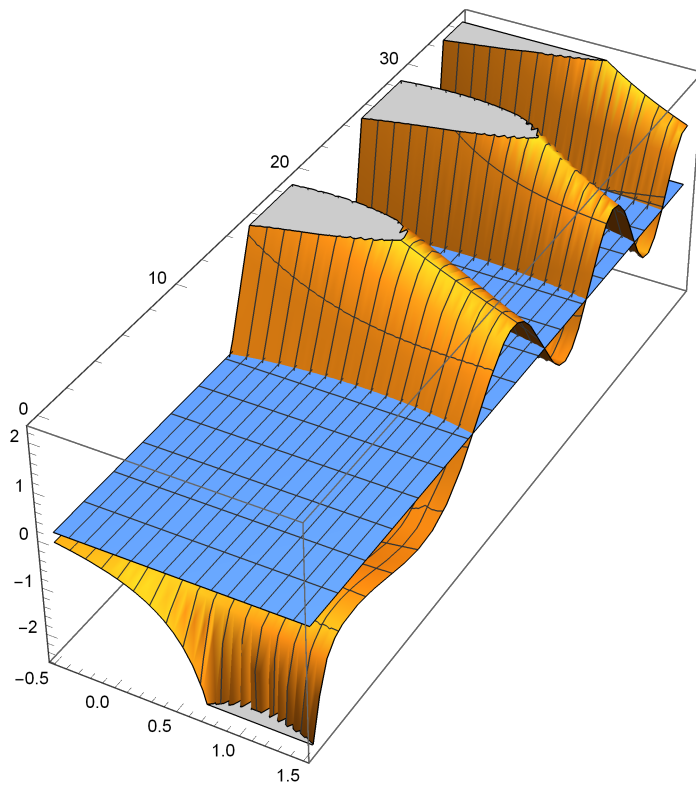


Fig. 15-4

Three-dimensional graphs will be hereafter drawn using the plane $z = 0$ as the auxiliary plane. Using topography as an analogy, the plane $z = 0$ (colored blue) is suited for the surface of the sea, and the candidate sites for the non-trivial zeros of the zeta function are along the coast lines.

Three – dimensional graph of $Z = \text{Im} \left(\tilde{\zeta}_5(x + i y) \right)$ and $Z = 0$, $x \in \left[-\frac{1}{2}, \frac{3}{2}\right]$, $y \in [0, 35]$

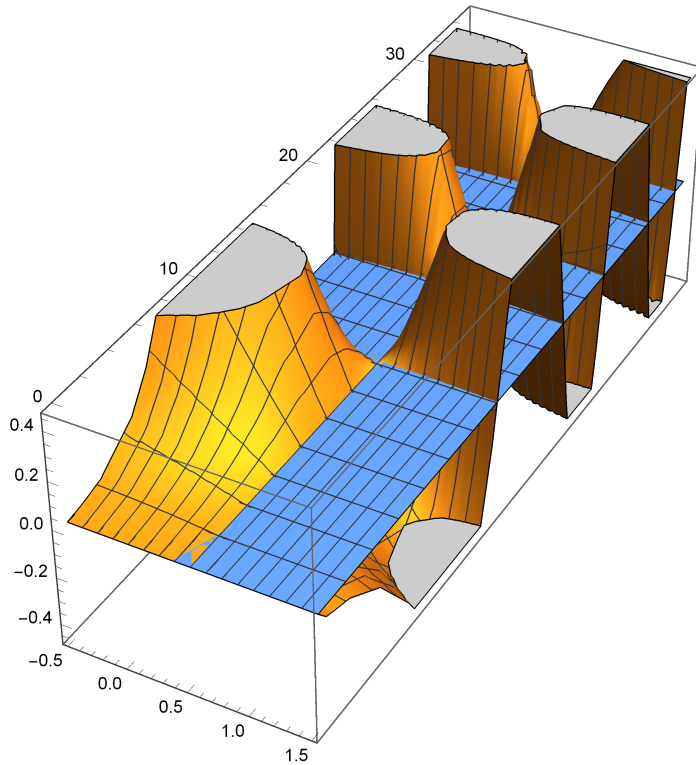


Fig. 15-5

On the sea’s complicated surface, the critical line is clearly visible.

The intersections of the coast line and the critical line in the two graphs above are the non-trivial zeros.

As reference, the same graphs above (Fig. 15-4 and Fig. 15-5) can also be drawn using only the well-known built-in Mathematica functions.

The Mathematica inputs are shown as follows:

$$\text{zetaxx}[\theta_] := \text{Abs} \left[\frac{2 \pi^{\frac{\theta}{2}}}{(1 - \theta) \text{Gamma} \left[1 + \frac{\theta}{2} \right]} \right] \text{Zeta}[\theta] / \frac{2 \pi^{\frac{\theta}{2}}}{(1 - \theta) \text{Gamma} \left[1 + \frac{\theta}{2} \right]};$$

$$\text{Plot3D} \left[\left\{ \text{Re}[\text{zetaxx}[x + i y]], 0 \right\}, \left\{ x, -\frac{1}{2}, \frac{3}{2} \right\}, \left\{ y, 0, 35 \right\}, \text{BoxRatios} \rightarrow \{1, 3, 1\} \right]$$

$$\text{Plot3D} \left[\left\{ \text{Im}[\text{zetaxx}[x + i y]], 0 \right\}, \left\{ x, -\frac{1}{2}, \frac{3}{2} \right\}, \left\{ y, 0, 35 \right\}, \text{BoxRatios} \rightarrow \{1, 3, 1\} \right]$$

Graph of $z = \text{Re} \left(\zeta_5 \left(\frac{1}{2} + i y \right) \right)$ and $z = \text{Im} \left(\zeta_5 \left(\frac{1}{2} + i y \right) \right)$, $y \in [0, 35]$

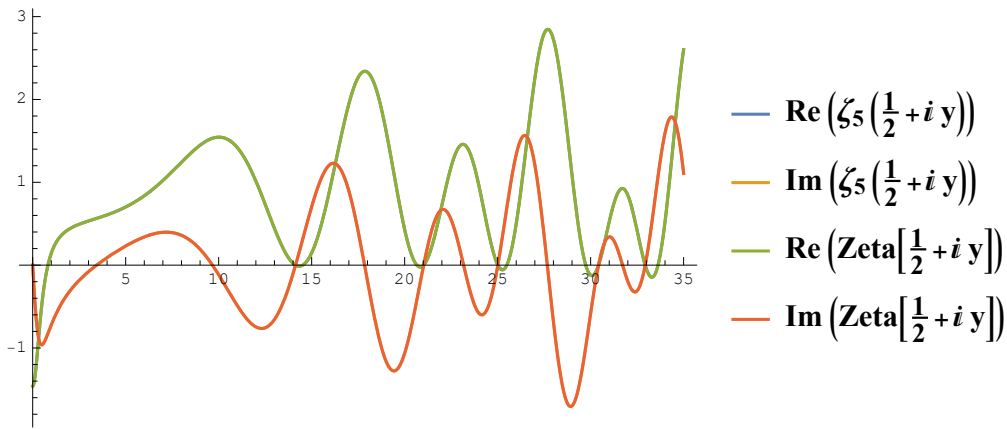


Fig. 15-6

To evaluate the error, real and imaginary values of $\text{Zeta}[1/2 + i y]$, i.e., Mathematica’s built-in zeta function, are superposed on the graph. A five degree of approximation is appropriate for the interval $[1/2+i 0, 1/2+i 35]$. The built-in zeta function should be superposed if necessary for all graphs.

Graph of $z = \text{Re} \left(\zeta_8 \left(\frac{1}{2} + i y \right) \right)$ and $z = \text{Im} \left(\zeta_8 \left(\frac{1}{2} + i y \right) \right)$, $y \in [35, 70]$

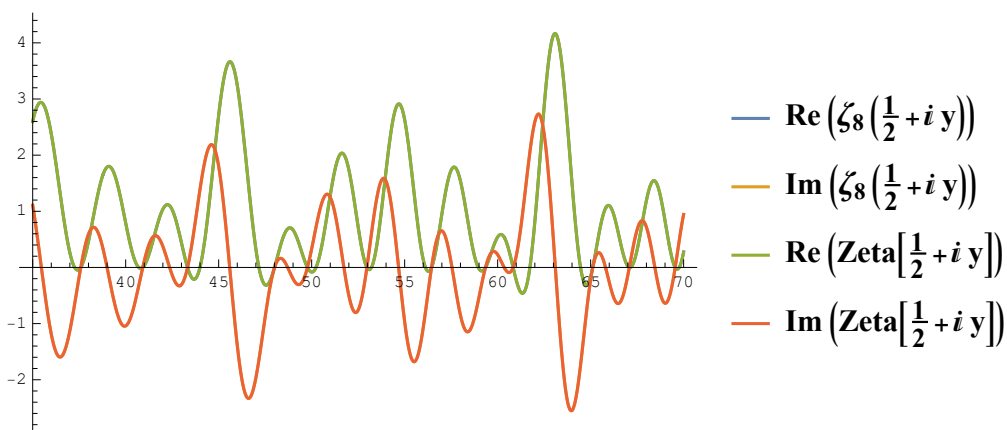


Fig. 15-7

An eight degree of approximation is appropriate for the interval $[1/2+i 35, 1/2+i 70]$.

I change to the function $(\text{zeta} - \hat{\text{zeta}})_{12}(\theta)$ for drawing in the interval $[1/2+i 70, 1/2+i 105]$ for accuracy.

Graph of $z = \text{Re}(\hat{\zeta}_{12}(\frac{1}{2} + iy))$ and $z = \text{Im}(\hat{\zeta}_{12}(\frac{1}{2} + iy))$, $y \in [70, 105]$

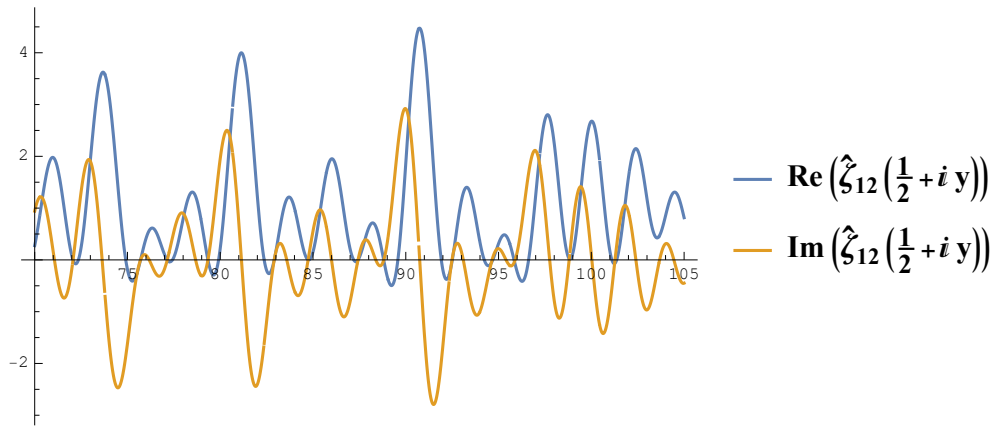


Fig. 15-8

It would be short on accuracy in the interval $[1/2 + i 95, 1/2 + i 105]$. The next graph is prepared to confirm this error.

Graph of $z = \text{Re}(\hat{\zeta}_{12}(\frac{1}{2} + iy))$, $z = \text{Im}(\hat{\zeta}_{12}(\frac{1}{2} + iy))$,
 $z = \text{Re}(\text{Zeta}[\frac{1}{2} + iy])$, and $z = \text{Im}(\text{Zeta}[\frac{1}{2} + iy])$, $y \in [95, 105]$

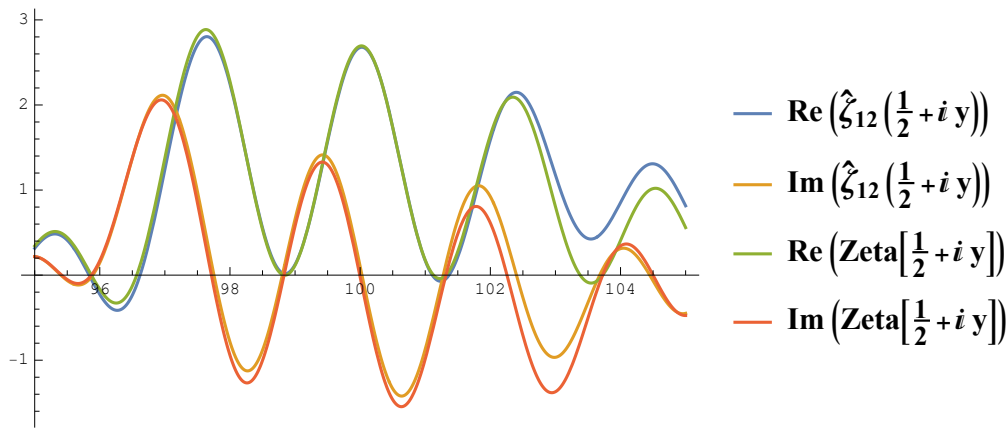


Fig. 15-9

From the above four graphs, I can recognize that in total, 29 non-trivial zeros of the zeta function lie on the interval $[1/2 + i 0, 1/2 + i 100]$.

Graph of $z = \zeta_5(x)$ and $z = \text{Zeta}[x]$, $x \in [-13, 10]$

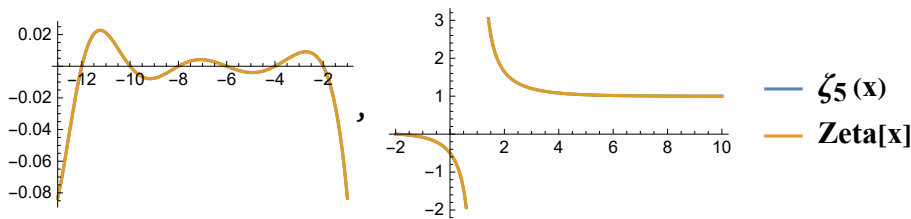


Fig. 15-10

I can see that the pole and the trivial zeros of the zeta function on the real axis are accurate.

Graph of $Z = \text{Re}(\tilde{\eta}_8(iy))$, $Z = \text{Im}(\tilde{\eta}_8(iy))$, and $Z = -3|\zeta_8(\frac{1}{2} + iy)|$, $y \in [0, 52]$

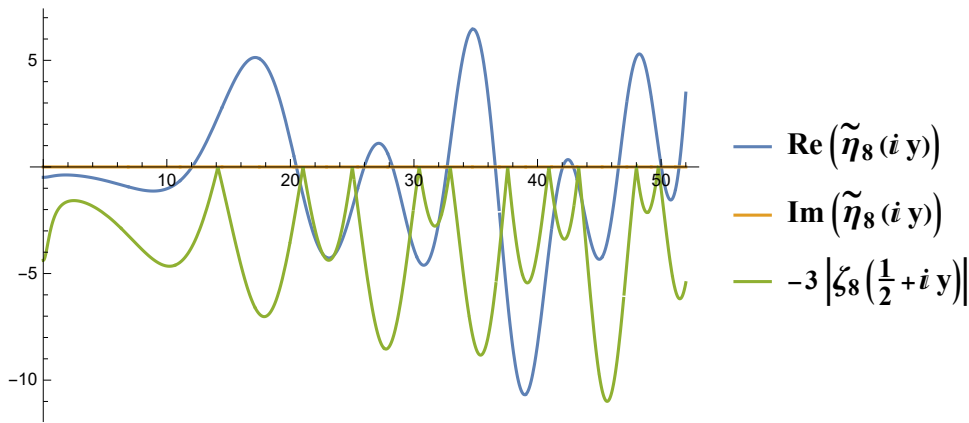


Fig. 15-11

Both the functions

$$\tilde{\eta}_8(iy) \text{ and } -3\left|\zeta_8\left(\frac{1}{2} + iy\right)\right|$$

are projected onto the same surface. I can also see the relation neither too close nor too far away between the imaginary parts of the zeta function's non-trivial zeros and those of the eta function. Since the gamma factor of the eta function is taken as an absolute value, the imaginary part of the function (eta - tilde) (θ) is zero anywhere on the imaginary axis. Function (eta - tilde) (θ) is observed to take real values on the imaginary axis; moreover, 11 non-trivial zeros of the eta function lie on the interval $[i0, i52]$.

Here are the details.

$$(15 - 19) \left\{ \text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-1.151464 \times 10^{-9}, 4.99152 \times 10^{-10}\}.$$

$$(15 - 20) \left\{ \text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{3.39148 \times 10^{-9}, -1.69660 \times 10^{-9}\}.$$

$$(15 - 21) \left\{ \text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-1.074615 \times 10^{-9}, 2.36501 \times 10^{-9}\}.$$

$$(15 - 22) \left\{ \text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{28\,269\,450\,283}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{28\,269\,450\,283}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{2.75628 \times 10^{-9}, -9.95323 \times 10^{-10}\}.$$

$$(15 - 23) \left\{ \text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{32\,685\,214\,209}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_9\left(i\left(\frac{32\,685\,214\,209}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-5.02927 \times 10^{-9}, 3.53524 \times 10^{-9}\}.$$

$$(15 - 24) \left\{ \text{Re}\left(\tilde{\eta}_{10}\left(i\left(\frac{36\,583\,986\,392}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_{10}\left(i\left(\frac{36\,583\,986\,392}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{7.85814 \times 10^{-9}, -4.74443 \times 10^{-9}\}.$$

$$(15 - 25) \left\{ \text{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,044\,079\,278}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,044\,079\,278}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-8.94336 \times 10^{-10}, 2.39905 \times 10^{-9}\}.$$

$$(15 - 26) \left\{ \text{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,901\,222\,688}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \text{Re}\left(\tilde{\eta}_{11}\left(i\left(\frac{42\,901\,222\,688}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{1.46300 \times 10^{-9}, -1.56310 \times 10^{-9}\}.$$

$$(15-27) \left\{ \operatorname{Re} \left(\tilde{\eta}_{11} \left(i \left(\frac{46\,556\,753\,041}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\eta}_{11} \left(i \left(\frac{46\,556\,753\,041}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-4.30492 \times 10^{-9}, 5.06314 \times 10^{-9}\}.$$

$$(15-28) \left\{ \operatorname{Re} \left(\tilde{\eta}_{11} \left(i \left(\frac{50\,021\,715\,160}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\eta}_{11} \left(i \left(\frac{50\,021\,715\,160}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{4.64200 \times 10^{-9}, -2.54722 \times 10^{-9}\}.$$

$$(15-29) \left\{ \operatorname{Re} \left(\tilde{\eta}_{11} \left(i \left(\frac{51\,457\,512\,850}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\tilde{\eta}_{11} \left(i \left(\frac{51\,457\,512\,850}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-5.47820 \times 10^{-9}, 3.82695 \times 10^{-9}\}.$$

Three - dimensional graph of $z = \operatorname{Re} \left(\tilde{\eta}_5(x + i y) \right)$ and $z = 0$, $x \in [-1, 1]$, $y \in [0, 35]$

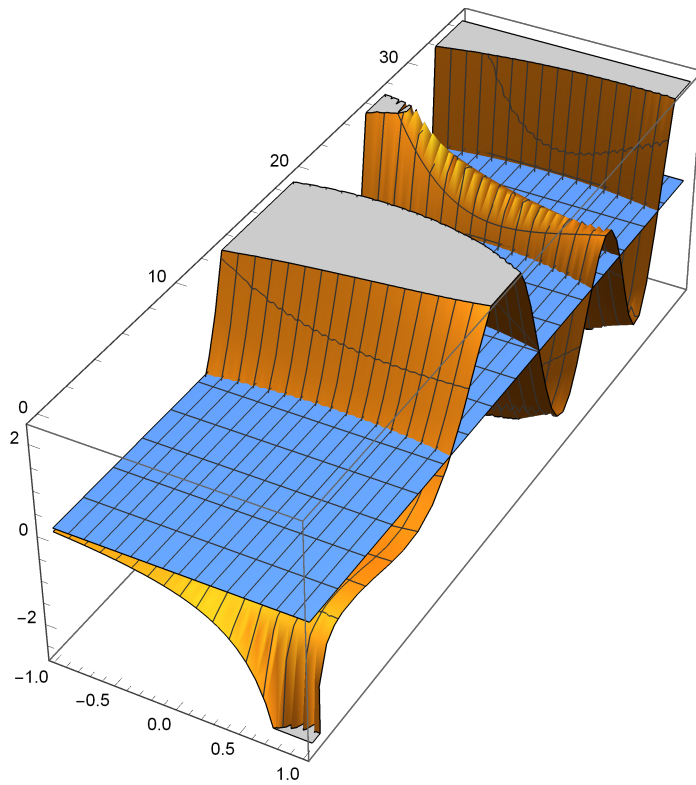


Fig. 15-12

The coast lines are candidate sites for the non-trivial zeros of the eta function.

Three – dimensional graph of $Z = \text{Im}(\tilde{\eta}_5(x + i y))$ and $Z = 0$, $x \in [-1, 1]$, $y \in [0, 35]$

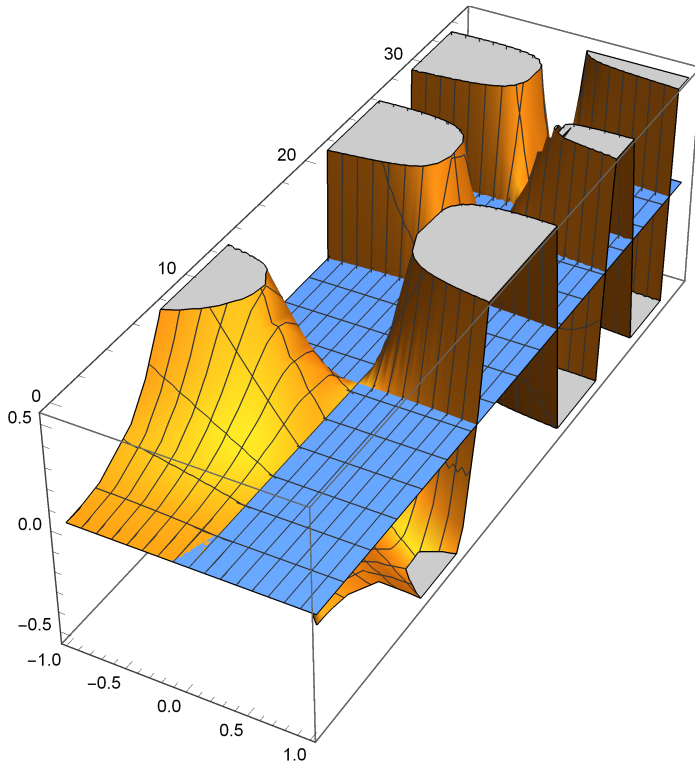


Fig. 15-13

On the intricate surface of the sea, the imaginary axis is clearly visible. The intersections of the coast lines and the imaginary axis in the following two graphs are the non-trivial zeros. Here, the imaginary axis corresponds to the critical line of the eta function.

Graph of $Z = \text{Re}(\eta_5(i y))$ and $Z = \text{Im}(\eta_5(i y))$, $y \in [0, 35]$

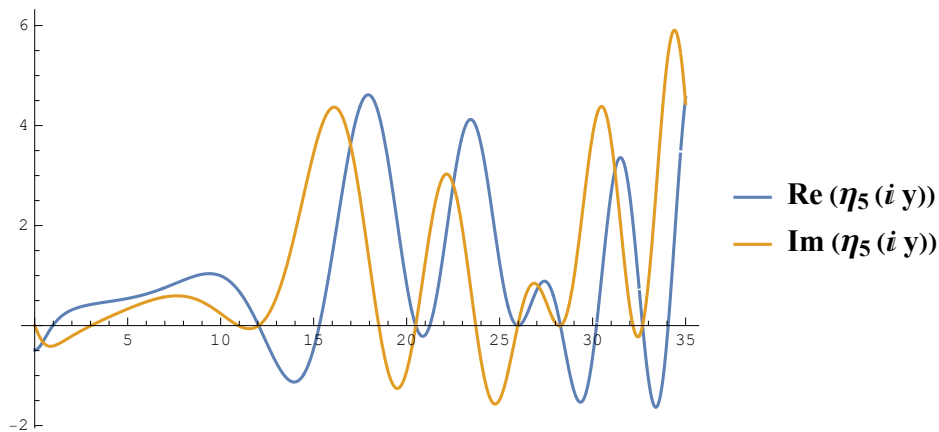


Fig. 15-14

The eta function on the imaginary axis has a similar shape as that of the zeta function on the critical line.

Graph of $Z = \eta_5(x)$ and $Z = \text{Zeta}[x]$, $x \in [-13, 10]$

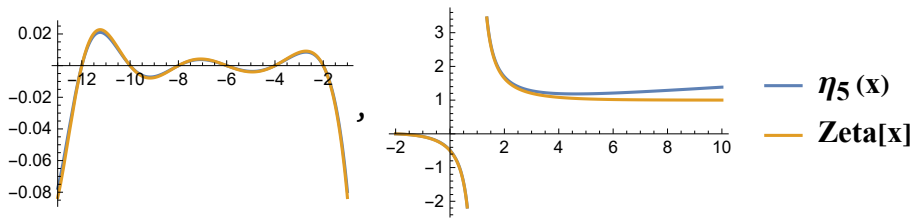


Fig. 15-15

On the real axis, I can also see the pole and the trivial zeros of the eta function.
 On the real axis, there is a little difference between the eta and the zeta functions.

16. Similarities and differences between the zeta and the eta functions, and other knowledge

16-1. Similarities and differences between the zeta and the eta functions

The explicit formulae for the zeta and the eta functions are shown again.

$$(16-1) \quad \zeta(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + (7-\theta)(2\pi p)^2) K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 2(2\pi p)^2)(2\pi p) K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

$$(16-2) \quad \eta(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

An easily recognized common point here is that they have the same gamma factor.
 Thus, having only a simple pole at $\theta = 1$ is one of the common points.

The residue of the eta function at $\theta = 1$ is as follows:

$$(16-3) \quad \text{Res}(1; \eta(\theta)) = (\theta-1)\eta(\theta)|_{\theta=1} = (\theta-1)\zeta(\theta)|_{\theta=1} = \text{Res}(1; \zeta(\theta)) = 1.$$

The calculation is performed for the second term of equation (16-3).

$$(16-4) \quad (\theta-1)\eta(\theta)|_{\theta=1} = \frac{4\pi^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-1}(p) p^{\frac{1}{2}} ((2\pi p) - 3)(2\pi p)^2 K_{\frac{1}{2}}(2\pi p) \\ = \frac{8\pi^{\frac{1}{2}}}{\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-1}(p) p^{\frac{1}{2}} ((2\pi p) - 3)(2\pi p)^2 \sqrt{\frac{\pi}{2(2\pi p)}} e^{-2\pi p} \\ = 4 \sum_{p=1}^{\infty} \sigma_{-1}(p) ((2\pi p) - 3)(2\pi p)^2 e^{-2\pi p}.$$

Thus, the following infinite series that gives the algebraic number 1/4 is obtained:

$$(16-5) \quad \sum_{p=1}^{\infty} \sigma_{-1}(p) ((2\pi p) - 3)(2\pi p)^2 e^{-2\pi p} = \frac{1}{4}.$$

In addition, another common point is that they have trivial zeros for any even number of -2 or less.

That is

$$(16-6) \quad \zeta(-2k) = \eta(-2k) = 0 \quad (k \in \mathbb{N}).$$

When k is assumed to be any positive integer again, the value of $\zeta(1-2k)$ corresponds to the value of $\zeta(2k)$ for the zeta function due to the point symmetry of the Riemann Xi function. In contrast, in the case of the eta function, the value of $\eta(-2k-1)$ corresponds to the value of $\eta(2k+1)$ due to the origin symmetry of the Chi function.

Furthermore, the mathematical phrase can define any relation between of $\eta(-2k-1)$ and $\eta(2k+1)$. The modified equation is performed to calculate the function value on the real axis of the eta function (16-2), which is obtained earlier (14-4).

$$(16-7) \quad \eta(\theta) = \frac{2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left((7-\theta)(2\pi p)^2 (\pi p)^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi p) - (2\pi p)^4 (\pi p)^{\frac{\theta-2}{2}} K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

When θ is any odd number of 3 or more, the following integral representation is used:

$$(16-8) \quad \eta(\theta) = \frac{2}{(1-\theta)\Gamma\left(1+\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left((7-\theta)(2\pi p)^2 \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx - (2\pi p)^4 \int_0^{\infty} x^{\theta-3} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

For equation (16-2), $-\theta$ is substituted for θ , which is obtained earlier as equation (14-7).

$$(16-9) \quad \eta(-\theta) = \frac{2\pi^{-\frac{\theta}{2}}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((7-\theta)(2\pi p)^2 K_{\frac{\theta}{2}}(2\pi p) - 2(2\pi p)^3 K_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

The modification is performed for equation (16-9).

$$(16-10) \quad \eta(-\theta) = \frac{2\pi^{-\theta}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left((7-\theta)(2\pi p)^2 (\pi p)^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\pi p) - (2\pi p)^4 (\pi p)^{\frac{\theta-2}{2}} K_{\frac{\theta-2}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

When θ is any odd number of 1 or more, the following integral representation is used:

$$(16-11) \quad \eta(-\theta) = \frac{2\pi^{-\theta}}{(1+\theta)\Gamma\left(1-\frac{\theta}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) \left((7-\theta)(2\pi p)^2 \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx - (2\pi p)^4 \int_0^{\infty} x^{\theta-3} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \quad (\theta \in \mathbb{C} \setminus \{-1\}).$$

For equation (16-11), 1 is substituted for θ .

$$\begin{aligned}
 (16-12) \quad \eta(-1) &= \frac{2\pi^{-1}}{2\Gamma\left(1-\frac{1}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(6(2\pi p)^2 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (2\pi p)^4 \int_0^{\infty} x^{-2} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= \frac{1}{\pi\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-1}(p) \left(6(2\pi p)^2 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi p} - (2\pi p)^4 \cdot \frac{\sqrt{\pi}}{2\pi p} e^{-2\pi p} \right).
 \end{aligned}$$

The representation for $\eta(-1)$ and its exact value are shown.

$$(16-13) \quad \eta(-1) = -\frac{1}{\pi} \sum_{p=1}^{\infty} \sigma_{-1}(p) ((2\pi p) - 3) (2\pi p)^2 e^{-2\pi p} = -\frac{1}{4\pi}.$$

For equation (16-11), 3 is substituted for θ .

$$\begin{aligned}
 (16-14) \quad \eta(-3) &= \frac{2\pi^{-3}}{4\Gamma\left(1-\frac{3}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(4(2\pi p)^2 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (2\pi p)^4 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= -\frac{1}{4\pi^3\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(4(2\pi p)^2 \cdot \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} - (2\pi p)^4 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi p} \right).
 \end{aligned}$$

The representation for $\eta(-3)$ and its approximate value are shown.

$$(16-15) \quad \eta(-3) = \frac{1}{8\pi^3} \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p)^2 - 2(2\pi p) - 2) (2\pi p)^2 e^{-2\pi p} \simeq 0.0077338931815.$$

In addition, in the equation (16-8), 3 is substituted for θ .

$$\begin{aligned}
 (16-16) \quad \eta(3) &= \frac{2}{-2\Gamma\left(1+\frac{3}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(4(2\pi p)^2 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (2\pi p)^4 \int_0^{\infty} x^0 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= -\frac{4}{3\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-3}(p) \left(4(2\pi p)^2 \cdot \frac{(2\pi p)+1}{4} \sqrt{\pi} e^{-2\pi p} - (2\pi p)^4 \cdot \frac{\sqrt{\pi}}{2} e^{-2\pi p} \right) \\
 &= \frac{2}{3} \sum_{p=1}^{\infty} \sigma_{-3}(p) ((2\pi p)^2 - 2(2\pi p) - 2) (2\pi p)^2 e^{-2\pi p} \simeq 1.2789292363.
 \end{aligned}$$

Equations (16-15) and (16-16) are integrated to obtain the relation between $\eta(3)$ and $\eta(-3)$.

$$(16-17) \quad \eta(3) = \frac{16\pi^3}{3} \eta(-3).$$

For equation (16-11), 5 is substituted for θ .

$$\begin{aligned}
 (16-18) \quad \eta(-5) &= \frac{2\pi^{-5}}{6\Gamma\left(1-\frac{5}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(2(2\pi p)^2 \int_0^{\infty} x^4 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (2\pi p)^4 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= \frac{1}{4\pi^5 \sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(2(2\pi p)^2 \cdot \frac{(2\pi p)^2 + 3(2\pi p) + 3}{8} \sqrt{\pi} e^{-2\pi p} \right. \\
 &\quad \left. - (2\pi p)^4 \cdot \frac{(2\pi p) + 1}{4} \sqrt{\pi} e^{-2\pi p} \right) \\
 &= -\frac{1}{16\pi^5} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left((2\pi p)^3 - 3(2\pi p) - 3 \right) (2\pi p)^2 e^{-2\pi p} \simeq -0.0036345168378.
 \end{aligned}$$

For equation (16-8), 5 is substituted for θ .

$$\begin{aligned}
 (16-19) \quad \eta(5) &= \frac{2}{-4\Gamma\left(1+\frac{5}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(2(2\pi p)^2 \int_0^{\infty} x^4 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\
 &\quad \left. - (2\pi p)^4 \int_0^{\infty} x^2 e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\
 &= -\frac{4}{15\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left(2(2\pi p)^2 \cdot \frac{(2\pi p)^2 + 3(2\pi p) + 3}{8} \sqrt{\pi} e^{-2\pi p} \right. \\
 &\quad \left. - (2\pi p)^4 \cdot \frac{(2\pi p) + 1}{4} \sqrt{\pi} e^{-2\pi p} \right) \\
 &= \frac{1}{15} \sum_{p=1}^{\infty} \sigma_{-5}(p) \left((2\pi p)^3 - 3(2\pi p) - 3 \right) (2\pi p)^2 e^{-2\pi p} \simeq 1.1863826102.
 \end{aligned}$$

Equations (16-18) and (16-19) are integrated to obtain the relation between $\eta(5)$ and $\eta(-5)$.

$$(16-20) \quad \eta(5) = -\frac{16\pi^5}{15} \eta(-5).$$

Hereafter, the calculations can be performed similarly.

In cases where $\theta = 7, 9, 11, 13, 15,$ and 17 , each representation by the elementary functions and the approximate value are shown all together.

$$\begin{aligned}
 (16-21) \quad \eta(7) &= \frac{512\pi^7}{4725} \eta(-7) \\
 &= \frac{2}{315} \sum_{p=1}^{\infty} \sigma_{-7}(p) \left((2\pi p)^2 + 3(2\pi p) + 3 \right) (2\pi p)^4 e^{-2\pi p} \simeq 1.2460084172.
 \end{aligned}$$

$$\begin{aligned}
 (16-22) \quad \eta(9) &= -\frac{128\pi^9}{19845} \eta(-9) \\
 &= \frac{1}{1890} \sum_{p=1}^{\infty} \sigma_{-9}(p) \left((2\pi p)^5 + 7(2\pi p)^4 + 25(2\pi p)^3 + 60(2\pi p)^2 \right. \\
 &\quad \left. + 105(2\pi p) + 105 \right) (2\pi p)^2 e^{-2\pi p} \simeq 1.3360607156.
 \end{aligned}$$

$$(16-23) \quad \eta(11) = \frac{4096\pi^{11}}{16372125} \eta(-11)$$

$$\begin{aligned}
 &= \frac{2}{51975} \sum_{p=1}^{\infty} \sigma_{-11}(p) \left((2\pi p)^6 + 12(2\pi p)^5 + 75(2\pi p)^4 + 315(2\pi p)^3 + 945(2\pi p)^2 \right. \\
 &\quad \left. + 1890(2\pi p) + 1890 \right) (2\pi p)^2 e^{-2\pi p} \simeq 1.4336203440. \\
 (16-24) \quad \eta(13) &= -\frac{4096\pi^{13}}{602026425} \eta(-13) \\
 &= \frac{1}{405405} \sum_{p=1}^{\infty} \sigma_{-13}(p) \left((2\pi p)^7 + 18(2\pi p)^6 + 168(2\pi p)^5 + 1050(2\pi p)^4 + 4725(2\pi p)^3 \right. \\
 &\quad \left. + 15120(2\pi p)^2 + 31185(2\pi p) + 31185 \right) (2\pi p)^2 e^{-2\pi p} \simeq 1.5315820258. \\
 (16-25) \quad \eta(15) &= \frac{262144\pi^{15}}{1917454163625} \eta(-15) \\
 &= \frac{2}{14189175} \sum_{p=1}^{\infty} \sigma_{-15}(p) \left((2\pi p)^8 + 25(2\pi p)^7 + 322(2\pi p)^6 + 2772(2\pi p)^5 + 17325(2\pi p)^4 \right. \\
 &\quad \left. + 79695(2\pi p)^3 + 259875(2\pi p)^2 + 540540(2\pi p) + 540540 \right) (2\pi p)^2 e^{-2\pi p} \simeq 1.6272471919. \\
 (16-26) \quad \eta(17) &= -\frac{16384\pi^{17}}{7761123995625} \eta(-17) \\
 &= \frac{1}{137837700} \sum_{p=1}^{\infty} \sigma_{-17}(p) \left((2\pi p)^9 + 33(2\pi p)^8 + 558(2\pi p)^7 + 6300(2\pi p)^6 + 51975(2\pi p)^5 \right. \\
 &\quad \left. + 322245(2\pi p)^4 + 1486485(2\pi p)^3 + 4864860(2\pi p)^2 \right. \\
 &\quad \left. + 10135125(2\pi p) + 10135125 \right) (2\pi p)^2 e^{-2\pi p} \simeq 1.7195887751.
 \end{aligned}$$

In the domain of $\theta > 1$ on the real axis, the zeta function is strictly monotonically decreasing. Conversely, the eta function has a minimal value at $\theta > 1$ in the same domain. Although the eta function is strictly monotonically decreasing in $1 < \theta < K$, it turns to be strictly monotonically increasing in $\theta > K$.

The range of the minimal value and K value (that gives the minimal value) are shown.

$$(16-27) \quad 1.183937412932 < \eta(K) < 1.183937412933 \quad (4.62270 < K < 4.62271).$$

The next graph shows the non-trivial zeros of both the eta and zeta functions.

Graph of the non-trivial zeros of the eta function and those of the zeta function

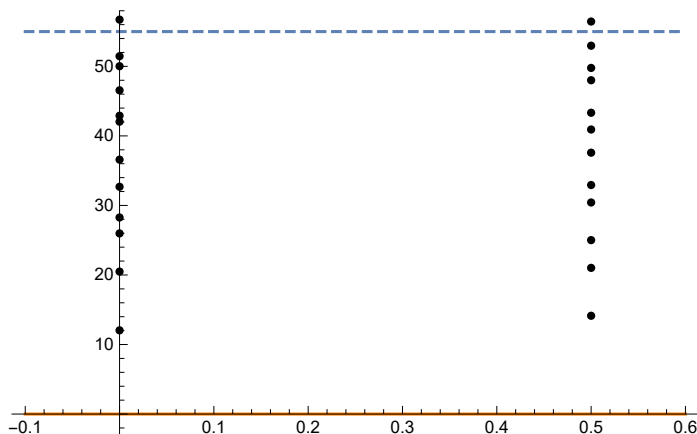


Fig. 16-1

The dots on the imaginary axis indicate the eta function's non-trivial zeros, while those on the critical line indicate the zeta function's non-trivial zeros.

16-2. The general representation for the eta function for any odd number of 3 or more

For equation (16-8), any odd number of 3 or more is substituted for θ .

$$(16-28) \quad \eta(2k+1) =$$

$$\begin{aligned} & \frac{1}{2k\Gamma\left(1+\frac{2k+1}{2}\right)} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left((2k-6)(2\pi p)^2 \int_0^{\infty} x^{2k} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ & \quad \left. + (2\pi p)^4 \int_0^{\infty} x^{2(k-1)} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \\ & = \frac{2^{k+1}}{k(2k+1)!!\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(2(k-3)(2\pi p)^2 \int_0^{\infty} x^{2k} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right. \\ & \quad \left. + (2\pi p)^4 \int_0^{\infty} x^{2(k-1)} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx \right) \quad (k \in \mathbb{N}). \end{aligned}$$

The general representation for the two integrals of equation (16-28) is given by the description in the subsection 19-4 i.e.,

$$(16-29) \quad \int_0^{\infty} x^{2k} e^{-\left(x^2 + \frac{(\pi p)^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^{\mu} \sqrt{\pi} e^{-2\pi p} \quad (p, k \in \mathbb{N}).$$

Where

$$(16-30) \quad a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \quad (k \in \mathbb{N}, \mu = 0, 1, \dots, k).$$

And the coefficients deserving special mention are as follows:

$$(16-31) \quad a_{k,k} = 1 \quad (k \in \mathbb{N}).$$

$$(16-32) \quad a_{k,1} = a_{k,0} = (2k-1)!! \quad (k \in \mathbb{N}).$$

The above general representation for the two integrals is introduced into equation (16-28).

$$(16-33) \quad \eta(2k+1) =$$

$$\begin{aligned} & \frac{2^{k+1}}{k(2k+1)!!\sqrt{\pi}} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(2(k-3)(2\pi p)^2 \cdot \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^{\mu} \sqrt{\pi} e^{-2\pi p} \right. \\ & \quad \left. + (2\pi p)^4 \cdot \frac{1}{2^k} \sum_{\mu=0}^{k-1} a_{k-1,\mu} (2\pi p)^{\mu} \sqrt{\pi} e^{-2\pi p} \right) \\ & = \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \\ & \quad \left((k-3) \sum_{\mu=0}^k a_{k,\mu} (2\pi p)^{\mu} + \sum_{\mu=2}^{k+1} a_{k-1,\mu-2} (2\pi p)^{\mu} \right) (2\pi p)^2 e^{-2\pi p} \quad (k \in \mathbb{N}). \end{aligned}$$

Therefor,

$$(16-34) \eta(2k+1) = \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(\mathbf{a}_{k-1,k-1} (2\pi p)^{k+1} + \sum_{\mu=2}^k (\mathbf{a}_{k-1,\mu-2} + (k-3)\mathbf{a}_{k,\mu}) (2\pi p)^\mu + (k-3)\mathbf{a}_{k,1} (2\pi p) + (k-3)\mathbf{a}_{k,0} \right) (2\pi p)^2 e^{-2\pi p} \quad (k \in \mathbb{N}).$$

From the definition of $\mathbf{a}_{k,\mu}$,

$$(16-35) \mathbf{a}_{k-1,k-1} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \Big|_{k=k-1, \mu=k-1} = 1 \quad (k \in \mathbb{N}).$$

$$(16-36) (k-3)\mathbf{a}_{k,1} = (k-3)\mathbf{a}_{k,0} = (k-3)(2k-1)!! \quad (k \in \mathbb{N}).$$

The above results are introduced into equation (16-34).

$$(16-37) \eta(2k+1) = \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left((2\pi p)^{k+1} + \sum_{\mu=2}^k (\mathbf{a}_{k-1,\mu-2} + (k-3)\mathbf{a}_{k,\mu}) (2\pi p)^\mu + (k-3)(2k-1)!! (2\pi p) + (k-3)(2k-1)!! \right) (2\pi p)^2 e^{-2\pi p} \quad (k \in \mathbb{N}).$$

Here, new coefficients $d_{k,\mu}$ are used for the general representation for the eta function for any odd number of 3 or more.

The coefficients $d_{k,\mu}$ are determined by the following equation of definition:

$$(16-38) \eta(2k+1) := \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left((2\pi p)^{k+1} + \sum_{\mu=2}^k d_{k,\mu} (2\pi p)^\mu + (k-3)(2k-1)!! (2\pi p) + (k-3)(2k-1)!! \right) (2\pi p)^2 e^{-2\pi p} \quad (k \in \mathbb{N} \setminus \{1\}).$$

The coefficients $d_{k,k+1}$, $d_{k,1}$, and $d_{k,0}$ are immediately determined as follows:

$$(16-39) \mathbf{d}_{k,k+1} = 1 \quad (k \in \mathbb{N}).$$

$$(16-40) \mathbf{d}_{k,1} = \mathbf{d}_{k,0} = (k-3)(2k-1)!! \quad (k \in \mathbb{N}).$$

The coefficients $d_{k,\mu}$ are calculated as follows:

$$(16-41) \mathbf{d}_{k,\mu} = \mathbf{a}_{k-1,\mu-2} + (k-3)\mathbf{a}_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \Big|_{k=k-1, \mu=\mu-2} + (k-3) \cdot \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} = \frac{\mu^2 + 5\mu - 6 - 2k(\mu+2) + 2k^2}{2^{k-\mu+1} \mu! (k-\mu+1)!} (2k-\mu)! \quad (k \in \mathbb{N} \setminus \{1\}, \mu = 2, 3, \dots, k).$$

For equation (16-41), 1 is substituted for k, and 2 is substituted for μ .

$$(16-42) \mathbf{d}_{1,2} = 1.$$

This result is equivalent to the result of substituting 1 for k in equation (16-39).

For equation (16-40), 1 is substituted for k, and 0 and 1 are substituted for μ .

$$(16-43) \{\mathbf{d}_{1,1}, \mathbf{d}_{1,0}\} = \{-2, -2\}.$$

This results are equivalent to the results of substituting 1 for k in equations (16-40).

Thus, the right side of equation (16-41) holds even for the case when $k = 1$.

$$(16-44) \quad d_{k,\mu} = \frac{\mu^2 + 5\mu - 6 - 2k(\mu + 2) + 2k^2}{2^{k-\mu+1} \mu! (k-\mu+1)!} (2k-\mu)! \quad (k \in \mathbb{N}, \mu = 2, 3, \dots, k).$$

Combining the above results, I arrive at the conclusions in this subsection.

As a result, the general representation for the eta function for any odd number of 3 or more is obtained.

$$(16-45) \quad \eta(2k+1) = \frac{2}{k(2k+1)!!} \sum_{p=1}^{\infty} \sigma_{-(2k+1)}(p) \left(\sum_{\mu=0}^{k+1} d_{k,\mu} (2\pi p)^\mu \right) (2\pi p)^2 e^{-2\pi p} \quad (k \in \mathbb{N}).$$

Where

$$(16-46) \quad d_{k,\mu} = \frac{\mu^2 + 5\mu - 6 - 2k(\mu + 2) + 2k^2}{2^{k-\mu+1} \mu! (k-\mu+1)!} (2k-\mu)! \quad (k \in \mathbb{N}, \mu = 0, 1, \dots, k+1).$$

And the coefficients deserving special mention are shown as follows:

$$(16-47) \quad d_{k,k+1} = 1 \quad (k \in \mathbb{N}).$$

$$(16-48) \quad d_{k,k} = \frac{(k-2)(k+3)}{2} \quad (k \in \mathbb{N}).$$

$$(16-49) \quad d_{k,1} = d_{k,0} = (k-3)(2k-1)!! \quad (k \in \mathbb{N}).$$

The coefficients of the infinite sum of the function $\eta(2k+1)$ are the same to those of the function $\zeta(2k+1)$ without a leading term.

Table of the coefficients $d_{k,\mu}$

k	$d_{k,9}$	$d_{k,8}$	$d_{k,7}$	$d_{k,6}$	$d_{k,5}$	$d_{k,4}$	$d_{k,3}$	$d_{k,2}$	$d_{k,1}$	$d_{k,0}$
1								1	-2	-2
2							1	0	-3	-3
3						1	3	3	0	0
4					1	7	25	60	105	105
5				1	12	75	315	945	1890	1890
6			1	18	168	1050	4725	15120	31185	31185
7		1	25	322	2772	17325	79695	259875	540540	540540
8	1	33	558	6300	51975	322245	1486485	4864860	10135125	10135125

Fig. 16-2

16-3. Observational results about relationships between prime numbers and the zeta function

This subsection describes the known relationships between the prime numbers and the zeta function that are necessary “[16].” The purpose is to compare the relationships between the prime numbers and the eta function. Let consider the second Chebyshev function “[19].”

$$(16-50) \quad \Psi(x) := \sum_{\substack{p^k \leq x \\ p: \text{prime}, k \in \mathbb{N}}} \log(p) \quad (x > 0).$$

The i-th prime number is assumed to be p_i . This function is a step function where the function value steps up by $\log(p_i)$ as the positive real variable x continuously increases and reaches the k -th power of the prime number p_i . Until it reaches the first prime number 2, the function value on the interval of $0 < x < 2$ is zero.

This function can be written using the Mangoldt function $\Lambda(n)$.

$$(16-51) \quad \Psi(x) = \sum_{n \leq x} \Lambda(n) \quad (x > 0).$$

Where

$$(16-52) \quad \Lambda(n) := \begin{cases} \log(p) & n = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Psi^*(x)$ be the modified function so that its value at the step-up point corresponds to the midpoint.

$$(16-53) \quad \Psi^*(x) := \frac{1}{2} \lim_{h \rightarrow 0} (\Psi(x+h) + \Psi(x-h)) \quad (x > 0).$$

The modified second Chebyshev function is described by the second Chebyshev function and the Mangoldt function as follows:

$$(16-54) \quad \Psi^*(x) = \begin{cases} \Psi(x) - \frac{1}{2} \Lambda(x) & x = 2, 3, 2^2, 5, 7, 2^3, 3^2, 11, 13, 2^4, 17, \dots, \\ \Psi(x) & \text{otherwise.} \end{cases}$$

The Riemann-von Mangoldt explicit formula states that the modified second Chebyshev function is described by the information on the zeta function as follows:

$$(16-55) \quad \Psi^*(x) = S(x) - \sum_{\rho: \text{non-trivial zero}} \frac{x^\rho}{\rho} \quad (x > 0).$$

Here, $s(x)$ is called the approximate term and is given by the following equations:

$$(16-56) \quad S(x) = x - \frac{\log(1-x^{-2})}{2} - \frac{\zeta'(0)}{\zeta(0)} \quad (x > 0).$$

$$(16-57) \quad \frac{\zeta'(0)}{\zeta(0)} = \log(2\pi) \approx 1.8378770664.$$

The periodic terms are the second term on the right side of equation (16-55), and the sum of the periodic terms crosses all non-trivial zeros of the zeta function.

The function $\Psi_N(x)$ is defined such that the infinite sum of periodic terms is restricted to the sum of N matching pairs. Here, the matching pair consists of a non-trivial zero and its complex conjugate.

$$(16-58) \quad \Psi_N(x) := S(x) - \sum_{m=1}^N \left(\frac{x^{1/2+i\rho_m}}{1/2+i\rho_m} + \frac{x^{1/2-i\rho_m}}{1/2-i\rho_m} \right) \quad (x > 0, N \in \mathbb{N}).$$

Here, ρ_m is the positive imaginary part of the m -th non-trivial zero of the zeta function to which the number is allocated in an order that is closer to the real axis.

The values of the 201 non-trivial zeros of the zeta function were calculated with 25-decimal accuracy for the following drawing "[15]." The numerical data will be presented at the end of this section.

Graph of the approximate term $S(x)$ and $\Psi_{200}(x)$ on the interval $[3/2, 30]$

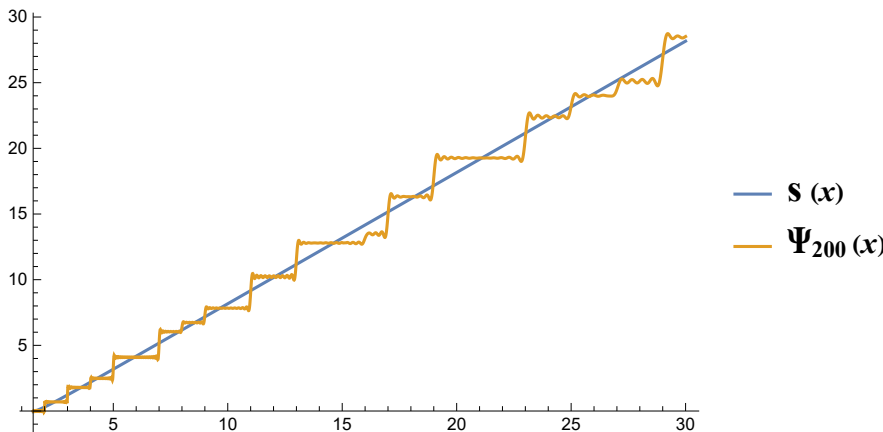


Fig. 16-3

Graph of the approximate term $S(x)$ and $\Psi_{200}(x)$ on the interval $[3/2, 120]$

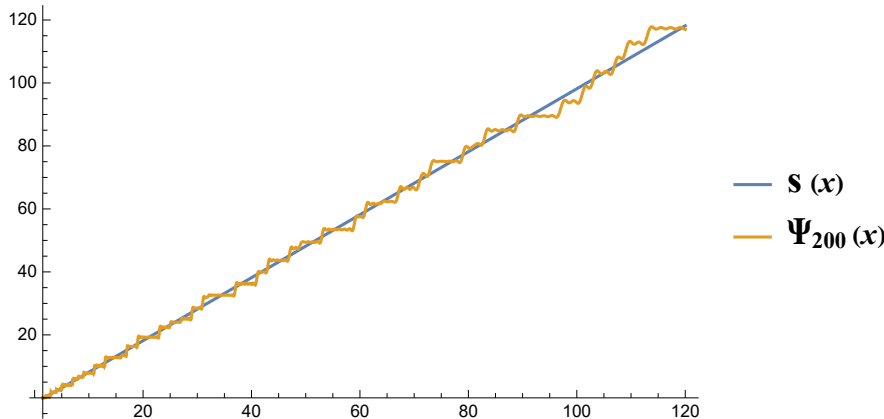


Fig. 16-4

It is shown that the step function is generated from the information on the non-trivial zeros of the zeta function.

As a main spectrum, the imaginary parts of the non-trivial zeros of the zeta function generate prime numbers. In principle, the points that the step function steps up (i.e., the positive integer power of the prime numbers) can be extracted by differentiating the periodic term of $\Psi_N(x)$ by x .

According to the known result, the positive integer powers of the prime numbers are given as the maximal values of the following function $\Phi_N(x)$, which is generated using the information on finite numbers of the imaginary parts of the non-trivial zeros of the zeta function:

$$(16-59) \quad \Phi_N(x) := -\frac{d}{dx} \sum_{m=1}^N \left(\frac{x^{1/2+i\rho_m}}{1/2+i\rho_m} + \frac{x^{1/2-i\rho_m}}{1/2-i\rho_m} \right)$$

$$= -\frac{2}{\sqrt{x}} \sum_{m=1}^N \cos(\rho_m \log(x)) \quad (x > 0, N \in \mathbb{N}).$$

Since the maximal values at the square or more of a sufficiently large prime number tend to be sufficiently small in comparison with the maximal value at the first power of the prime number, I use the expression “main spectrum.”

Graph of $\Phi_{50}(x)$, $\Phi_{100}(x)$, and $\Phi_{200}(x)$ on the interval $[3/2, 15]$

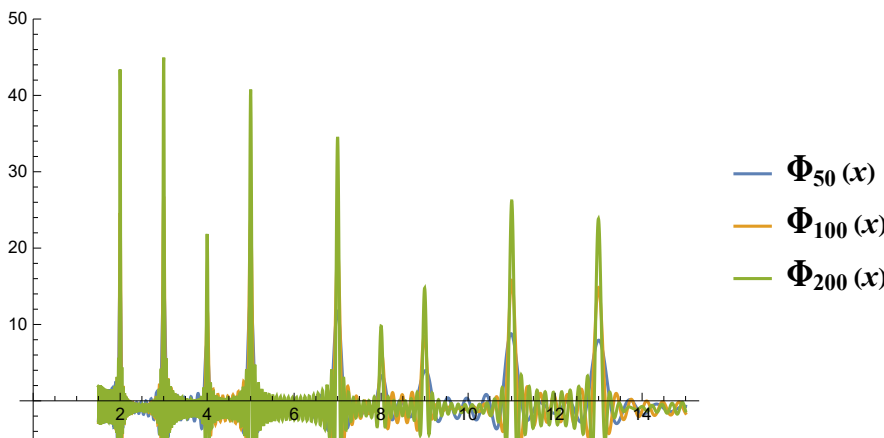


Fig. 16-5

The peaks are observed at the points where x is one of the prime numbers 2, 3, 5, 7, 11, and 13; 4 and 9 are the squares of prime numbers 2 and 3, respectively; and 8 is the cube of prime number 2.

Graph of $\Phi_{50}(x)$, $\Phi_{100}(x)$, and $\Phi_{200}(x)$ on the interval [15, 50]

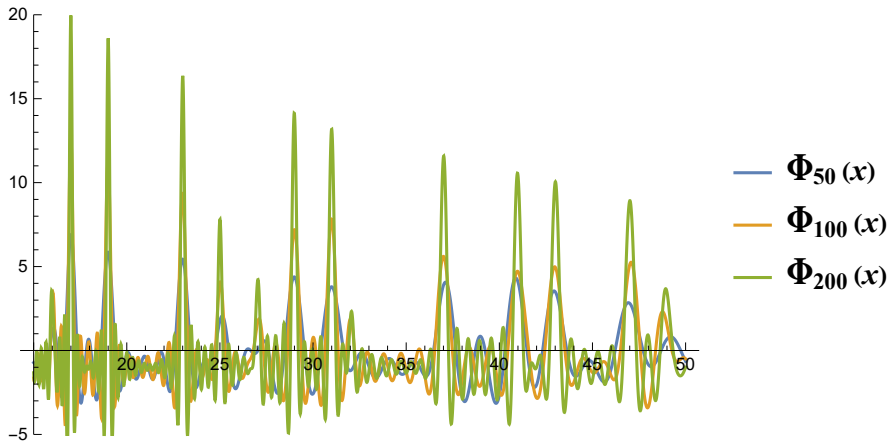


Fig. 16-6

The peaks are found at the points where x is one of the following prime numbers: 17, 19, 23, 29, 31, 37, 41, 43, and 47; 25 and 49 are the squares of prime numbers 5 and 7, respectively; 27 is the cube of prime number 3; and 16 and 32 are the fourth and fifth powers of prime number 2, respectively.

Graph of $\Phi_{50}(x)$, $\Phi_{100}(x)$, and $\Phi_{200}(x)$ on the interval [50, 100]

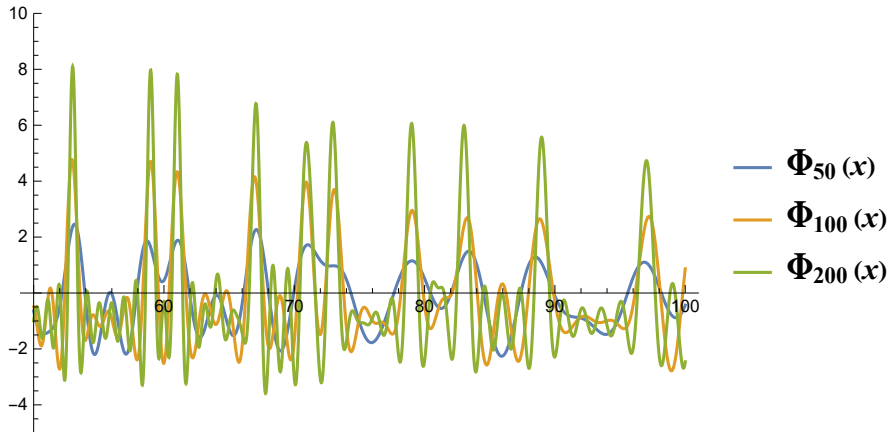


Fig. 16-7

Peaks are recognized where x is one of the following prime numbers: 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97. However, at the points where x is 64 (the sixth power of prime number 2) and 81 (the fourth power of prime number 3), the peak values are so small that their detection is difficult. The imaginary parts of the non-trivial zeros of the zeta function are proven to generate prime numbers as a main spectrum.

When I consider another relation between the prime numbers and the zeta function's non-trivial zeros, the following equation, whose peaks correspond to the imaginary parts of the non-trivial zeros, is also known as "[16]":

$$(16-60) \Omega_C(y) := - \sum_{\substack{p^n \leq C \\ p: \text{prime}, n \in \mathbb{N}}} \frac{\log(p)}{p^n} \cos(y \log(p^n)) \quad (y \in \mathbb{R}, C \in \mathbb{N}).$$

Here, the sum crosses all the integers for which the positive integer powers of the conditional prime number p are less than or equal to the sufficiently large arbitrary positive integer C .

Assuming the k -th prime number as p_k , the conditional partial sum is given as follows:

$$(16-61) \omega_C(y, m, n) := - \sum_{k=1}^m \frac{\log(p_k)}{(p_k)^n} \cos(y \log((p_k)^n)) \quad (C, m, n \in \mathbb{N}, y \in \mathbb{R}, (p_m)^n \leq C).$$

For the positive integer C , 3 numbers (1260, 110 880, and 10 810 800) are selected to display graphs. Some prime numbers p_k for the calculations are shown as follows:

$$(16-62) \{p_1, p_2, p_3, p_4, p_5, p_{205}, p_{10\,522}, p_{714\,823}\} = \{2, 3, 5, 7, 11, 1259, 110\,879, 10\,810\,763\}.$$

When $C = 1260$,

$$(16-63) \Omega_{1260}(y) = \omega_{1260}(y, 205, 1) + \omega_{1260}(y, 11, 2) + \omega_{1260}(y, 4, 3) + \omega_{1260}(y, 3, 4) \\ + \sum_{n=5}^6 \omega_{1260}(y, 2, n) + \sum_{n=7}^{10} \omega_{1260}(y, 1, n) \quad (y \in \mathbb{R}).$$

When $C = 110\,880$,

$$(16-64) \Omega_{110\,880}(y) = \omega_{110\,880}(y, 10\,522, 1) + \omega_{110\,880}(y, 67, 2) + \omega_{110\,880}(y, 15, 3) \\ + \omega_{110\,880}(y, 7, 4) + \omega_{110\,880}(y, 4, 5) + \sum_{n=6}^7 \omega_{110\,880}(y, 3, n) + \sum_{n=8}^{10} \omega_{110\,880}(y, 2, n) \\ + \sum_{n=11}^{16} \omega_{110\,880}(y, 1, n) \quad (y \in \mathbb{R}).$$

When $C = 10\,810\,800$,

$$(16-65) \Omega_{10\,810\,800}(y) = \omega_{10\,810\,800}(y, 714\,823, 1) + \omega_{10\,810\,800}(y, 462, 2) + \omega_{10\,810\,800}(y, 47, 3) \\ + \omega_{10\,810\,800}(y, 16, 4) + \omega_{10\,810\,800}(y, 9, 5) + \omega_{10\,810\,800}(y, 6, 6) + \sum_{n=7}^8 \omega_{10\,810\,800}(y, 4, n) \\ + \sum_{n=9}^{10} \omega_{10\,810\,800}(y, 3, n) + \sum_{n=11}^{14} \omega_{10\,810\,800}(y, 2, n) + \sum_{n=15}^{23} \omega_{10\,810\,800}(y, 1, n) \quad (y \in \mathbb{R}).$$

Graph of $\Omega_C(y)$ ($C = 1260, 110\,880, 10\,810\,800$) and $-\frac{1}{4} |\zeta_5(\frac{1}{2} + iy)|$ on the interval $[2, 35]$

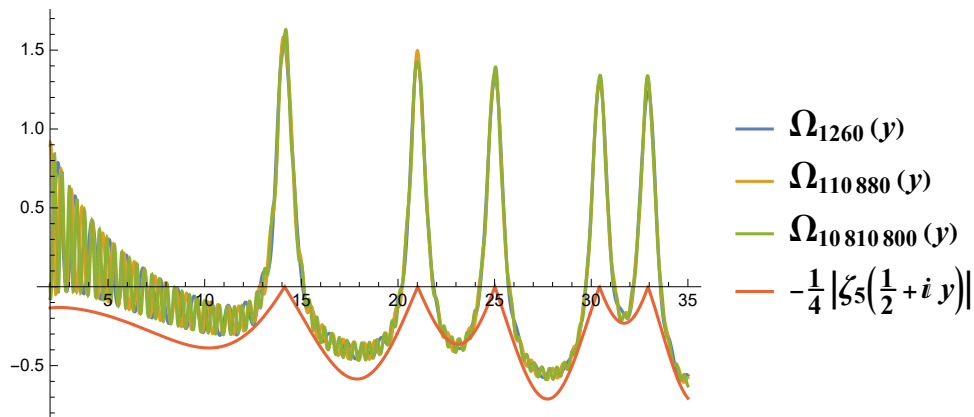


Fig. 16-8

To clearly show the imaginary parts of the non-trivial zeros of the zeta function, the function

$$-\frac{1}{4} \left| \zeta_5 \left(\frac{1}{2} + i y \right) \right|$$

is superposed on the above graph. Hereafter, functions showing the imaginary parts of the non-trivial zeros of the zeta functions or those of the eta functions are superposed on all graphs as needed.

The imaginary parts of the non-trivial zeros of the zeta function are proven to exist under each peak, i.e., the maximal values of the CapitalOmega functions.

Graph of $\Omega_c(y)$ ($c = 1260, 110\,880, 10\,810\,800$) and $-\frac{1}{4} \left| \zeta_5 \left(\frac{1}{2} + i y \right) \right|$ on the interval $[35, 70]$

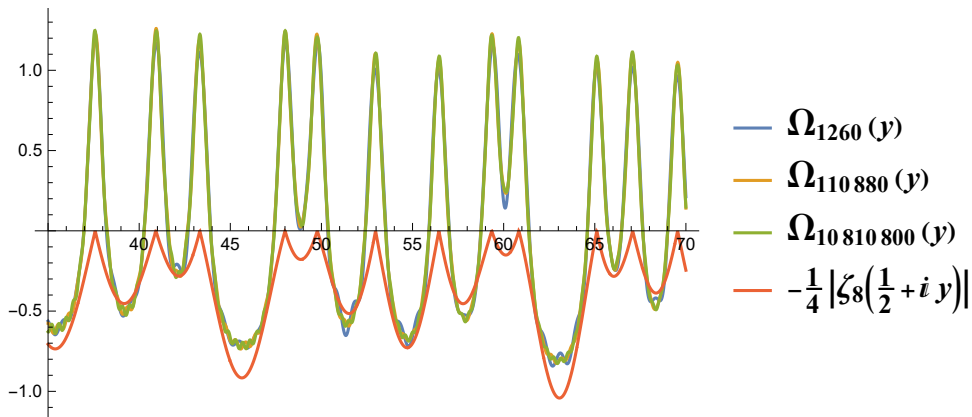


Fig. 16-9

I demonstrated that the imaginary parts of the zeta function’s non-trivial zeros are generated as a spectrum using only prime number information.

16-4. Observational results about relationships between prime numbers and the eta function without a proof

I define the following equation as an analogy to the equation (16-55):

$$(16 - 66) \quad \tilde{\Psi}^*(x) := \tilde{S}(x) - \sum_{\substack{\eta(\tau)=0 \\ \tau: \text{non-trivial zero}}} \frac{(\sqrt{x})^\tau}{\tau} \quad (x > 0).$$

In comparison to equations (16-56) and (16-57), the positive real variable x is replaced with $x^{1/2}$ and the non-trivial zeros of the zeta function are replaced with those of the eta function.

$$(16 - 67) \quad \tilde{S}(x) := \sqrt{x} - \frac{\log(1-x^{-1})}{2} - \frac{\eta'(0)}{\eta(0)} \quad (x > 0).$$

$$(16 - 68) \quad \frac{\eta'(0)}{\eta(0)} \simeq 1.8609727754.$$

The sum of the periodic terms crosses all non-trivial zeros of the eta function.

As an analogy to the equation (16-58), I define the following equation:

$$(16 - 69) \quad \tilde{\Psi}_N(x) := \tilde{S}(x) - \sum_{m=1}^N \left(\frac{(\sqrt{x})^{i\tau_m}}{i\tau_m} + \frac{(\sqrt{x})^{-i\tau_m}}{-i\tau_m} \right) \quad (x > 0, N \in \mathbb{N}).$$

Here, τ_m is the positive imaginary part of the eta function’s m -th non-trivial zero, to which the number is allocated in order that is closer to the real axis.

The values of 201 non-trivial zeros of the eta function were obtained with 25-decimal accuracy following the drawing. Specifically, as part of the search for the non-trivial zeros of the eta function, the sign changing was investigated at a step width of 0.1 (finely adjusted if necessary) using $(\eta - \text{tilde})_\lambda(\theta)$, which takes the real value on the imaginary axis. At the end of this section, the numerical data will be presented.

Graph of the approximate term $\tilde{S}(x)$ and $\tilde{\Psi}_{200}(x)$ on the interval $[3/2, 30]$

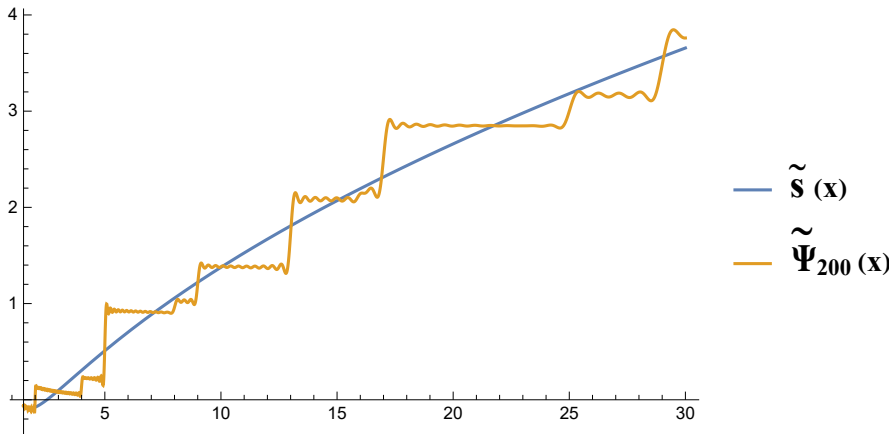


Fig. 16-10

The step-up is not recognized at the non-Pythagorean prime points of 3, 7, 11, 19, and 23 at all.

Graph of the approximate term $\tilde{S}(x)$ and $\tilde{\Psi}_{200}(x)$ on the interval $[3/2, 120]$

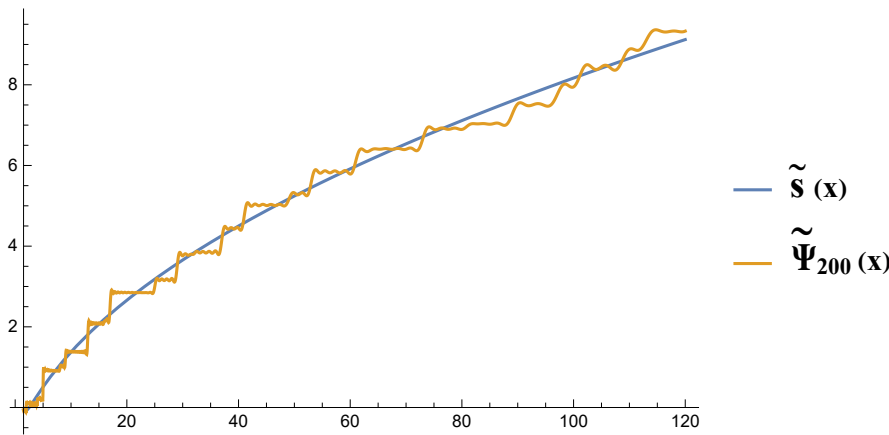


Fig. 16-11

The information on the non-trivial zeros of the eta function is used to produce the different step function.

In principle, the points that the step function steps up can be extracted by differentiating the periodic terms of $(\text{CapitalPsi} - \text{tilde})_N(x)$ by x .

As a counterpart to the equation (16-59), I define the following equation:

$$(16 - 70) \tilde{\Phi}_N(x) := -\frac{d}{dx} \sum_{m=1}^N \left(\frac{(\sqrt{x})^{i\tau_m}}{i\tau_m} + \frac{(\sqrt{x})^{-i\tau_m}}{-i\tau_m} \right)$$

$$= -\frac{1}{x} \sum_{m=1}^N \cos\left(\frac{1}{2} \tau_m \log(x)\right) \quad (x > 0, N \in \mathbb{N}).$$

In contrast to the equation (16-59), the essential correction is that ρ_m is replaced by $\tau_m/2$.

Graph of $\tilde{\Phi}_{50}(x)$, $\tilde{\Phi}_{100}(x)$, and $\tilde{\Phi}_{200}(x)$ on the interval $[3/2, 15]$

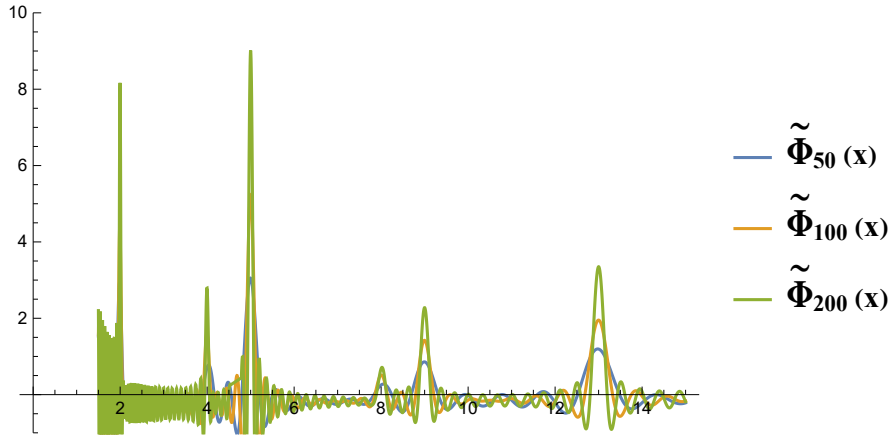


Fig. 16-12

The peaks are recognized at the points where x is the Pythagorean primes 5 and 13; 2, 4, and 8 are the first, second, and third powers of prime number 2; and 9 is the square of prime number 3.

Graph of $\tilde{\Phi}_{50}(x)$, $\tilde{\Phi}_{100}(x)$, and $\tilde{\Phi}_{200}(x)$ on the interval $[15, 50]$

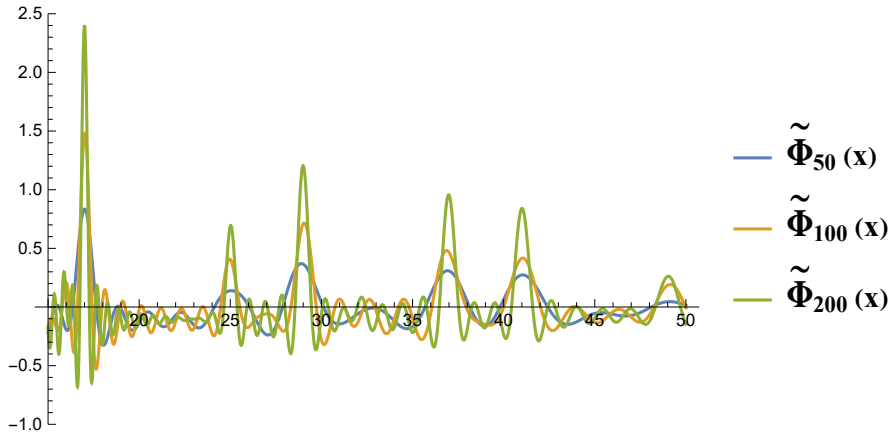


Fig. 16-13

The peaks are recognized at the points where x is one of the following Pythagorean primes: 17, 29, 37, and 41; 16 is the fourth power of prime number 2; and 25 and 49 are the squares of prime numbers 5 and 7, respectively. However, the peaks are not recognized at the points where x is 27 (the cube of prime number 3) and 32 (fifth power of prime number 2).

Graph of $\tilde{\Phi}_{50}(x)$, $\tilde{\Phi}_{100}(x)$, and $\tilde{\Phi}_{200}(x)$ on the interval [50, 100]

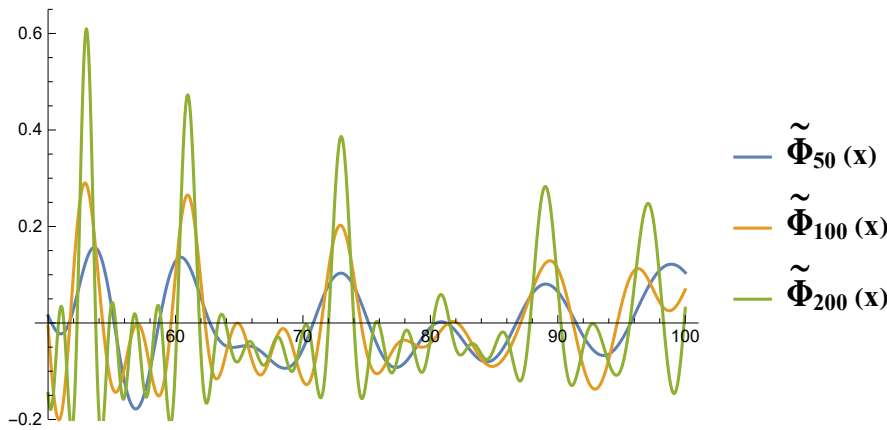


Fig. 16-14

The peaks are recognized at the points where x is one of the Pythagorean primes 53, 61, 73, 89, and 97; 81 is the fourth power of prime number 3. However, the recognition of the peak is difficult when x is 64 (sixth power of prime number 2). It is obvious that the imaginary parts of the eta function's non-trivial zeros generate Pythagorean primes, squares of non-Pythagorean primes, and even prime 2 as a main spectrum.

Based on the research results so far, let define the following function:

$$(16-71) \tilde{\Omega}_{\mathbf{c}}(y) := -\frac{1}{2} \sum_{\substack{2^n \leq \mathbf{c} \\ n \in \mathbb{N}}} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) - \sum_{\substack{q^n \leq \mathbf{c} \\ n \in \mathbb{N} \\ \mathbf{q}: \text{Pythagorean prime}}} \frac{\log(q)}{q^n} \cos\left(\frac{y}{2} \log(q^n)\right) - \sum_{\substack{r^{2n} \leq \mathbf{c} \\ n \in \mathbb{N} \\ \mathbf{r}: \text{non-Pythagorean prime} \setminus \{2\}}} \frac{\log(r)}{r^{2n}} \cos\left(\frac{y}{2} \log(r^{2n})\right) \quad (y \in \mathbb{R}, \mathbf{c} \in \mathbb{N}).$$

Assuming the k -th Pythagorean prime as q_k , the conditional partial sum of the second term on the right side of equation (16-71) is defined as follows:

$$(16-72) \tilde{\omega}_{\mathbf{c}}(y, m, n) := - \sum_{k=1}^m \frac{\log(q_k)}{(q_k)^n} \cos\left(\frac{y}{2} \log((q_k)^n)\right) \quad (\mathbf{c}, m, n \in \mathbb{N}, y \in \mathbb{R}, (q_m)^n \leq \mathbf{c}).$$

Here, the sum crosses all integers whose positive integer powers corresponding to the conditional Pythagorean prime q are less than or equal to the sufficiently large arbitrary positive integer C .

For the positive integer C , 3 numbers (5040, 554400, and 61261200) are selected to display graphs.

Some Pythagorean primes q_k for the calculations are shown as follows:

$$(16-73) \{q_1, q_2, q_3, q_4, q_5, q_{331}, q_{22799}, q_{1815875}\} = \{5, 13, 17, 29, 37, 5021, 554377, 61261169\}.$$

Assuming the k -th non-Pythagorean prime except even prime 2 as r_k , the conditional partial sum of the third term on the right side of equation (16-71) is as follows:

$$(16-74) \hat{\omega}_{\mathbf{c}}(y, m, n) := - \sum_{k=1}^m \frac{\log(r_k)}{(r_k)^{2n}} \cos\left(\frac{y}{2} \log((r_k)^{2n})\right) \quad (\mathbf{c}, m, n \in \mathbb{N}, y \in \mathbb{R}, (r_m)^n \leq \mathbf{c}).$$

Here, the sum crosses all integers whose positive even powers of the conditional non-Pythagorean prime r are less than or equal to the sufficiently large arbitrary positive integer C .

Some non-Pythagorean primes r_k for the calculations are shown as follows:

$$(16 - 75) \{r_1, r_2, r_3, r_4, r_5, r_{10}, r_{69}, r_{499}\} = \{3, 7, 11, 19, 23, 67, 743, 7823\}.$$

When $C = 5040$,

$$(16 - 76) \tilde{\Omega}_{5040}(y) = -\frac{1}{2} \sum_{n=1}^{12} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) + \tilde{\omega}_{5040}(y, 331, 1) + \tilde{\omega}_{5040}(y, 8, 2) \\ + \tilde{\omega}_{5040}(y, 3, 3) + \tilde{\omega}_{5040}(y, 2, 4) + \sum_{n=5}^7 \tilde{\omega}_{5040}(y, 1, n) \\ + \hat{\omega}_{5040}(y, 10, 1) + \hat{\omega}_{5040}(y, 2, 2) + \hat{\omega}_{5040}(y, 1, 3) \quad (y \in \mathbb{R}).$$

When $C = 554\,400$,

$$(16 - 77) \tilde{\Omega}_{554\,400}(y) = -\frac{1}{2} \sum_{n=1}^{19} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) + \tilde{\omega}_{554\,400}(y, 22\,799, 1) \\ + \tilde{\omega}_{554\,400}(y, 62, 2) + \tilde{\omega}_{554\,400}(y, 9, 3) + \tilde{\omega}_{554\,400}(y, 3, 4) + \tilde{\omega}_{554\,400}(y, 2, 5) \\ + \sum_{n=6}^8 \tilde{\omega}_{554\,400}(y, 1, n) + \hat{\omega}_{554\,400}(y, 69, 1) + \hat{\omega}_{554\,400}(y, 5, 2) + \hat{\omega}_{554\,400}(y, 2, 3) \\ + \sum_{n=4}^6 \hat{\omega}_{554\,400}(y, 1, n) \quad (y \in \mathbb{R}).$$

When $C = 61\,261\,200$,

$$(16 - 78) \tilde{\Omega}_{61\,261\,200}(y) = -\frac{1}{2} \sum_{n=1}^{25} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) + \tilde{\omega}_{61\,261\,200}(y, 1\,815\,875, 1) \\ + \tilde{\omega}_{61\,261\,200}(y, 489, 2) + \tilde{\omega}_{61\,261\,200}(y, 36, 3) + \tilde{\omega}_{61\,261\,200}(y, 9, 4) + \tilde{\omega}_{61\,261\,200}(y, 4, 5) \\ + \tilde{\omega}_{61\,261\,200}(y, 3, 6) + \sum_{n=7}^{11} \tilde{\omega}_{61\,261\,200}(y, 1, n) + \hat{\omega}_{61\,261\,200}(y, 499, 1) + \hat{\omega}_{61\,261\,200}(y, 13, 2) \\ + \hat{\omega}_{61\,261\,200}(y, 4, 3) + \hat{\omega}_{61\,261\,200}(y, 2, 4) + \sum_{n=5}^8 \hat{\omega}_{61\,261\,200}(y, 1, n) \quad (y \in \mathbb{R}).$$

Graph of $\tilde{\Omega}_C(y)$ ($C = 5040, 554\,400, 61\,261\,200$) and $-\frac{1}{16} |\eta_5(iy)|$ on the interval $[2, 35]$

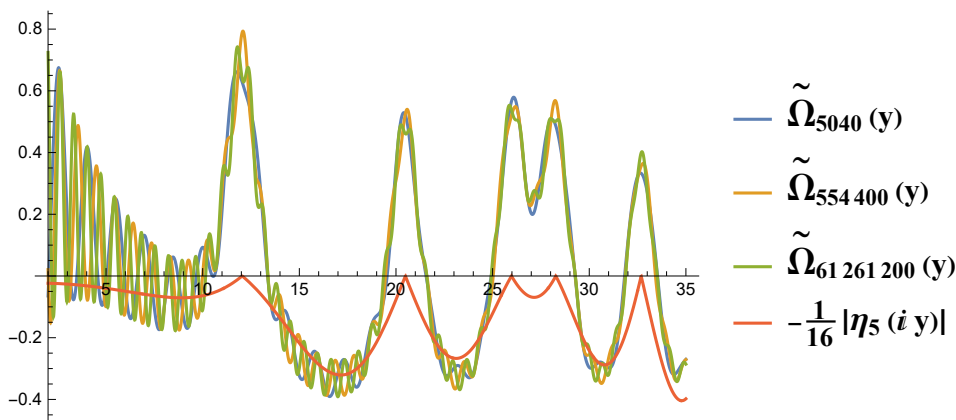


Fig. 16-15

Graph of $\tilde{\Omega}_c(y)$ ($c = 5040, 554\,400, 61\,261\,200$) and $-\frac{1}{16} |\eta_8(iy)|$ on the interval $[35, 70]$

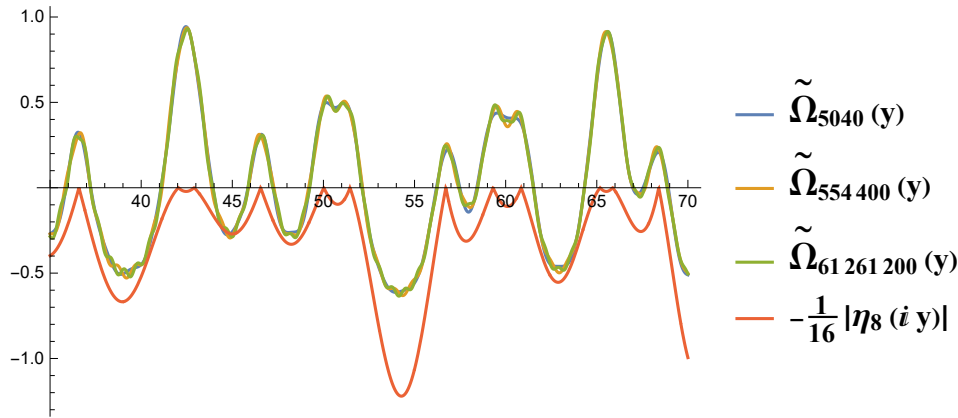


Fig. 16-16

Below each peak, there are one or two imaginary parts of the eta function's non-trivial zeros. It is shown that the imaginary parts of the non-trivial zeros of the eta function are generated as a quasi-spectrum using the information on Pythagorean primes, the squares of non-Pythagorean primes, and even prime 2.

For further exploration, I newly define the following function:

$$(16 - 79) \quad \bar{\Omega}_c(y) := -\frac{1}{2} \sum_{\substack{2^n \leq c \\ n \in \mathbb{N}}} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) - \sum_{\substack{r^{2n-1} \leq c \\ n \in \mathbb{N}}} \frac{\log(r)}{r^{2n-1}} \cos\left(\frac{y}{2} \log(r^{2n-1})\right) \quad (y \in \mathbb{R}, c \in \mathbb{N}).$$

r : non-Pythagorean prime $\setminus \{2\}$

By using this, I obtain the following obvious relation:

$$(16 - 80) \quad \tilde{\Omega}_c(y) = \tilde{\Omega}_c(2y) + \bar{\Omega}_c(2y) \quad (y \in \mathbb{R}, c \in \mathbb{N}).$$

The conditional partial sum of the second term on the right side of equation (16-80) is defined as follows:

$$(16 - 81) \quad \bar{\omega}_c(y, m, n) := -\sum_{k=1}^m \frac{\log(r_k)}{(r_k)^{2n-1}} \cos\left(\frac{y}{2} \log((r_k)^{2n-1})\right) \quad (c, m, n \in \mathbb{N}, y \in \mathbb{R}, (r_m)^{2n-1} \leq c).$$

When $C = 5040$,

$$(16 - 82) \quad \bar{\Omega}_{5040}(y) = -\frac{1}{2} \sum_{n=1}^{12} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) + \bar{\omega}_{5040}(y, 343, 1) + \bar{\omega}_{5040}(y, 3, 2) + \sum_{n=3}^4 \bar{\omega}_{5040}(y, 1, n) \quad (y \in \mathbb{R}).$$

When $C = 554\,400$,

$$(16 - 83) \quad \bar{\Omega}_{554400}(y) = -\frac{1}{2} \sum_{n=1}^{19} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) + \bar{\omega}_{554400}(y, 22\,855, 1) + \bar{\omega}_{554400}(y, 12, 2) + \bar{\omega}_{554400}(y, 3, 3) + \sum_{n=4}^6 \bar{\omega}_{554400}(y, 1, n) \quad (y \in \mathbb{R}).$$

When $C = 61\,261\,200$,

$$\begin{aligned}
 (16 - 84) \bar{\Omega}_{61\,261\,200}(y) &= -\frac{1}{2} \sum_{n=1}^{25} \frac{\log(2)}{2^n} \cos\left(\frac{y}{2} \log(2^n)\right) + \bar{\omega}_{61\,261\,200}(y, 1\,816\,644, 1) \\
 &+ \bar{\omega}_{61\,261\,200}(y, 40, 2) + \bar{\omega}_{61\,261\,200}(y, 6, 3) + \bar{\omega}_{61\,261\,200}(y, 3, 4) + \bar{\omega}_{61\,261\,200}(y, 2, 5) \\
 &+ \sum_{n=6}^8 \bar{\omega}_{61\,261\,200}(y, 1, n) \quad (y \in \mathbb{R}).
 \end{aligned}$$

Graph of $\bar{\Omega}_C(y)$ ($C = 5040, 554\,400, 61\,261\,200$) and $-\frac{1}{16} |\eta_5(iy)|$ on the interval $[2, 35]$

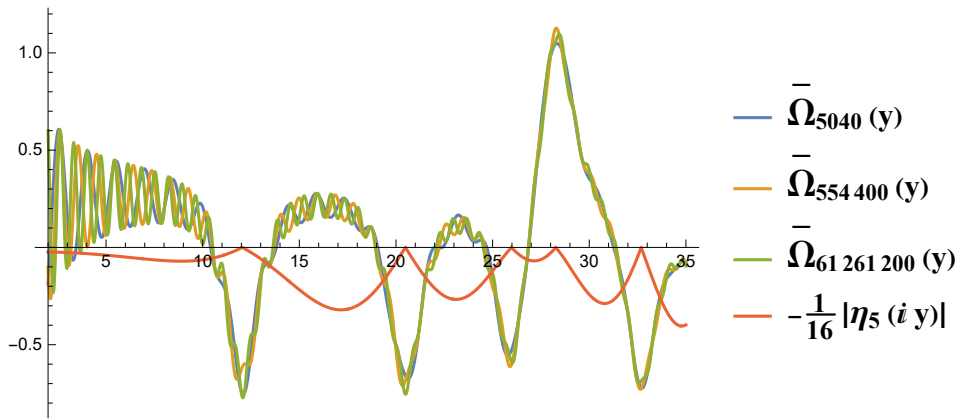


Fig. 16-17

Graph of $\bar{\Omega}_C(y)$ ($C = 5040, 554\,400, 61\,261\,200$) and $-\frac{1}{16} |\eta_8(iy)|$ on the interval $[35, 70]$

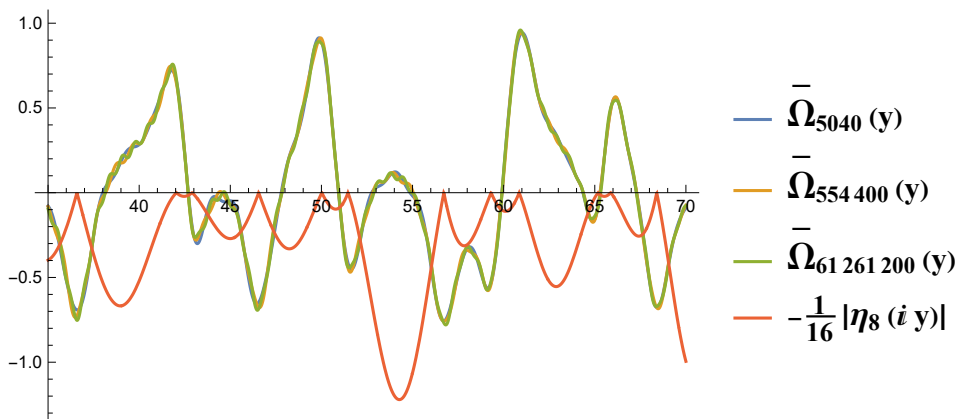


Fig. 16-18

It is shown shortly after that the CapitalOmega-tilde and the CapitalOmega-dash functions are important when considering their difference and sum.

Graph of $\tilde{\Omega}_c(y) - \bar{\Omega}_c(y)$ ($c = 5040, 554\,400$),
 $-\frac{1}{4} |\text{Zeta}[\frac{1}{2} + i \frac{y}{2}]|$, and $-\frac{1}{16} |\eta_5(i y)|$ on the interval $[2, 35]$

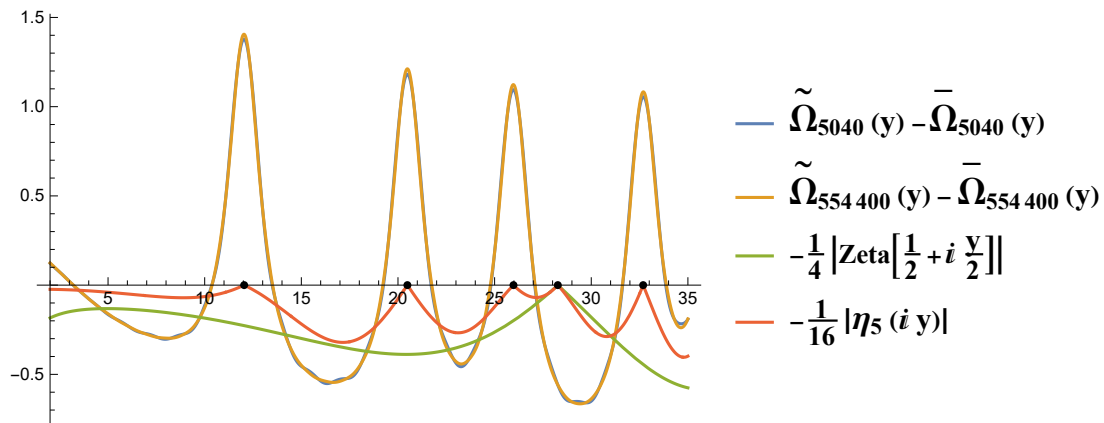


Fig. 16-19

Graph of $\tilde{\Omega}_c(y) - \bar{\Omega}_c(y)$ ($c = 5040, 554\,400$),
 $-\frac{1}{4} |\text{Zeta}[\frac{1}{2} + i \frac{y}{2}]|$, and $-\frac{1}{16} |\eta_8(i y)|$ on the interval $[35, 70]$

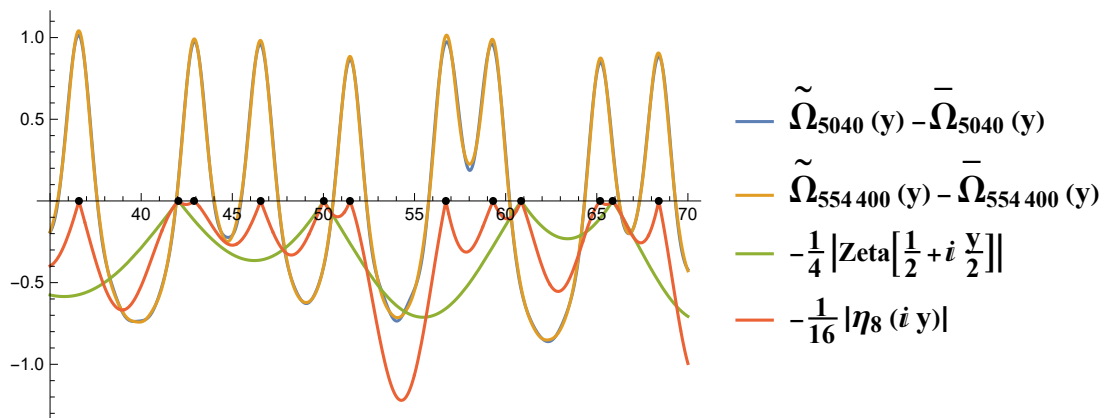


Fig. 16-20

The dots on the horizontal axis indicate the imaginary parts of the eta function's non-trivial zeros. Below each peak, there is always one imaginary part of the eta function's non-trivial zero. In the observation range, the imaginary parts of the non-trivial zeros of the modified zeta function $\zeta(\theta/2 + 1/4)$ belong to the subset of those of the eta function. In other words, twice the imaginary part of each zeta function's non-trivial zero belongs to the subset of imaginary parts of the eta function's non-trivial zeros. Twice the imaginary parts of the zeta function's non-trivial zeros are far from the y-coordinates of the peaks, respectively.

Graph of $\tilde{\Omega}_{\mathbf{c}}(2y) + \bar{\Omega}_{\mathbf{c}}(2y)$ ($\mathbf{c} = 5040, 554\,400$), $-\frac{1}{4}|\zeta[\frac{1}{2} + iy]|$, and $-\frac{1}{16}|\eta_8(i2y)|$ on the interval $[2, 35]$

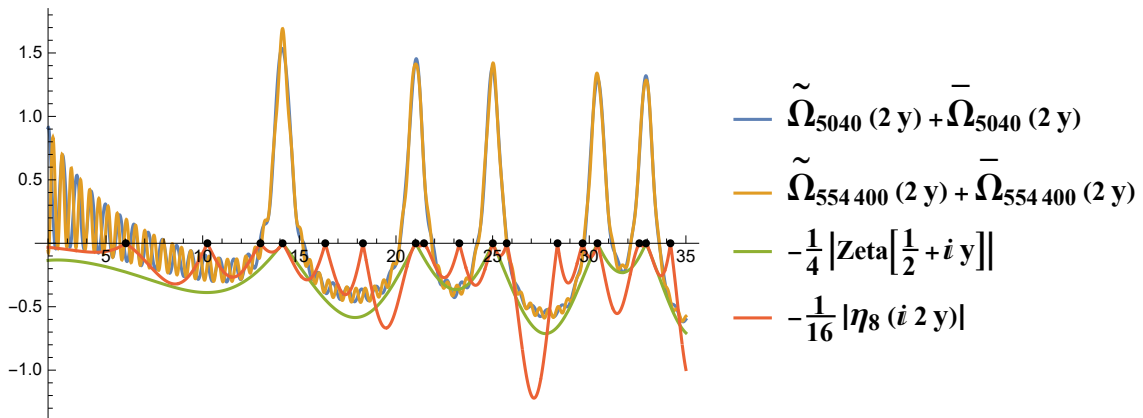


Fig. 16-21

Graph of $\tilde{\Omega}_{\mathbf{c}}(2y) + \bar{\Omega}_{\mathbf{c}}(2y)$ ($\mathbf{c} = 5040, 554\,400$) and $-\frac{1}{4}|\zeta[\frac{1}{2} + iy]|$ on the interval $[35, 70]$

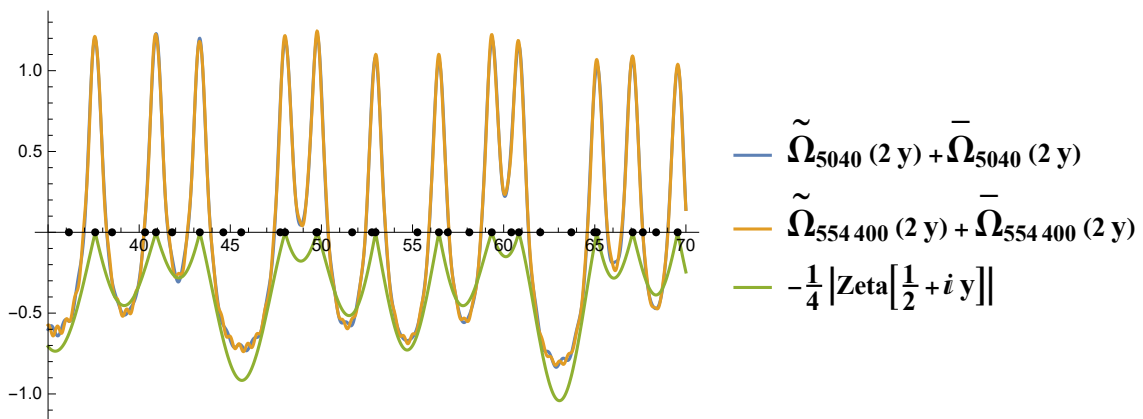


Fig. 16-22

The dots on the horizontal axis indicate the imaginary parts of the non-trivial zeros of the modified eta function $\eta(2\theta)$.

Below each peak, there is always one imaginary part of the non-trivial zero of the modified eta function. Furthermore, it is also the imaginary part of the non-trivial zero of the zeta function.

List of 201 non-trivial zeros of the zeta function (Positive imaginary part, 25-decimal accuracy), Part1

k	P_{5k+1}	P_{5k+2}	P_{5k+3}	P_{5k+4}	P_{5k+5}	$5k+1 \sim 5k+5$
0	14.13472514173469379045725	21.02203963877155499262848	25.01085758014568876521379	30.42487612585951321031190	32.93506158773918969066237	1 ~ 5
1	37.58617815882567125721776	40.91871901214749518739813	43.32707328091499991949612	48.00515088116715972794247	49.77383247767230218191678	6 ~ 10
2	52.97032147771446064414730	56.446247697065339480436776	59.34704400260235307965365	60.83177852460980984425990	65.11254404808160666087505	11 ~ 15
3	67.07981052949417371447883	69.54640171117397925292686	72.06715767448190758252211	75.7046906908393516832692	77.14484006887480537268266	16 ~ 20
4	79.33757502024936792276359	82.91038085408603018316484	84.73549298051705010573531	87.42527461312522940653167	88.80911120765446542368235	21 ~ 25
5	92.49189927053848429625973	94.65134404051988696659793	95.87063422824530975874103	98.8311942181936922332442	101.3178510057313912287854	26 ~ 30
6	103.7255380404783394163984	105.4466230523260944936708	107.1686111842764075151234	111.0295355431696745246565	111.8746591769926370856121	31 ~ 35
7	114.3202209154527127658909	116.2266803208575543821608	118.7907828659762173229791	121.3701250024206450189455	122.9468292935525882008175	36 ~ 40
8	124.2568185543457671847320	127.516683795964951242793	129.5787041999560509857680	131.087688509326567235664	133.497737202997864501305	41 ~ 45
9	134.756509753738713313261	138.1160420545334432001916	139.7362089521213889504500	141.1237074040211237619404	143.1118458076206327394051	46 ~ 50
10	146.0009824867653185474025	147.4227653425596020495212	150.0535204207848803514325	150.9252576122414667618525	153.0246938111988961982565	51 ~ 55
11	156.1129092942378675697502	157.5975918175940598875305	158.8499881714204987241750	161.1889641375960273194373	163.0307096871819872433110	56 ~ 60
12	165.5370691879004188300389	167.1844399781745134409578	169.0945154155688214895059	169.9119764794116989666998	173.4115365195915529598461	61 ~ 65
13	174.7541915233657258133788	176.441434297104188888926	178.3774077760999772858309	179.9164840202569961393400	182.2070784843664619154070	66 ~ 70
14	184.8744678483875088009606	185.5987836777074714665277	187.2289225835018519916415	189.4161586560169370848523	192.0266563607137865472836	71 ~ 75
15	193.0797266038457040474022	195.265396679529255214632	196.8764818409583169486223	198.0153096762519124249199	201.2647519437037887330161	76 ~ 80
16	202.4935945141405342776867	204.189671803104543307164	205.3946972021632860252124	207.9062388878062098615020	209.5765097168562598528356	81 ~ 85
17	211.6908625953653075639075	213.3479193597126661906391	214.5470447834914232229442	216.1695385082637002658696	219.0675963490213789856773	86 ~ 90
18	220.7149188393140033691156	221.430705554693387520975	224.0070002546043352117289	224.9833246695822875037825	227.4214442796792913104614	91 ~ 95
19	229.3374133055253481077601	231.2501887004991647738062	231.9872352531802486037717	233.6934041789085006407045	236.5242296658162058024755	96 ~ 100

Fig. 16-23

List of 201 non-trivial zeros of the zeta function (Positive imaginary part, 25-decimal accuracy), Part2

k	ρ_{k+1}	ρ_{k+2}	ρ_{k+3}	ρ_{k+4}	ρ_{k+5}	$5k+1 \sim 5k+5$
20	237.7698204809252040032366	239.5554775733276287402689	241.0491577962165864128379	242.8232719342226000168265	244.0708984970781582368165	101~105
21	247.1369900748974994675510	248.1019900601484592567621	249.5736896447072091923298	251.0149477950160011429542	253.0699867479994771945990	106~110
22	255.3062564549140227530865	256.3807136944344777893584	258.6104394915313682089831	259.8744069896780003506728	260.8050845045968701859312	111~115
23	263.5738939048701322330816	265.5578518388763202924773	266.61497378151010724957201	267.9219150828240594403790	269.9704490239976035946935	116~120
24	271.4940556416449990181794	273.4596091884032870457143	275.5874926493438412487407	276.4520495031329386798873	278.2507435298419544927483	121~125
25	279.2292509277451892284099	282.4651147650520962330272	283.2111857332338674204938	284.8359639809047241331576	286.6674453630028842928476	126~130
26	287.9119205014221871552541	289.5798549292188341527380	291.8462913290673958355131	293.558434139356283556767	294.9653696192655421750664	131~135
27	295.5732548789582923884608	297.9792770619434152099297	299.8403260537213129600271	301.6493254621941836234701	302.6967495896069170517515	136~140
28	304.8643713408572977001487	305.7289126020368092892228	307.2194961281700547894100	310.1094631467018988047862	311.1651415303560032709427	141~145
29	312.4278011806008919804860	313.9852857111589229790490	315.4756160894757338683961	317.7348059423701803956455	318.8531042563165979066892	146~150
30	321.1601343091135782919215	322.1445586724829322988374	323.4669695575120505062120	324.8628660517396132649801	327.4439012619054573434693	151~155
31	329.0330716804809340336147	329.9532397282338663438921	331.4744678286634243756618	333.6453785248698505849617	334.2113548332443832324034	156~160
32	336.8418504283906847946548	338.3399928508066118862573	339.8582167253635401923266	341.042261110465604825978	342.0548775103635854514038	161~165
33	344.6617029402523370441812	346.3478705660099473959365	347.272677844204844757971	349.5162608706961441231556	350.4084193491920991876720	166~170
34	351.8786490253592804367134	353.4889004887188067836038	356.0175749772649473179604	357.1513022520396248096029	357.9526851016322737551289	171~175
35	359.7437549531144487992920	361.2893616958046503902913	363.3313305789738347473344	364.7360241140889937162621	366.2127102883313168610771	176~180
36	367.9935754817403033261833	368.9684380957343898915769	370.0509192121060003396512	373.0619283721128384491194	373.8648739109085697447564	181~185
37	375.8259127667393341079077	376.3240922106680521171908	378.4366802499654797240910	379.872975346532466510241	381.4844686171865249196625	186~190
38	383.4435294495564877043458	384.9561168148636871037516	385.8613008459742291805620	387.222990222879809759485	388.8461283542322546008094	191~195
39	391.4560835636380457705782	392.24508333395190967490152	393.42774738444340259366990	395.5828700109937209708777	396.3818542225921869319995	196~200
40	397.9187362096142433869812	Not calculated	Not calculated	Not calculated	Not calculated	201~205

Fig. 16-24

List of 201 non-trivial zeros of the eta function (Positive imaginary part, 25-decimal accuracy), Part I

k	T_{5k+1}	T_{5k+2}	T_{5k+3}	T_{5k+4}	T_{5k+5}	$5k+1 \sim 5k+5$
0	12.04189780939519330980400	20.48754060833310910427551	25.97619602462484501490622	28.26945028346938758091450	32.68521420917444438995372	1 ~ 5
1	36.58398639224706967705201	42.04407927754310998525696	42.90122268796692099440190	46.556753040091906306363912	50.02171516029137752642758	6 ~ 10
2	51.4571285017745513453018	56.7192686860506557130322	59.31276802918630544361981	60.84975225171902642062380	65.18437305423431026160309	11 ~ 15
3	65.87012317547837938132474	68.39991501842629382608959	72.28576091660627566113163	75.17235631765134251443553	77.02384628343738258755301	16 ~ 20
4	80.64534813338108836068679	81.83743802429499037479625	83.61416924000912467431504	86.65414656182999903899224	89.23578211732460678696410	21 ~ 25
5	91.19916879338313349187540	95.48312456187828250156269	96.01030176233431945388495	99.44625864756517213313914	99.54766495534460436383357	26 ~ 30
6	103.37218690577410568790676	105.5376415356094585300702	105.9406429554289212882946	110.5350871693984496934365	112.8924953941267896087355	31 ~ 35
7	113.8687481104045937736034	116.2334142213478359545247	118.6940880052047061593073	120.8434278980156693460372	121.6635570492196196883198	36 ~ 40
8	124.0172645715355389038612	127.4292822375708662470366	129.9523411461919986972118	130.2250880961632133217501	134.1396210589883474289577	41 ~ 45
9	135.2738417270921367961100	136.7317690076688459224675	139.0928034223479585058537	140.31717598176042241227426	144.1343153489638151650442	46 ~ 50
10	144.3109699487637624293831	147.5352710429717866723093	150.2862432948662228115960	151.4093813981678665366538	153.3926064068603981291333	51 ~ 55
11	154.2896801377496107453653	157.6199966286418260073836	158.674730040498735845272	160.4202624767332778302961	162.4279032537663023154697	56 ~ 60
12	165.8207617081720603663297	167.333312028941433025666	169.4634807275632572164352	169.4709859610341002114706	173.1553203367805288204211	61 ~ 65
13	174.8505492262504388130633	175.259436239175793780789	177.6182224152689308473647	179.6022632333916226519388	182.6994076293951469478620	66 ~ 70
14	184.4749998209085160920084	184.9837985411169685925195	188.3332391719200426141061	189.3026880810397739331959	191.7412684564906195174821	71 ~ 75
15	192.272022325611163705490	193.9234831588349671552155	197.5106008315090553372079	197.6623884363873844666488	200.2697734061335705803846	76 ~ 80
16	202.6357020114627824575709	204.2827616537792276535200	206.576150763380054031867	207.4510760809566788327968	208.6665396885349090099017	81 ~ 85
17	210.8932461046521889873417	213.388917817916834592008	214.3372223685528150302467	215.3804139502934004162589	218.5198859478563849825795	86 ~ 90
18	220.9992163528581895609698	222.0590710863393490493129	223.7493183539852741712242	224.7356334880243599143332	227.6295910979853470791262	91 ~ 95
19	228.6404418309054255317819	230.2847392952986558070189	232.386409316924379792603	232.4533606417151087643216	237.0751087576937233156002	96 ~ 100

Fig. 16-25

List of 201 non-trivial zeros of the eta function (Positive imaginary part, 25-decimal accuracy), Part2

k	T_{5k+1}	T_{5k+2}	T_{5k+3}	T_{5k+4}	T_{5k+5}	$5k+1 \sim 5k+5$
20	237.5815657319524346459583	238.9059197524181807164837	241.4625872373207538673698	242.7402500048412918378911	244.8949227581374238393590	101~105
21	245.8936585871051164016349	247.5890975206390121975876	248.51363710868913343694640	251.5370391198311875249928	252.5975520498828181958003	106~110
22	255.0333677591929902485586	255.9188153661265987661118	259.1574083999121019715361	259.7712467172909386524701	262.1753770618653134471327	111~115
23	262.1871575081679921504687	264.2871532019756537610026	266.9954744059951729002610	267.4883629279496263828475	269.5130195067477426626522	116~120
24	270.9816745051140044951081	273.0946245345651167786625	276.2320841090668864003831	276.9145890193925153301808	277.5003554092228802590954	121~125
25	279.4724179042427779009001	282.2474148080422475238807	282.5072650195035682561334	284.7888350443955645966940	286.2236916152412654788102	126~130
26	286.658125483254640231872	289.9563325554227243635735	292.0019649735310370948050	293.0440107698166769799168	294.8455306851192040990424	131~135
27	295.8690616243618340696205	298.3769154346050727633361	300.1070408415697607028649	300.5927180030862429160528	301.8505152244829335237050	136~140
28	303.9239753409673814558369	306.049387622397923965131	307.3992252470305210510120	309.1509828574398346725859	311.3004973264868717362307	141~145
29	312.225818588475751395004	315.195183651881197750610	315.4966106158058566381491	317.4100422510587022498614	317.6999765428409974483500	146~150
30	320.4729681935420846675554	322.3779282751920550388747	322.8142939223134146905627	325.1320933791980792525484	326.0614193743639744866221	151~155
31	329.4623292331537938398151	330.8028385717263081892296	331.0741383758008376600778	333.5077583382168542293586	334.3688799563490268819155	156~160
32	336.088841563419489214857	338.1890308511376429790117	339.8239529588233979333997	340.1022623750533804296962	341.4693339891576109129115	161~165
33	344.5609635432426762367639	346.8230730391831059196922	346.8859576722700314339416	349.5083830467314516267575	349.8301761708011468088310	166~170
34	352.8828685954208377777853	353.1946042838361348859912	355.4024251491475516160485	356.7247498179713112188241	356.7548155521999545716619	171~175
35	359.8329680405139922786801	361.1386207705710627955890	363.2298274634087799618162	364.4141569687329238308141	365.8333366579379344544134	176~180
36	368.2300647507367775313421	369.748935696750176019213	370.7479932154009498344296	371.1975673554149429330554	374.1375211813888190829118	181~185
37	374.4578451670037039832831	376.5427486645415015718075	378.8323173120338741697046	378.9834629842570850271637	380.742375215115454866728	186~190
38	384.053127214275730945673	384.7222757521126368148321	386.1594532076914080948044	387.5941458522933619891087	388.4637664587217984439057	191~195
39	390.5307933590584706429264	392.2640113093685696467059	393.729636819166338972445	394.2268924431528170755268	396.0306193525038248498398	196~200
40	397.6129514722938134226983	Not calculated	Not calculated	Not calculated	Not calculated	201~205

Fig. 16-26

17. The equivalent function to the Riemann-Siegel $Z(t)$

The function $\tilde{\zeta}\left(\frac{1}{2} + it\right)$, which takes real values on the critical line, is equivalent to the Riemann-Siegel $Z(t)$ function “[6].”

$$(17-1) \quad \tilde{\zeta}\left(\frac{1}{2} + it\right) = Z(t) \quad (t \in \mathbb{R}).$$

Therefore, the new explicit formula for the Riemann-Siegel $Z(t)$ function is given as follows:

$$(17-2) \quad Z(t) = \left| \frac{2\pi^{\frac{1}{4}}}{\frac{1-it}{2} \Gamma\left(\frac{5+it}{4}\right)} \right| \sum_{p=1}^{\infty} \sigma_{-\frac{1+it}{2}}(p) p^{\frac{1+it}{4}} \left(-\frac{1+4t^2}{4} + \frac{13-it}{2} (2\pi p)^2 \right) K_{\frac{1+it}{4}}(2\pi p) - \left(\frac{1-it}{2} + 2(2\pi p)^2 \right) (2\pi p) K_{\frac{3-it}{4}}(2\pi p) \quad (t \in \mathbb{R}).$$

The rule in subsection 6-3 is applied to the above equation.

$$(17-3) \quad Z_{\lambda}(t) = \left| \frac{2\pi^{\frac{1}{4}}}{\frac{1-it}{2} \Gamma\left(\frac{5+it}{4}\right)} \right| \sum_{p=1}^{\lambda} \sigma_{-\frac{1+it}{2}}(p) p^{\frac{1+it}{4}} \left(-\frac{1+4t^2}{4} + \frac{13-it}{2} (2\pi p)^2 \right) K_{\frac{1+it}{4}}(2\pi p) - \left(\frac{1-it}{2} + 2(2\pi p)^2 \right) (2\pi p) K_{\frac{3-it}{4}}(2\pi p) \quad (t \in \mathbb{R}, \lambda \in \mathbb{N}).$$

Graph of $\text{Re}(Z_{11}(t))$, $\text{Im}(Z_{11}(t))$, and $\text{RiemannSiegelZ}[t]$, $t \in [0, 70]$

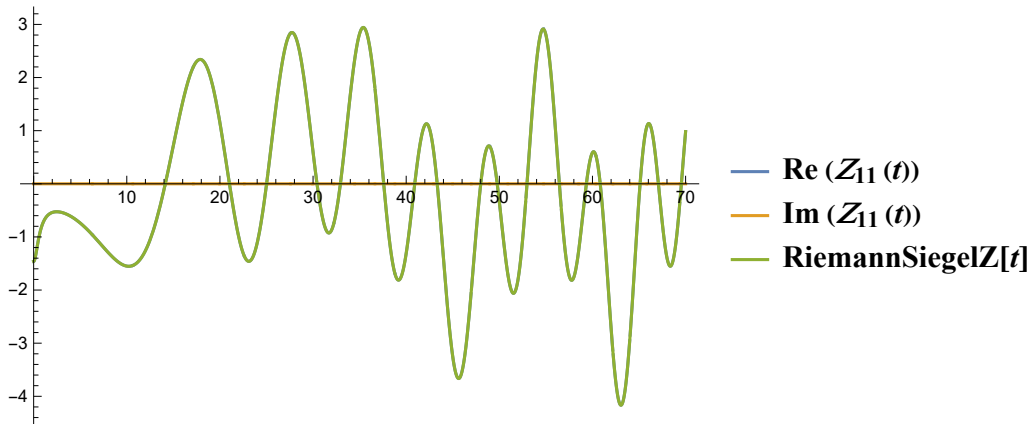


Fig. 17-1

To confirm the equivalence, the graph is superposed with Mathematica’s built-in Riemann-Siegel $Z(t)$ function.

The following equation defines the Riemann-Siegel theta function:

$$(17-4) \quad Z(t) := e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \in \mathbb{R} \quad (t \in \mathbb{R}).$$

On the critical line, the Riemann-Siegel theta function takes a real value. From the defining equation,

$$(17-5) \quad e^{i\theta(t)} = \frac{\frac{1-it}{2} \Gamma\left(\frac{5+it}{4}\right)}{2\pi^{\frac{1+it}{4}}} \left| \frac{2\pi^{\frac{1+it}{4}}}{\frac{1-it}{2} \Gamma\left(\frac{5+it}{4}\right)} \right| = \pi^{-\frac{it}{2}} \sqrt{\frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{it}{2}\right)}} \quad (t \in \mathbb{R}).$$

Taking a logarithm and dividing by i in both sides of the above equation,

$$(17-6) \quad \vartheta(t) + 2k\pi = -\frac{\log(\pi)}{2}t - \frac{i}{2} \log\left(\frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{it}{2}\right)}\right) \quad (k \in \mathbb{Z}, t \in \mathbb{R}).$$

Here, the integer k is chosen such that the Riemann-Siegel theta function is continuous.

Using the following proposition formula, the discontinued points of the second term on the right side of the above equation are obtained:

$$(17-7) \quad \operatorname{Re}\left(\frac{\Gamma\left(\frac{1}{4} + \frac{if_m}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{if_m}{2}\right)}\right) + 1 = 0 \implies \log\left(\frac{\Gamma\left(\frac{1}{4} + \frac{if_m}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{if_m}{2}\right)}\right) = \log(-1) = i\pi \quad (m \in \mathbb{Z} \setminus \{0\}).$$

Specifically, f_m can be determined as the zeros in the premise formula of the proposition. The following are some f_m examples:

$$(17-8) \quad \{f_1, f_2, f_3, f_4, f_{35}, f_{36}\} \simeq \{8.58494, 12.3904, 15.6226, 18.5534, 80.6619, 82.3566\}.$$

$$(17-9) \quad f_{-m} = -f_m \quad (m \in \mathbb{N}).$$

For describing the partial Riemann-Siegel theta function, the following unit step function is proposed:

$$(17-10) \quad U(x) := \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Hence, the partial Riemann-Siegel theta function is obtained as follows:

$$(17-11) \quad \Theta_n^Z(t) = -\frac{\log(\pi)}{2}t - \frac{i}{2} \log\left(\frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{it}{2}\right)}\right) + \pi \sum_{m=-n-1}^n (m+1 - U(m)) (U(t-f_m) - U(t-f_{m+1}))$$

$(n \in \mathbb{N}, t \in \mathbb{R}, |t| < f_{n+1}).$

Here, f_0 is specially defined for simple expression.

$$(17-12) \quad f_0 := 0.$$

The partial Riemann-Siegel theta function is an odd function.

$$(17-13) \quad \Theta_n^Z(t) = -\Theta_n^Z(-t) \quad (n \in \mathbb{N}, t \in \mathbb{R}, |t| < f_{n+1}).$$

The Riemann-Siegel theta function is determined as the infinite limit of an integer n .

$$(17-14) \quad \vartheta(t) = \lim_{n \rightarrow \infty} \Theta_n^Z(t) \quad (t \in \mathbb{R}).$$

Graph of RiemannSiegelTheta[t] and $\Theta_5^Z(t)$, $t \in [0, 30]$

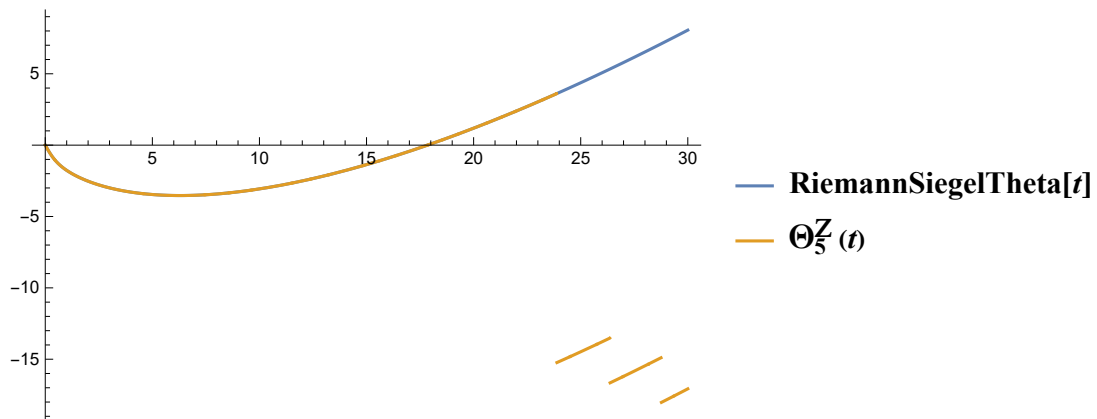


Fig. 17-2

The graph is superposed with Mathematica's built-in Riemann-Siegel theta function.

The built-in Riemann-Siegel theta function is continuous on the whole interval, although the partial Riemann-Siegel theta function is discontinuous.

Graph of $\text{Re} \left(e^{i \Theta_{35}^Z(t)} \text{Zeta} \left[\frac{1}{2} + i t \right] \right)$, $\text{Im} \left(e^{i \Theta_{35}^Z(t)} \text{Zeta} \left[\frac{1}{2} + i t \right] \right)$, $\text{Re} \left(\tilde{\zeta}_{11} \left(\frac{1}{2} + i t \right) \right)$,
and $\text{Im} \left(\tilde{\zeta}_{11} \left(\frac{1}{2} + i t \right) \right)$, $t \in [0, 70]$

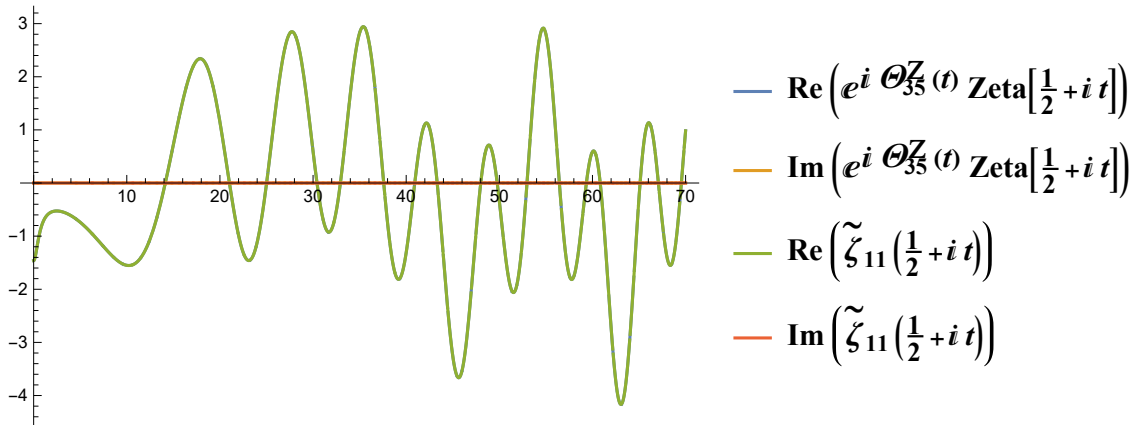


Fig. 17-3

For the eta function, the theta-hat function is defined by the following equation:

$$(17-15) \quad E(t) := e^{i \hat{\theta}(t)} \eta(it) = \tilde{\eta}(it) \in \mathbb{R} \quad (t \in \mathbb{R}).$$

According to the defining equation,

$$(17-16) \quad e^{i \hat{\theta}(t)} = \frac{(1-it)\Gamma\left(1+\frac{it}{2}\right)}{2\pi^{\frac{it}{2}}} \left| \frac{2\pi^{\frac{it}{2}}}{(1-it)\Gamma\left(1+\frac{it}{2}\right)} \right| = \pi^{-\frac{it}{2}} \frac{1-it}{\sqrt{1+t^2}} \sqrt{\frac{\Gamma\left(1+\frac{it}{2}\right)}{\Gamma\left(1-\frac{it}{2}\right)}} \quad (t \in \mathbb{R}).$$

Taking a logarithm and dividing by i on both sides of the above equation,

$$(17-17) \quad \hat{\theta}(t) + 2k\pi = -\frac{\log(\pi)}{2}t - i \log\left(\frac{1-it}{\sqrt{1+t^2}}\right) - \frac{i}{2} \log\left(\frac{\Gamma\left(1+\frac{it}{2}\right)}{\Gamma\left(1-\frac{it}{2}\right)}\right) \quad (k \in \mathbb{Z}, t \in \mathbb{R}).$$

Here, the integer k is chosen such that the theta-hat function is continuous.

Using the following proposition, the discontinued points of the third term on the right side of the above equation are obtained:

$$(17-18) \quad \text{Re} \left(\frac{\Gamma\left(1+\frac{ig_m}{2}\right)}{\Gamma\left(1-\frac{ig_m}{2}\right)} \right) + 1 = 0 \implies \log \left(\frac{\Gamma\left(1+\frac{ig_m}{2}\right)}{\Gamma\left(1-\frac{ig_m}{2}\right)} \right) = \log(-1) = i\pi \quad (m \in \mathbb{Z} \setminus \{0\}).$$

Specifically, g_m can be determined as the zeros in the premise formula of the proposition. The following are such examples of g_m :

$$(17-19) \quad \{g_1, g_2, g_3, g_4, g_{35}, g_{36}\} \simeq \{6.87947, 11.0772, 14.4675, 17.4912, 80.0252, 81.7234\}.$$

$$(17-20) \quad g_{-m} = -g_m \quad (m \in \mathbb{N}).$$

The partial Theta function for the eta function is obtained as follows:

$$(17-21) \quad \Theta_n^E(t) = -\frac{\log(\pi)}{2}t - i \log\left(\frac{1-it}{\sqrt{1+t^2}}\right) - \frac{i}{2} \log\left(\frac{\Gamma(1+\frac{it}{2})}{\Gamma(1-\frac{it}{2})}\right) \\ + \pi \sum_{m=-n-1}^n (m+1-U(m))(U(t-g_m)-U(t-g_{m+1})) \quad (n \in \mathbb{N}, t \in \mathbb{R}, |t| < g_{n+1}).$$

Here, g_0 is specially defined for simple expression.

$$(17-22) \quad g_0 := 0.$$

The partial Theta function for the eta function is an odd function.

$$(17-23) \quad \Theta_n^E(t) = -\Theta_n^E(-t) \quad (n \in \mathbb{N}, t \in \mathbb{R}, |t| < g_{n+1}).$$

The theta-hat function is also determined as the infinite limit of an integer n .

$$(17-24) \quad \hat{\vartheta}(t) = \lim_{n \rightarrow \infty} \Theta_n^E(t) \quad (t \in \mathbb{R}).$$

Graph of $\Theta_{35}^E(t)$ and $\Theta_5^E(t)$, $t \in [0, 30]$

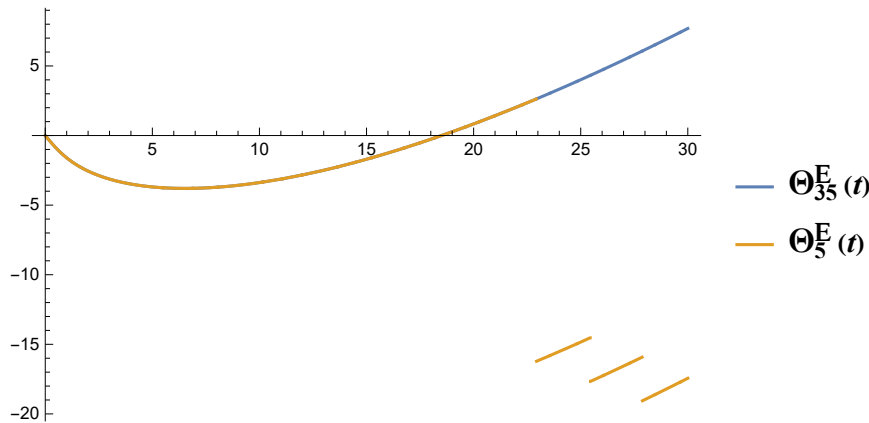


Fig. 17-4

The function $(\text{CapitalTheta} - E)_{35}(t)$ is continuous in the observational range, although the function $(\text{CapitalTheta} - E)_5(t)$ is discontinuous.

Graph of $\text{Re}(e^{i \Theta_{35}^E(t)} \eta_{11}(it))$, $\text{Im}(e^{i \Theta_{35}^E(t)} \eta_{11}(it))$,

$\text{Re}(\tilde{\eta}_{11}(it))$, and $\text{Im}(\tilde{\eta}_{11}(it))$, $t \in [0, 70]$

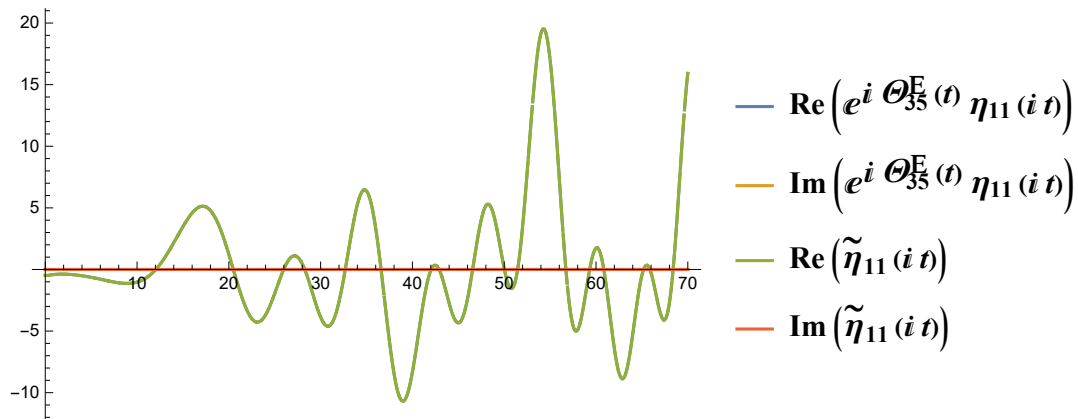


Fig. 17-5

18. Conjectures

This section will go through five kinds of conjectures.

First Conjecture

Pythagorean primes, the squares of non-Pythagorean primes, and even prime 2 would be generated as a main spectrum using information on the imaginary parts of the eta function's non-trivial zeros.

The following functions are developed to describe the next topic of the first conjecture:

$$(18-1) \quad \hat{\Omega}_c(y) := \tilde{\Omega}_c(y) - \bar{\Omega}_c(y) = - \sum_{\substack{p^n \leq c \\ p: \text{prime}, p \equiv 1 \pmod{4} \\ n \in \mathbb{N}}} \frac{\log(p)}{p^n} \cos\left(\frac{y}{2} \log(p^n)\right) \\ - \sum_{\substack{p^n \leq c \\ p: \text{prime}, p \equiv 3 \pmod{4} \\ n \in \mathbb{N}}} (-1)^n \frac{\log(p)}{p^n} \cos\left(\frac{y}{2} \log(p^n)\right) \quad (y \in \mathbb{R}, c \in \mathbb{N}).$$

The peaks of the functions $(\text{CapitalOmega} - \text{tilde})_c(y)$ would be separated into two types.

Peaks of the first type are provided by the functions

$(\text{CapitalOmega} - \text{tilde})_c(2y) + (\text{CapitalOmega} - \text{dash})_c(2y)$, and they would correspond to twice the imaginary parts of the zeta function's non-trivial zeros.

Specifically, almost half of the imaginary parts of the non-trivial zeros of the modified eta function $\eta(2\theta)$ would be generated as a spectrum using the information on all prime numbers.

Conversely, peaks of the second type are given by the functions $(\text{CapitalOmega} - \text{hat})_c(y)$, and they do not correspond to twice the imaginary parts of the zeta function's non-trivial zeros. As a spectrum, almost half the imaginary parts of the non-trivial zeros of the eta function $\eta(\theta)$ would be generated using the information on all odd prime numbers.

It can be seen that the squares of non-Pythagorean primes except prime 2 play the same role as Pythagorean primes.

I believe the following facts are reasons for it: Over the field of complex integers, both Pythagorean primes and the squares of non-Pythagorean primes are not prime numbers, while non-Pythagorean primes are prime numbers.

Pythagorean primes are uniquely represented by the sum of two squares over the standard field of real integers, but the squares of non-Pythagorean primes are represented in at least one way by the sum of three squares. In contrast, non-Pythagorean primes are represented by the sum of three or four squares. Here, even prime 2 have been omitted for simplicity.

$$(18-2) \quad \mathbf{q}_i = k^2 + l^2 \quad (k, l, i \in \mathbb{N}, \exists k < \exists l).$$

$$(18-3) \quad \mathbf{r}_i^2 = k^2 + l^2 + m^2 \quad (k, l, m, i \in \mathbb{N}, \exists k \leq \exists l \leq \exists m).$$

The following are examples of squares of non-Pythagorean primes except prime 2:

$$(18-4) \quad \mathbf{r}_1^2 = 3^2 = 1^2 + 2^2 + 2^2,$$

$$\mathbf{r}_2^2 = 7^2 = 2^2 + 3^2 + 6^2,$$

$$\mathbf{r}_3^2 = 11^2 = 2^2 + 6^2 + 9^2 = 6^2 + 6^2 + 7^2,$$

$$\mathbf{r}_4^2 = 19^2 = 1^2 + 6^2 + 18^2 = 6^2 + 6^2 + 17^2 = 6^2 + 10^2 + 15^2,$$

$$\mathbf{r}_5^2 = 23^2 = 3^2 + 6^2 + 22^2 = 3^2 + 14^2 + 18^2 = 6^2 + 13^2 + 18^2,$$

$$\mathbf{r}_6^2 = 31^2 = 5^2 + 6^2 + 30^2 = 6^2 + 14^2 + 27^2 = 6^2 + 21^2 + 22^2 = 14^2 + 18^2 + 21^2.$$

Second Conjecture

Twice the imaginary part of any non-trivial zero of the zeta function would belong to the subset of imaginary parts of the non-trivial zeros of the eta function.

Specifically, the following proposition would hold:

$$(18-5) \quad \zeta\left(\frac{1}{2} + i\rho_m\right) = 0 \implies \eta(i2\rho_m) = 0 \quad (m \in \mathbb{N}).$$

This conjecture supports the existence of an infinite number of non-trivial zeros of the eta function.

The converse proposition would not be necessarily true.

$$(18-6) \quad \eta(i\tau_m) = 0 \quad (m \in \mathbb{N}) \\ \implies \zeta\left(\frac{1}{2} + i\frac{\tau_1}{2}\right) \neq 0, \zeta\left(\frac{1}{2} + i\frac{\tau_2}{2}\right) \neq 0, \zeta\left(\frac{1}{2} + i\frac{\tau_3}{2}\right) \neq 0, \zeta\left(\frac{1}{2} + i\frac{\tau_4}{2}\right) = 0, \text{ etc.}$$

The second conjecture can be confirmed graphically in the narrow domain.

Graph of $\text{Re}(\tilde{\eta}_{11}(i2y))$, $\text{Re}(\tilde{\zeta}_8(\frac{1}{2} + iy))$, $\text{Im}(\tilde{\eta}_{11}(i2y))$, and $\text{Im}(\tilde{\zeta}_8(\frac{1}{2} + iy))$, $y \in [0, 45]$

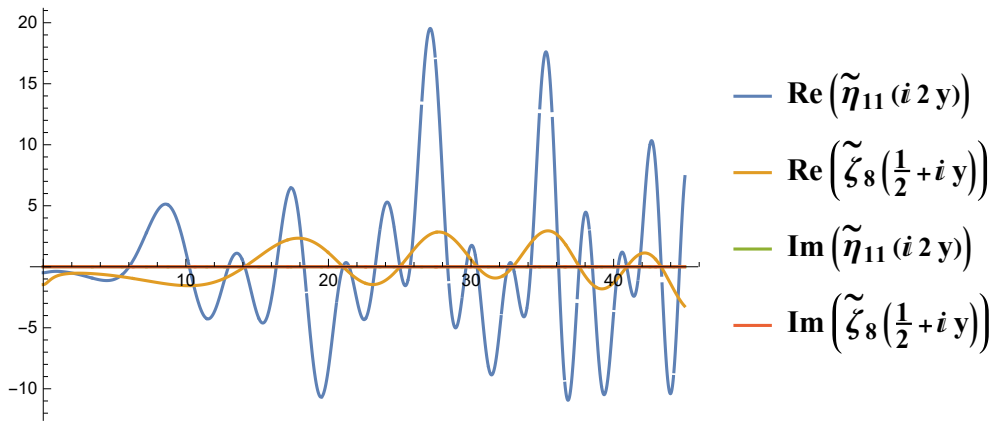


Fig. 18-1

A variable interval is constructed to describe the next topic of the second conjecture.

On the imaginary axis, the variable interval is defined as follows:

$$(18-7) \quad U(m) := \{iy : \tau_m > 0, \eta(i\tau_m) = 0, 0 \leq y \leq \tau_m\} \quad (m \in \mathbb{N}).$$

From the observational results, the following equation would hold:

$$(18-8) \quad \lim_{m \rightarrow \infty} \frac{M_1(m)}{M(m)} = \frac{1}{2}.$$

Where

$$(18-9) \quad M(m) := \#\{\tau_m : \tau_m > 0, \eta(i\tau_m) = 0, i\tau_m \in U(m)\} = m \quad (m \in \mathbb{N}).$$

$$(18-10) \quad M_1(m) := \#\{\rho_n : n \in \mathbb{N}, \rho_n > 0, \zeta(1/2 + i\rho_n) = 0, i2\rho_n \in U(m)\} \quad (m \in \mathbb{N}).$$

Here the symbol # express the number of elements of set.

Assuming the second conjecture is correct, the eta function's non-trivial zeros can be classified into two groups:

$$(18-11) \quad \begin{cases} \eta(i\hat{\tau}_k) = 0 & , \zeta\left(\frac{1}{2} + i\hat{\tau}_k\right) \neq 0 \quad (k \in \mathbb{N}), \\ \eta(i2\rho_l) = 0 & , \zeta\left(\frac{1}{2} + i\rho_l\right) = 0 \quad (l \in \mathbb{N}). \end{cases}$$

The Hadamard product representation of the Chi function is modified as follows:

$$(18-12) \quad \chi(\theta) = \chi(0) \prod_{k=1}^{\infty} \left(1 + \frac{\theta^2}{(\hat{\tau}_k)^2}\right) \prod_{l=1}^{\infty} \left(1 + \frac{\theta^2}{(2\rho_l)^2}\right) \quad (\theta \in \mathbb{C}).$$

Conversely, the Hadamard product representation of the Riemann Xi function is transformed as follows:

$$(18-13) \quad \xi(\theta) = \xi(0) \prod_{m=1}^{\infty} \frac{(2\rho_m)^2 + (2\theta-1)^2}{(2\rho_m)^2 + 1} = \frac{1}{2} \prod_{m=1}^{\infty} \frac{1 + \frac{(2\theta-1)^2}{(2\rho_m)^2}}{1 + \frac{1}{(2\rho_m)^2}} \quad (\theta \in \mathbb{C}).$$

Therefore

$$(18-14) \quad \prod_{m=1}^{\infty} \left(1 + \frac{(2\theta-1)^2}{(2\rho_m)^2}\right) = 2 \xi(\theta) \prod_{m=1}^{\infty} \left(1 + \frac{1}{(2\rho_m)^2}\right) \quad (\theta \in \mathbb{C}).$$

Replace the complex variable θ with $(1+\theta)/2$.

$$(18-15) \quad \prod_{m=1}^{\infty} \left(1 + \frac{\theta^2}{(2\rho_m)^2}\right) = 2 \xi\left(\frac{1+\theta}{2}\right) \prod_{m=1}^{\infty} \left(1 + \frac{1}{(2\rho_m)^2}\right) \quad (\theta \in \mathbb{C}).$$

Using this result,

$$(18-16) \quad \chi(\theta) = 2 \chi(0) \prod_{l=1}^{\infty} \left(1 + \frac{1}{(2\rho_l)^2}\right) \prod_{k=1}^{\infty} \left(1 + \frac{\theta^2}{(\hat{\tau}_k)^2}\right) \xi\left(\frac{1+\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

Replace the complex variable θ with $-\theta$.

$$(18-17) \quad \chi(-\theta) = 2 \chi(0) \prod_{l=1}^{\infty} \left(1 + \frac{1}{(2\rho_l)^2}\right) \prod_{k=1}^{\infty} \left(1 + \frac{\theta^2}{(\hat{\tau}_k)^2}\right) \xi\left(\frac{1-\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

I define the Chi-hat function as follows:

$$(18-18) \quad \hat{\chi}(\theta) := 2 \chi(\theta) \xi\left(\frac{1-\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

Then the Chi-hat function satisfies the following functional equation:

$$(18-19) \quad \hat{\chi}(\theta) = \hat{\chi}(-\theta) \quad (\theta \in \mathbb{C}).$$

Proceed with the calculation according to the definition.

$$(18-20) \quad \begin{aligned} \hat{\chi}(\theta) &= 2 \cdot \frac{\theta(\theta-1)}{2} \pi^{-\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}\right) \eta(\theta) \cdot \frac{1}{2} \left(\frac{1-\theta}{2}\right) \left(\frac{1-\theta}{2} - 1\right) \pi^{-\frac{1-\theta}{4}} \Gamma\left(\frac{1-\theta}{4}\right) \zeta\left(\frac{1-\theta}{2}\right) \\ &= (1-\theta)(1+\theta) \pi^{-\frac{1+\theta}{4}} \Gamma\left(1 + \frac{\theta}{2}\right) \Gamma\left(\frac{5-\theta}{4}\right) \zeta\left(\frac{1-\theta}{2}\right) \eta(\theta) \quad (\theta \in \mathbb{C}). \end{aligned}$$

I define the eta-hat function as follows:

$$(18-21) \quad \hat{\eta}(\theta) := -\frac{\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma\left(1 + \frac{\theta}{2}\right)} \hat{\chi}(\theta) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

Therefore

$$(18-22) \quad \hat{\eta}(\theta) = -(1+\theta) \pi^{-\frac{1-\theta}{4}} \Gamma\left(\frac{5-\theta}{4}\right) \zeta\left(\frac{1-\theta}{2}\right) \eta(\theta) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

In order to draw graphs, I modify the eta-hat function by two functions I have already obtained.

$$(18-23) \quad \hat{\eta}_{\lambda}(\theta) := -(1+\theta) \pi^{-\frac{1-\theta}{4}} \Gamma\left(\frac{5-\theta}{4}\right) \zeta\left(\frac{1-\theta}{2}\right) \eta_{\lambda}(\theta) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

$$(18-24) \quad \bar{\eta}_{\lambda}(\theta) := -(1+\theta) \pi^{-\frac{1-\theta}{4}} \Gamma\left(\frac{5-\theta}{4}\right) \zeta\left(\frac{1-\theta}{2}\right) \tilde{\eta}_{\lambda}(\theta) \quad (\theta \in \mathbb{C} \setminus \{1\}, \lambda \in \mathbb{N}).$$

Graph of $\cosh(2y/7) \operatorname{Re}(\hat{\eta}_8(iy))$ and $\cosh(2y/7) \operatorname{Im}(\hat{\eta}_8(iy))$ on the interval $y \in [0, 41.5]$

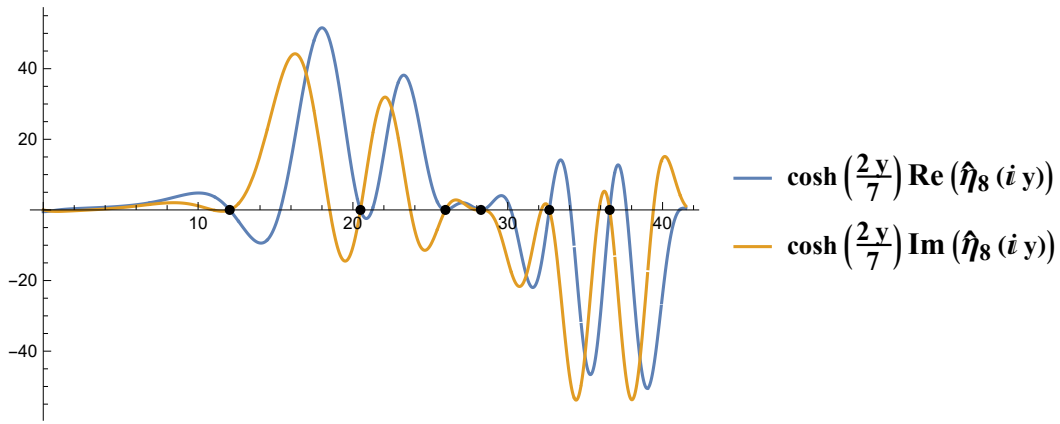


Fig. 18-2

The dots on the horizontal axis indicate the imaginary parts of the eta-hat function's non-trivial zeros. Since the eta-hat function is a rapidly decreasing function on the imaginary axis, the displayed curves are the eta-hat function multiplied by the rapidly increasing function $\cosh(2y/7)$. The real part of the eta-hat function is positive without changing sign in its vicinity except for the fourth non-trivial zero of the eta-hat function.

Graph of $\cosh(2y/7) \operatorname{Re}(\bar{\eta}_8(iy))$ and $\cosh(2y/7) \operatorname{Im}(\bar{\eta}_8(iy))$ on the interval $y \in [0, 41.5]$

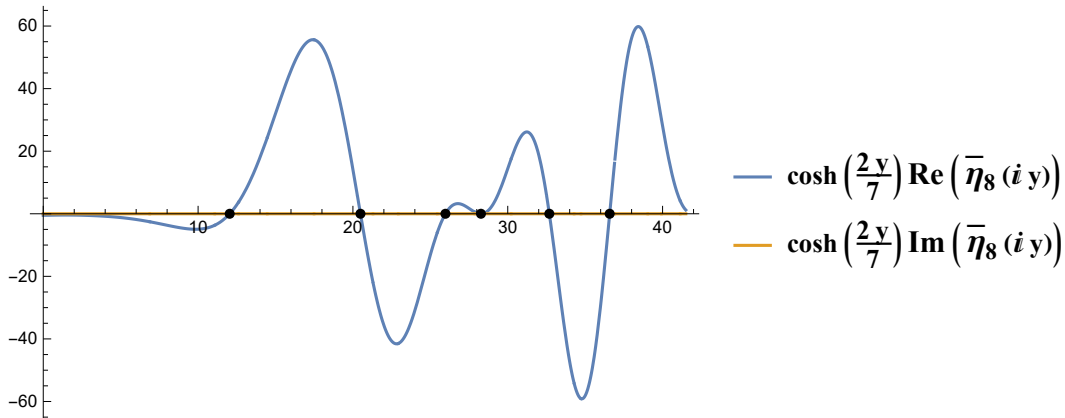


Fig. 18-3

The dots on the horizontal axis indicate the imaginary parts of the eta-dash function's non-trivial zeros. Since the eta-dash function is also a rapidly decreasing function, the displayed curve is the eta-dash function multiplied by the rapidly increasing function $\cosh(2y/7)$. The function value is positive without changing sign in its vicinity except for the fourth non-trivial zero of the eta-dash function. Here are the details.

$$(18-25) \left\{ \operatorname{Re}\left(\bar{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \operatorname{Re}\left(\bar{\eta}_8\left(i\left(\frac{12\,041\,897\,809}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-4.82204 \times 10^{-10}, 2.09032 \times 10^{-10}\}.$$

$$(18-26) \left\{ \operatorname{Re}\left(\bar{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \operatorname{Re}\left(\bar{\eta}_8\left(i\left(\frac{20\,487\,540\,608}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{2.22244 \times 10^{-10}, -1.11178 \times 10^{-10}\}.$$

$$(18-27) \left\{ \operatorname{Re}\left(\bar{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000} - 10^{-9}\right)\right)\right), \operatorname{Re}\left(\bar{\eta}_9\left(i\left(\frac{25\,976\,196\,025}{1\,000\,000\,000} + 10^{-9}\right)\right)\right) \right\} \simeq \{-6.34781 \times 10^{-12}, 1.39702 \times 10^{-11}\}.$$

$$(18-28) \left\{ \operatorname{Re} \left(\bar{\eta}_9 \left(i \left(2 \times \frac{141\,347\,251\,417}{10\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_9 \left(i \left(2 \times \frac{141\,347\,251\,417}{10\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{ 2.96614 \times 10^{-21}, 2.24625 \times 10^{-21} \}.$$

$$(18-29) \left\{ \operatorname{Re} \left(\bar{\eta}_{10} \left(i \left(\frac{32\,685\,214\,209}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{10} \left(i \left(\frac{32\,685\,214\,209}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{ 7.09716 \times 10^{-12}, -4.98882 \times 10^{-12} \}.$$

$$(18-30) \left\{ \operatorname{Re} \left(\bar{\eta}_{10} \left(i \left(\frac{36\,583\,986\,392}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{10} \left(i \left(\frac{36\,583\,986\,392}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{ -3.76781 \times 10^{-12}, 2.27485 \times 10^{-12} \}.$$

Graph of $\cosh(2y/7) \operatorname{Re}(\bar{\eta}_8(iy))$ and $\cosh(2y/7) \operatorname{Im}(\bar{\eta}_8(iy))$ on the interval $y \in [41.5, 43.5]$

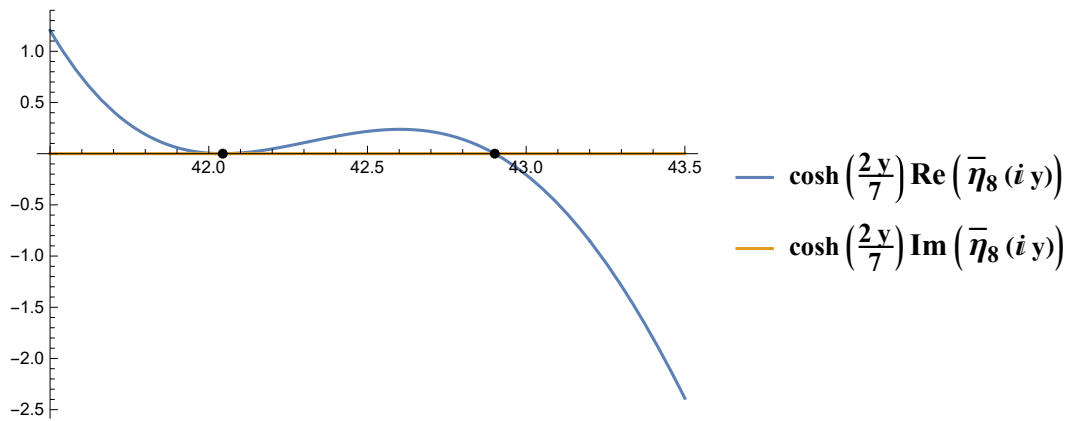


Fig. 18-4

The function value is also positive without changing sign in its vicinity except for the seventh non-trivial zero of the eta-dash function. Here are the details.

$$(18-31) \left\{ \operatorname{Re} \left(\bar{\eta}_{11} \left(i \left(2 \times \frac{210\,220\,396\,388}{10\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{11} \left(i \left(2 \times \frac{210\,220\,396\,388}{10\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{ 2.59974 \times 10^{-23}, 3.26487 \times 10^{-23} \}.$$

$$(18-32) \left\{ \operatorname{Re} \left(\bar{\eta}_{11} \left(i \left(\frac{42\,901\,222\,688}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{11} \left(i \left(\frac{42\,901\,222\,688}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{ 1.58941 \times 10^{-14}, -1.69816 \times 10^{-14} \}.$$

Graph of $\cosh(2y/7) \operatorname{Re}(\bar{\eta}_8(iy))$ and $\cosh(2y/7) \operatorname{Im}(\bar{\eta}_8(iy))$ on the interval $y \in [43.5, 52]$

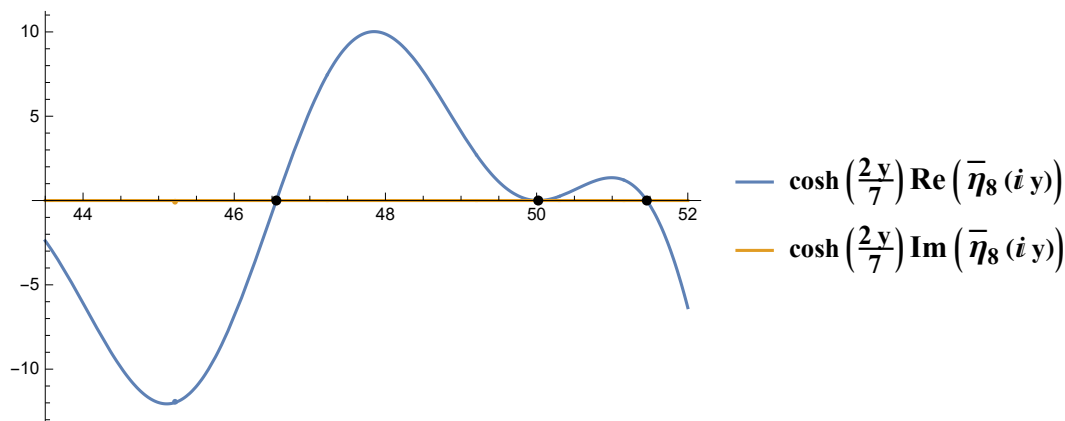


Fig. 18-5

The function value is also positive without changing sign in its vicinity except for the tenth non-trivial zero of the eta-dash function. Here are the details.

$$(18-33) \left\{ \operatorname{Re} \left(\bar{\eta}_{11} \left(i \left(\frac{46\,556\,753\,041}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{11} \left(i \left(\frac{46\,556\,753\,041}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-3.94975 \times 10^{-14}, 4.64542 \times 10^{-14}\}.$$

$$(18-34) \left\{ \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(2 \times \frac{250\,108\,575\,801}{10\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(2 \times \frac{250\,108\,575\,801}{10\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{5.41705 \times 10^{-24}, 3.75474 \times 10^{-24}\}.$$

$$(18-35) \left\{ \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(\frac{51\,457\,512\,850}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(\frac{51\,457\,512\,850}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{6.26619 \times 10^{-15}, -4.37742 \times 10^{-15}\}.$$

Graph of $\cosh(2y/7) \operatorname{Re}(\bar{\eta}_8(iy))$ and $\cosh(2y/7) \operatorname{Im}(\bar{\eta}_8(iy))$ on the interval $y \in [52, 63]$

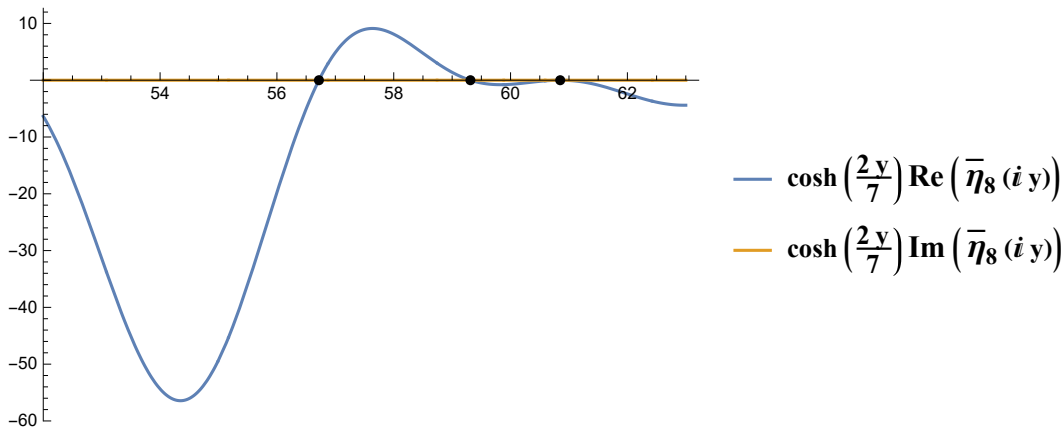


Fig. 18-6

The function value is negative without changing sign in its vicinity except for the fourteenth non-trivial zero of the eta-dash function. Here are the details.

$$(18-36) \left\{ \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(\frac{56\,719\,268\,686}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(\frac{56\,719\,268\,686}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-3.98984 \times 10^{-15}, 3.60511 \times 10^{-15}\}.$$

$$(18-37) \left\{ \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(\frac{59\,312\,768\,029}{1\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{12} \left(i \left(\frac{59\,312\,768\,029}{1\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{3.41677 \times 10^{-16}, -2.34359 \times 10^{-16}\}.$$

$$(18-38) \left\{ \operatorname{Re} \left(\bar{\eta}_{13} \left(i \left(2 \times \frac{304\,248\,761\,259}{10\,000\,000\,000} - 10^{-9} \right) \right) \right), \operatorname{Re} \left(\bar{\eta}_{13} \left(i \left(2 \times \frac{304\,248\,761\,259}{10\,000\,000\,000} + 10^{-9} \right) \right) \right) \right\} \simeq \{-9.04002 \times 10^{-26}, -1.25067 \times 10^{-25}\}.$$

The imaginary part of the eta-dash function’s m-th contact zero with the imaginary axis would be twice the imaginary part of the zeta function’s m-th non-trivial zero. This is a paraphrase of the second conjecture.

Third Conjecture

Variable intervals are constructed to describe the first topic of the third conjecture. On the imaginary axis, the variable intervals are defined as follows:

$$(18-39) \forall (m) := \left\{ iy : \rho_m > 0, \zeta(1/2 + iy) = 0, 0 \leq y \leq \rho_m \right\} \quad (m \in \mathbb{N}).$$

From the observational results, the following relations of inclusion would hold:

$$(18-40) \quad \mathbf{N}(m) \in \{m-1, m, m+1\} \quad (m \in \mathbb{N}).$$

Where

$$(18-41) \quad \mathbf{N}(m) := \#\{\tau_n : n \in \mathbb{N}, \tau_n > 0, \eta(i\tau_n) = 0, i\tau_n \in \mathbf{V}(m)\} \quad (m \in \mathbb{N}).$$

The second topic: concerning the relation between the imaginary parts of non-trivial zeros of the zeta function and those of the eta function, I noticed that it is neither too close nor too far away in the observational range.

The following relations between ρ_m and τ_m would be true in a narrow domain:

$$(18-42) \quad \tau_m < \rho_{m+1} < \tau_{m+3} \quad (m \in \mathbb{N}).$$

$$(18-43) \quad \rho_m < \tau_{m+2} < \rho_{m+3} \quad (m \in \mathbb{N}).$$

Fourth Conjecture

The following equation is defined to describe the fourth conjecture:

$$(18-44) \quad \lambda(\theta) := \zeta(\theta) + \eta(\theta) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

And the habitable zone is defined.

$$(18-45) \quad \text{Habitable zone} := \{\theta \in \mathbb{C} : 0 < \operatorname{Re}(\theta) < 1/2\}.$$

The explicit formula for the lambda function is given as follows:

$$(18-46) \quad \lambda(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma(1+\frac{\theta}{2})} \sum_{p=1}^{\infty} \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + 2(7-\theta)(2\pi p)^2) \mathbb{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 4(2\pi p)^2)(2\pi p) \mathbb{K}_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

The rule in subsection 6-3 (using the character k instead of λ) is applied to the above equation.

$$(18-47) \quad \lambda_k(\theta) = \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma(1+\frac{\theta}{2})} \sum_{p=1}^k \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + 2(7-\theta)(2\pi p)^2) \mathbb{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 4(2\pi p)^2)(2\pi p) \mathbb{K}_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}, k \in \mathbb{N}).$$

I define that the positive imaginary part of the m -th non-trivial zero of the lambda function, which is the number allocated in the order that is closer to the real axis, is α_m . Under this assumption,

$$(18-48) \quad \lambda(u_m \pm i\alpha_m) := 0 \quad (u_m, \alpha_m \in \mathbb{R}, m \in \mathbb{N}).$$

Here the real part of the m -th non-trivial zero of the lambda function is u_m .

The fourth conjecture is that: all non-trivial zeros of the lambda function would be present in the habitable zone.

The following function is defined for drawing a graph that just shows the lambda function's non-trivial zeros:

$$(18-49) \quad \tilde{\lambda}_k(\theta) := \left| \frac{2\pi^{\frac{\theta}{2}}}{(1-\theta)\Gamma(1+\frac{\theta}{2})} \right| \sum_{p=1}^k \sigma_{-\theta}(p) p^{\frac{\theta}{2}} \left((\theta(\theta-1) + 2(7-\theta)(2\pi p)^2) \mathbb{K}_{\frac{\theta}{2}}(2\pi p) \right. \\ \left. + ((\theta-1) - 4(2\pi p)^2)(2\pi p) \mathbb{K}_{\frac{2-\theta}{2}}(2\pi p) \right) \quad (\theta \in \mathbb{C} \setminus \{1\}).$$

3 D graph of $\text{Re} \left(\tilde{\lambda}_8 (x + i y) \right)$, Zero, and $4 \text{Im} \left(\tilde{\lambda}_8 (x + i y) \right)$,

$x \in [0, 1/2]$, $y \in [8, 40]$, Viewpoint \rightarrow Above

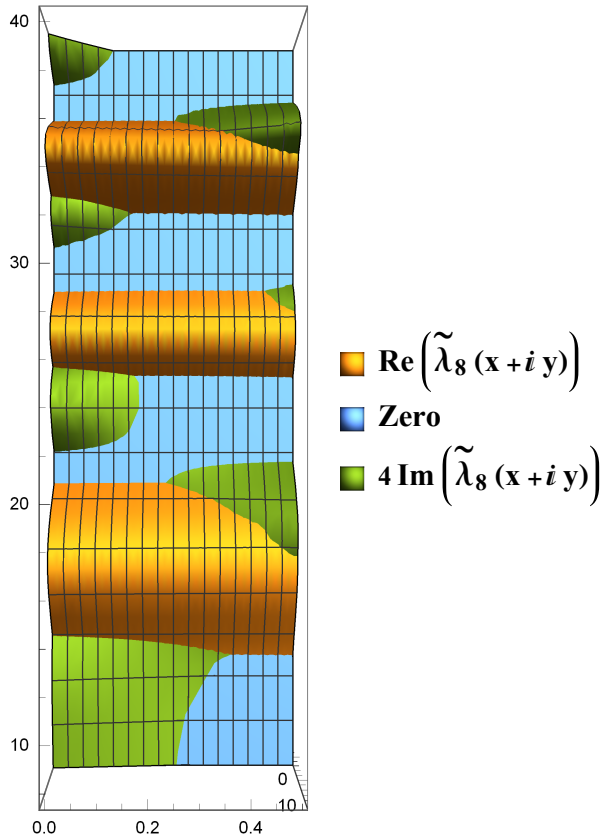


Fig. 18-7

The non-trivial zeros of the lambda function are the cross points of three colors.

The coefficient number 4 is used in front of the imaginary part to emphasize the non-trivial zeros.

Here are the details.

$$(18 - 50) \quad u_1 + i \alpha_1 \approx 0.36360066983 + i 12.970445119,$$

$$u_2 + i \alpha_2 \approx 0.23314410196 + i 20.645405240,$$

$$u_3 + i \alpha_3 \approx 0.16959589417 + i 25.432819891,$$

$$u_4 + i \alpha_4 \approx 0.41836013776 + i 29.250622535,$$

$$u_5 + i \alpha_5 \approx 0.17066664686 + i 32.737058741,$$

$$u_6 + i \alpha_6 \approx 0.23291116250 + i 36.864192387.$$

Calculation errors of non-trivial zeros in the lambda function are shown as follows:

$$(18 - 51) \quad \left| \lambda_8 \left(\frac{36360066983}{100000000000} + i \frac{12970445119}{1000000000} \right) \right| \approx 4.41687 \times 10^{-10}.$$

$$(18 - 52) \quad \left| \lambda_8 \left(\frac{23314410196}{100000000000} + i \frac{20645405240}{1000000000} \right) \right| \approx 8.02663 \times 10^{-7}.$$

$$(18 - 53) \quad \left| \lambda_9 \left(\frac{16959589417}{100000000000} + i \frac{25432819891}{1000000000} \right) \right| \approx 9.03159 \times 10^{-9}.$$

$$(18 - 54) \quad \left| \lambda_9 \left(\frac{41836013776}{100000000000} + i \frac{29250622535}{1000000000} \right) \right| \approx 1.31080 \times 10^{-8}.$$

$$(18 - 55) \quad \left| \lambda_9 \left(\frac{17066664686}{100000000000} + i \frac{32737058741}{1000000000} \right) \right| \approx 5.91107 \times 10^{-8}.$$

$$(18-56) \quad \left| \lambda_{11} \left(\frac{23\,291\,116\,250}{100\,000\,000\,000} + i \frac{36\,864\,192\,387}{1\,000\,000\,000} \right) \right| \approx 6.99155 \times 10^{-10} .$$

Fifth Conjecture

This is a conjecture regarding diverging series that diverge slower than the reciprocal series of the Pythagorean primes.

Remember that the sum of three squares can be used to express the square in non-Pythagorean primes except prime 2.

Consider adding the constraint that the sum of two from the smallest of the three squares is a prime number.

The following simultaneous Diophantine equations describe this:

$$(18-57) \quad \begin{cases} r_i^2 = x^2 + y^2 + z^2 & (x, y, z, i \in \mathbb{N}, x \leq y \leq z), \\ \sigma_0(x^2 + y^2) = 2 & (x, y \in \mathbb{N}, x < y). \end{cases}$$

Some non-Pythagorean primes have unique solutions, and all verified solutions are of this form.

$$(18-58) \quad r_i^2 = k_i^2 + l_i^2 + (r_i - 1)^2 \quad (\exists k_i, \exists l_i \in \mathbb{N}, i = 1, 2, 4, 6, \dots, 174\,491).$$

Since primes except even prime 2, represented by the sum of two squares are limited to Pythagorean primes, let it be the j-th Pythagorean prime q_j. Furthermore, I assume that there are infinitely many solutions with this form.

$$(18-59) \quad r_i^2 = q_j + (r_i - 1)^2 \quad ((i, j) = (1, 1), (2, 2), (4, 5), (6, 8), \dots).$$

Since the square of r_i is canceled on both sides,

$$(18-60) \quad 2 r_i - 1 = q_j \quad ((i, j) = (1, 1), (2, 2), (4, 5), (6, 8), \dots).$$

I obtain a rare relation between the non-Pythagorean primes and Pythagorean primes by adding only one condition.

After this, the rare relationship is shown in two graphs.

Graph of the points (2 r_i - 1, q_j) with the line y = x, x ∈ [0, 500]

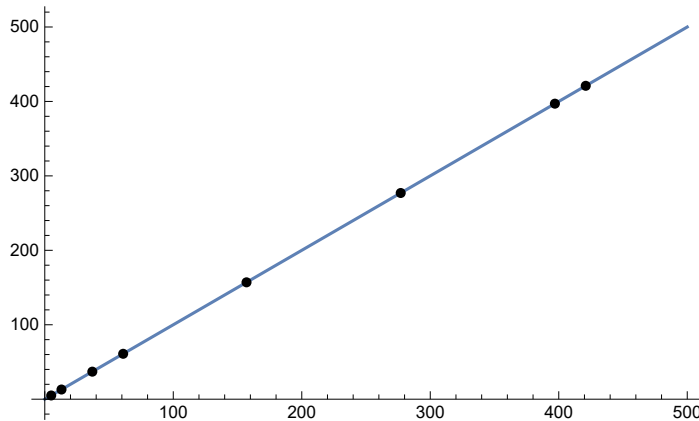


Fig. 18-8

The dot on the graph indicates the point (2 r_i - 1, q_j) is on the line y = x.

Here are the details.

$$(18-61) \quad q_1 = 5 = 2 \times 3 - 1 = 2 r_1 - 1,$$

$$q_2 = 13 = 2 \times 7 - 1 = 2 r_2 - 1,$$

$$q_5 = 37 = 2 \times 19 - 1 = 2 r_4 - 1,$$

$$q_8 = 61 = 2 \times 31 - 1 = 2 r_6 - 1,$$

$$\begin{aligned}
 \mathbf{q}_{17} &= 157 = 2 \times 79 - 1 = 2 \mathbf{r}_{12} - 1, \\
 \mathbf{q}_{27} &= 277 = 2 \times 139 - 1 = 2 \mathbf{r}_{18} - 1, \\
 \mathbf{q}_{37} &= 397 = 2 \times 199 - 1 = 2 \mathbf{r}_{24} - 1, \\
 \mathbf{q}_{40} &= 421 = 2 \times 211 - 1 = 2 \mathbf{r}_{25} - 1 .
 \end{aligned}$$

Graph of the points $(2 \mathbf{r}_i - 1, \mathbf{q}_i)$ with the line $y = x, x \in [10000000, 10010000]$

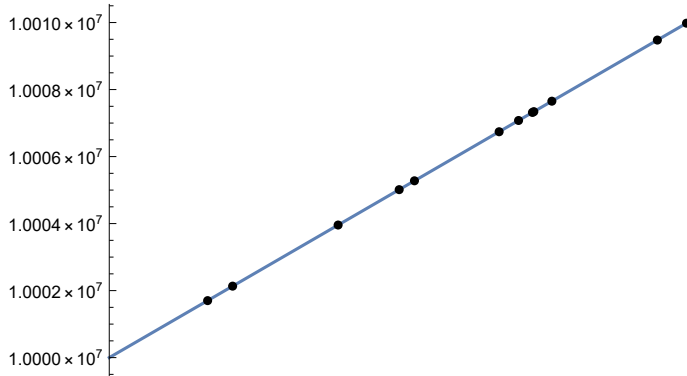


Fig. 18-9

There is no regularity in distribution even for very high numbers.

On the contrary, adjacent non-Pythagorean primes are found.

Here are the details.

$$\begin{aligned}
 (18 - 62) \quad \mathbf{q}_{332\,228} &= 10\,001\,701 = 2 \times 5\,000\,851 - 1 = 2 \mathbf{r}_{174\,346} - 1, \\
 \mathbf{q}_{332\,245} &= 10\,002\,133 = 2 \times 5\,001\,067 - 1 = 2 \mathbf{r}_{174\,354} - 1, \\
 \mathbf{q}_{332\,293} &= 10\,003\,957 = 2 \times 5\,001\,979 - 1 = 2 \mathbf{r}_{174\,390} - 1, \\
 \mathbf{q}_{332\,319} &= 10\,005\,013 = 2 \times 5\,002\,507 - 1 = 2 \mathbf{r}_{174\,403} - 1, \\
 \mathbf{q}_{332\,327} &= 10\,005\,277 = 2 \times 5\,002\,639 - 1 = 2 \mathbf{r}_{174\,407} - 1, \\
 \mathbf{q}_{332\,372} &= 10\,006\,741 = 2 \times 5\,003\,371 - 1 = 2 \mathbf{r}_{174\,434} - 1, \\
 \mathbf{q}_{332\,384} &= 10\,007\,077 = 2 \times 5\,003\,539 - 1 = 2 \mathbf{r}_{174\,442} - 1, \\
 \mathbf{q}_{332\,392} &= 10\,007\,317 = 2 \times 5\,003\,659 - 1 = 2 \mathbf{r}_{174\,445} - 1, \\
 \mathbf{q}_{332\,394} &= 10\,007\,341 = 2 \times 5\,003\,671 - 1 = 2 \mathbf{r}_{174\,446} - 1, \\
 \mathbf{q}_{332\,402} &= 10\,007\,653 = 2 \times 5\,003\,827 - 1 = 2 \mathbf{r}_{174\,453} - 1, \\
 \mathbf{q}_{332\,451} &= 10\,009\,477 = 2 \times 5\,004\,739 - 1 = 2 \mathbf{r}_{174\,481} - 1, \\
 \mathbf{q}_{332\,462} &= 10\,009\,981 = 2 \times 5\,004\,991 - 1 = 2 \mathbf{r}_{174\,491} - 1 .
 \end{aligned}$$

Up to 10010000, there are 28202 conditional Pythagorean primes, but 56089 Sophie Germain primes. Here, Sophie Germain prime is a prime number that is prime again by adding 1 to it's double.

It is unknown whether there are infinitely many Sophie Germain primes “[9].”

Let now define the m -th conditional Pythagorean prime as twice the m -th conditional non-Pythagorean prime minus 1. They are denoted as \hat{q}_m -hat and \hat{r}_m -hat, respectively.

Under the premise, there are infinitely many pairs of \hat{q}_m -hat and \hat{r}_m -hat.

$$(18 - 63) \quad \hat{\mathbf{q}}_m = 2 \hat{\mathbf{r}}_m - 1 \quad (m \in \mathbb{N}).$$

Let $S_q(m)$ be the reciprocal sum up to the m -th conditional Pythagorean prime.

$$(18-64) \quad S_q(m) := \sum_{i=1}^m \hat{q}_i^{-1} = \frac{1}{5} + \frac{1}{13} + \frac{1}{37} + \frac{1}{61} + \frac{1}{157} + \frac{1}{277} + \frac{1}{397} + \cdots + \hat{q}_m^{-1}.$$

Similarly, let $S_r(m)$ be the reciprocal sum up to the m -th conditional non-Pythagorean prime.

$$(18-65) \quad S_r(m) := \sum_{i=1}^m \hat{r}_i^{-1} = \frac{1}{3} + \frac{1}{7} + \frac{1}{19} + \frac{1}{31} + \frac{1}{79} + \frac{1}{139} + \frac{1}{199} + \cdots + \hat{r}_m^{-1}.$$

The following inequality holds for any positive integer:

$$(18-66) \quad S_r(m) > S_q(m) \quad (m \in \mathbb{N}).$$

The following reciprocal series of conditional Pythagorean primes would be a divergent series “[11]”:

$$(18-67) \quad \frac{1}{5} + \frac{1}{13} + \frac{1}{37} + \frac{1}{61} + \frac{1}{157} + \frac{1}{277} + \frac{1}{397} + \cdots + \hat{q}_m^{-1} + \cdots = +\infty.$$

The following reciprocal series of conditional non-Pythagorean primes would also be a divergent series:

$$(18-68) \quad \frac{1}{3} + \frac{1}{7} + \frac{1}{19} + \frac{1}{31} + \frac{1}{79} + \frac{1}{139} + \frac{1}{199} + \cdots + \hat{r}_m^{-1} + \cdots = +\infty.$$

19. Additional knowledge

19-1. Proof of the absolute convergence of the integral

$$\int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C})$$

At first, the function of three real variables is defined as follows:

$$(19-1) \quad u(x, t, \alpha) := x^{t-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} \quad (x, \alpha > 0, t \in \mathbb{R}).$$

The variable x integrates the above function into the open interval $(0, \infty)$.

$$(19-2) \quad \int_0^{\infty} u(x, t, \alpha) dx = \int_0^{\infty} x^{t-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, t \in \mathbb{R}).$$

The left-sided limit of the integrand is zero as follows:

$$(19-3) \quad \lim_{x \rightarrow +0} u(x, t, \alpha) = \lim_{x \rightarrow +0} e^{-x^2} \cdot \lim_{x \rightarrow +0} x^{t-1} e^{-\frac{\alpha^2}{x^2}} \\ = \lim_{y \rightarrow \infty} y^{1-t} e^{-\alpha^2 y^2} = 0 \quad (\alpha > 0, t \in \mathbb{R}).$$

The right-sided limit of the integrand is also zero, as follows:

$$(19-4) \quad \lim_{x \rightarrow \infty} u(x, t, \alpha) = \lim_{x \rightarrow \infty} e^{-\frac{\alpha^2}{x^2}} \cdot \lim_{x \rightarrow \infty} x^{t-1} e^{-x^2} \\ = \lim_{y \rightarrow +0} e^{-\alpha^2 y^2} \cdot \lim_{x \rightarrow \infty} x^{t-1} e^{-x^2} = 0 \quad (\alpha > 0, t \in \mathbb{R}).$$

The integrand is partially differentiated by the variable x .

$$(19-5) \quad \frac{\partial}{\partial x} u(x, t, \alpha) = ((t-1)x^{t-2} - 2x^t + 2\alpha^2 x^{t-4}) e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)}$$

$$= -(2x^4 + (1-t)x^2 - 2\alpha^2)x^{t-4} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} \quad (x, \alpha > 0, t \in \mathbb{R}).$$

And I assume that the integrand's differentiation is zero at $x = \beta$ ($\beta > 0$).

$$(19-6) \quad \frac{\partial}{\partial x} u(x, t, \alpha) \Big|_{x=\beta} = 0 \quad (x, \alpha, \beta > 0, t \in \mathbb{R}).$$

The dependent variable β is then determined as the solution of the following quartic equation:

$$(19-7) \quad 2\beta^4 + (1-t)\beta^2 - 2\alpha^2 = 0 \quad (\alpha, \beta > 0, t \in \mathbb{R}).$$

The quartic equation is defined as the quadratic equation of the variable β^2 . Based on the formula for solving the quadratic equations,

$$(19-8) \quad \beta^2 = \frac{-(1-t) + \sqrt{(1-t)^2 + 16\alpha^2}}{4} \quad (\alpha > 0, t \in \mathbb{R}).$$

The quartic equation has the unique solution β in the open interval $(0, \infty)$.

$$(19-9) \quad \beta = \frac{1}{2} \sqrt{\sqrt{(1-t)^2 + 16\alpha^2} - (1-t)} > 0 \quad (\alpha > 0, t \in \mathbb{R}).$$

Table of the increase and decrease for the integrand

x	+0	β	∞
$\frac{\partial}{\partial x} u(x, t, \alpha)$		+	0	-	
$u(x, t, \alpha)$	0	↗	maximum	↘	0

Fig. 19-1

The open interval of the improper integral is divided into two semi-open intervals on the right side of equation (19-2) as follows:

$$(19-10) \quad \int_0^\infty u(x, t, \alpha) dx = \int_0^1 x^{t-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx + \int_1^\infty x^{t-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, t \in \mathbb{R}).$$

For the second term of the right side, I perform the variable transformation $x = y^{-1}$.

$$(19-11) \quad \int_1^\infty x^{t-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \int_1^0 \left(\frac{1}{y}\right)^{t-1} e^{-\left(\frac{1}{y^2} + \alpha^2 y^2\right)} (-y^{-2}) dy = \int_0^1 y^{-1-t} e^{-\left(\frac{1}{y^2} + \alpha^2 y^2\right)} dy \quad (\alpha > 0, t \in \mathbb{R}).$$

The following inequality is obtained by applying the Schwarz's inequality to the equation (19-10):

$$(19-12) \quad \int_0^\infty |u(x, t, \alpha)| dx \leq \int_0^1 x^{t-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx + \int_0^1 y^{-1-t} e^{-\left(\frac{1}{y^2} + \alpha^2 y^2\right)} dy \quad (\alpha > 0, t \in \mathbb{R}).$$

The function of three real variables $v(x, t, \alpha)$ is defined as the integrand of the result of equation (19-11) for convenience.

$$(19-13) \quad v(x, t, \alpha) := x^{-1-t} e^{-\left(\frac{1}{x^2} + \alpha^2 x^2\right)} \quad (x, \alpha > 0, t \in \mathbb{R}).$$

Hereafter, I will find the upper limit of the improper integral of the function $|u(x, t, \alpha)|$.

I begin by considering the two-dimensional $\alpha - t$ phase space.

The boundary curve divides the space's right-half-plane into two regions.

To generate the curve's representation, 1 is substituted for β on the left side of equation (19-8).

The result is

$$(19-14) \quad t = 3 - 2\alpha^2 \quad (\alpha > 0).$$

Of course, the value of β is equal to 1 throughout the boundary curve.

Under the condition of $0 < \beta \leq 1$,

$$(19-15) \quad t \leq 3 - 2\alpha^2 \quad (\alpha > 0).$$

The improper integral's upper limit is calculated as the sum of two square areas.

$$(19-16) \quad \int_0^\infty |u(x, t, \alpha)| dx \leq 1 \cdot u(\beta, t, \alpha) + 1 \cdot v(1, t, \alpha) = \\ \beta^{t-1} e^{-\left(\beta^2 + \frac{\alpha^2}{\beta^2}\right)} + e^{-(\alpha^2+1)} \quad (\alpha > 0, t \leq 3 - 2\alpha^2).$$

Under the condition of $\beta > 1$ i.e.,

$$(19-17) \quad t > 3 - 2\alpha^2 \quad (\alpha > 0).$$

The upper limit of the improper integral is also calculated as the sum of two square areas.

$$(19-18) \quad \int_0^\infty |u(x, t, \alpha)| dx \leq 1 \cdot u(1, t, \alpha) + 1 \cdot v\left(\frac{1}{\beta}, t, \alpha\right) = \\ e^{-(\alpha^2+1)} + \left(\frac{1}{\beta}\right)^{-1-t} e^{-\left(\beta^2 + \frac{\alpha^2}{\beta^2}\right)} = \beta^{t+1} e^{-\left(\beta^2 + \frac{\alpha^2}{\beta^2}\right)} + e^{-(\alpha^2+1)} \quad (\alpha > 0, t > 3 - 2\alpha^2).$$

The results of equations (19-16) and (19-18) show that the improper integral absolutely converges.

Finally, I expand the domain of definition of the variable t from a real number to a complex number.

To clearly express this, I change the character from t to θ .

Herein, I take the absolute value of the improper integral.

$$(19-19) \quad \left| \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \right| \leq \int_0^\infty |x^{\theta-1}| e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \\ \int_0^\infty x^{-1+\text{Re}(\theta)} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C}).$$

The right side in inequality (19-19) is the same to substituting the real value $\text{Re}(\theta)$ for t in the equation (19-2).

Therefore, the proof of the assertion in the title of this subsection is complete.

19-2. Proof of the recurrence formula for the integral

$$\int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C})$$

I apply the integration by parts to the integral excluding the case when $\theta = 0$ as follows:

$$(19-20) \quad \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \int_0^\infty \left(\frac{x^\theta}{\theta}\right)' e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \\ = \left[\frac{x^\theta}{\theta} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} \right]_0^\infty - \int_0^\infty \frac{x^\theta}{\theta} \left(-2x + \frac{2\alpha^2}{x^3}\right) e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx$$

$$= \frac{2}{\theta} \int_0^\infty x^{\theta+1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx - \frac{2\alpha^2}{\theta} \int_0^\infty x^{\theta-3} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C} \setminus \{0\}).$$

Thus, the integral of the second term on the right side of the result is written as follows:

$$(19-21) \int_0^\infty x^{\theta-3} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{\alpha^2} \left(\int_0^\infty x^{\theta+1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx - \frac{\theta}{2} \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \right) \quad (\alpha > 0, \theta \in \mathbb{C} \setminus \{0\}).$$

In case of $\theta = 0$, I perform the variable transformation $x = y^{-1}$ for the left side of the above equation.

$$(19-22) \int_0^\infty x^{-3} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \int_\infty^0 \left(\frac{1}{y}\right)^{-3} e^{-\left(\frac{1}{y^2} + \alpha^2 y^2\right)} (-y^{-2}) dy = \int_0^\infty y e^{-\left(\frac{1}{y^2} + \alpha^2 y^2\right)} dy \quad (\alpha > 0).$$

I perform the variable transformation $y = x/\alpha$, one more time.

$$(19-23) \int_0^\infty y e^{-\left(\frac{1}{y^2} + \alpha^2 y^2\right)} dy = \int_0^\infty \frac{x}{\alpha} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} \frac{1}{\alpha} dx = \frac{1}{\alpha^2} \int_0^\infty x e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0).$$

For the right side of equation (19-21), 0 is substituted directly for θ and the result is the same to the above result.

Thus, I can include the condition $\theta = 0$ for the equation (19-21).

$$(19-24) \int_0^\infty x^{\theta-3} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{\alpha^2} \left(\int_0^\infty x^{\theta+1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx - \frac{\theta}{2} \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \right) \quad (\alpha > 0, \theta \in \mathbb{C}).$$

The integral of the first term on the right side of equation (19-24) is written as follows:

$$(19-25) \int_0^\infty x^{\theta+1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \alpha^2 \int_0^\infty x^{\theta-3} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx + \frac{\theta}{2} \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C}).$$

For equation (19-25), 0 is substituted for α .

And for the integral of the left side, I perform the variable transformation $x = y^{1/2}$.

$$(19-26) \int_0^\infty x^{\theta+1} e^{-x^2} dx = \int_0^\infty y^{\frac{\theta+1}{2}} e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy = \frac{1}{2} \int_0^\infty y^{\frac{\theta}{2}} e^{-y} dy = \frac{1}{2} \Gamma\left(1 + \frac{\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

For equation (19-25), 0 is also substituted for α .

And for the integral of the right side, I perform the variable transformation $x = y^{1/2}$.

$$(19-27) \frac{\theta}{2} \int_0^{\infty} x^{\theta-1} e^{-x^2} dx =$$

$$\frac{\theta}{2} \int_0^{\infty} y^{\frac{\theta-1}{2}} e^{-y} \frac{1}{2} y^{-\frac{1}{2}} dy = \frac{\theta}{4} \int_0^{\infty} y^{\frac{\theta}{2}-1} e^{-y} dy = \frac{\theta}{4} \Gamma\left(\frac{\theta}{2}\right) = \frac{1}{2} \Gamma\left(1 + \frac{\theta}{2}\right) \quad (\theta \in \mathbb{C}).$$

Both sides of the equation (19-25) are the same when $\alpha = 0$ and the results have same poles at even numbers of -2 or less. As a result, the integral's recurrence formula is as follows:

$$(19-28) \int_0^{\infty} x^{\theta+1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx =$$

$$\alpha^2 \int_0^{\infty} x^{\theta-3} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx + \frac{\theta}{2} \int_0^{\infty} x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C}) \vee$$

$$(\alpha = 0, \theta \in \mathbb{C} \setminus \{-2n : n \in \mathbb{N}\}).$$

19-3. Representations for the integrals

$$\int_0^{\infty} x^{2p} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, p = -1) \vee (\alpha \geq 0, p = 0, 1, 2, \dots).$$

When $p = 0$,

I consider the integral

$$\int_0^{\infty} x^0 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha \geq 0).$$

At first, I show the equation of definition of the function $J(\alpha)$ for preparation “[10].”

$$(19-29) \mathbf{J}(\alpha) := \int_0^{\infty} e^{-\left(x - \frac{\alpha}{x}\right)^2} dx \quad (\alpha \in \mathbb{R}).$$

The function $J(\alpha)$ is also written as follows:

$$(19-30) \mathbf{J}(\alpha) = e^{2\alpha} \int_0^{\infty} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx > 0 \quad (\alpha \in \mathbb{R}).$$

The value of $J(0)$ is given follows:

$$(19-31) \mathbf{J}(0) = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad (\text{Gaussian integral}).$$

The value of left-sided limit of $J(\alpha)$ is given as follows:

$$(19-32) \lim_{\alpha \rightarrow -\infty} \mathbf{J}(\alpha) = \lim_{\alpha \rightarrow -\infty} e^{2\alpha} \cdot \lim_{\alpha \rightarrow -\infty} \int_0^{\infty} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \leq \lim_{\alpha \rightarrow -\infty} e^{2\alpha} \cdot \int_0^{\infty} e^{-x^2} dx = 0.$$

Assuming that the function $J(\alpha)$ has uniform convergence, the differentiation of $J(\alpha)$ is carried out for the integrand as partial differentiation.

$$(19-33) \frac{d}{d\alpha} \mathbf{J}(\alpha) = \int_0^{\infty} \frac{\partial}{\partial \alpha} e^{-\left(x - \frac{\alpha}{x}\right)^2} dx = \int_0^{\infty} -2\left(x - \frac{\alpha}{x}\right) \left(-\frac{1}{x}\right) e^{-\left(x - \frac{\alpha}{x}\right)^2} dx \quad (\alpha \in \mathbb{R}).$$

Therefore

$$(19-34) \frac{d}{d\alpha} \mathbf{J}(\alpha) = 2 \int_0^{\infty} e^{-\left(x - \frac{\alpha}{x}\right)^2} dx - 2\alpha \int_0^{\infty} e^{-\left(\frac{1}{y} - \alpha y\right)^2} dy \quad (\alpha \in \mathbb{R}).$$

When $\alpha < 0$, the differentiation of $J(\alpha)$ takes positive value as follows:

$$(19-35) \quad \frac{d}{d\alpha} J(\alpha) = 2 \int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx - 2\alpha \int_0^\infty e^{-\left(-\frac{\alpha}{x}+x\right)^2} \left(-\frac{dx}{\alpha}\right) = 4 \int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx > 0 \quad (\alpha < 0).$$

The right-sided limit of the differentiation of $J(\alpha)$ in the neighborhood of $\alpha = 0$ takes the fixed value as follows:

$$(19-36) \quad \frac{d}{d\alpha} J(\alpha) \Big|_{\alpha=-0} = 4 \int_0^\infty e^{-x^2} dx = 4 \cdot \frac{\sqrt{\pi}}{2} = 2\sqrt{\pi}.$$

When $\alpha > 0$, the differentiation of $J(\alpha)$ takes zero as follows:

$$(19-37) \quad \frac{d}{d\alpha} J(\alpha) = 2 \int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2} dx - 2\alpha \int_0^\infty e^{-\left(\frac{\alpha}{x}-x\right)^2} \frac{dx}{\alpha} = 0 \quad (\alpha > 0).$$

The left-sided limit of the differentiation of $J(\alpha)$ in the neighborhood of $\alpha = 0$ takes zero as follows:

$$(19-38) \quad \frac{d}{d\alpha} J(\alpha) \Big|_{\alpha=+0} = 0.$$

Therefore, when $\alpha \geq 0$, the function $J(\alpha)$ takes the constant as follows:

$$(19-39) \quad J(\alpha) = \text{Constant} = J(0) = \frac{\sqrt{\pi}}{2} \quad (\alpha \geq 0).$$

Table shows the increase and decrease in the function $J(\alpha)$

α	$-\infty$	0	∞
$\frac{d}{d\alpha} J(\alpha)$		+	discontinuous	0	
$J(\alpha)$	0	↗	$\frac{\sqrt{\pi}}{2}$	→	$\frac{\sqrt{\pi}}{2}$

Fig. 19-2

By referring to the equation (19-30),

$$(19-40) \quad J(\alpha) = e^{2\alpha} \int_0^\infty e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2} \quad (\alpha \geq 0).$$

Both the central and right sides are multiplied by $e^{-2\alpha}$ to obtain the formula in the case of $p = 0$.

$$(19-41) \quad \int_0^\infty x^0 e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2} e^{-2\alpha} \quad (\alpha \geq 0).$$

I show another proof of the above formula that makes no assumptions. If $\alpha > 0$,

For the right side of equation (19-30), I perform the variable transformation $x = (\alpha y)^{1/2}$.

$$(19-42) \quad J(\alpha) = e^{2\alpha} \int_0^\infty e^{-\alpha\left(y+\frac{1}{y}\right)} \sqrt{\alpha} \frac{1}{2} y^{-\frac{1}{2}} dy = \sqrt{\alpha} e^{2\alpha} \cdot \frac{1}{2} \int_0^\infty y^{\frac{1}{2}-1} e^{-\frac{2\alpha}{2}\left(y+\frac{1}{y}\right)} dy = \sqrt{\alpha} e^{2\alpha} K_{\frac{1}{2}}(2\alpha) \quad (\alpha > 0).$$

Now, recall the formula for the modified Bessel function of the second kind at the index $\nu = \pm 1/2$.

$$(19-43) \quad K_{\pm \frac{1}{2}}(Z) = \sqrt{\frac{\pi}{2Z}} e^{-Z} \quad (Z \in \mathbb{C}).$$

The formula (19-43) is introduced into the result of equation (19-42) to obtain the result as follows:

$$(19-44) \quad \mathbf{J}(\alpha) = \sqrt{\alpha} e^{2\alpha} \cdot \sqrt{\frac{\pi}{2(2\alpha)}} e^{-2\alpha} = \frac{\sqrt{\pi}}{2} \quad (\alpha > 0).$$

In case of $\alpha < 0$,

For the right side of equation (19-30), I perform the variable transformation $x = (-\alpha y)^{1/2}$.

$$(19-45) \quad \mathbf{J}(\alpha) = e^{2\alpha} \int_0^\infty e^{\alpha \left(y + \frac{1}{y}\right)} \sqrt{-\alpha} \frac{1}{2} y^{-\frac{1}{2}} dy =$$

$$\sqrt{-\alpha} e^{2\alpha} \cdot \frac{1}{2} \int_0^\infty y^{\frac{1}{2}-1} e^{-\frac{(-2\alpha)}{2} \left(y + \frac{1}{y}\right)} dy = \sqrt{-\alpha} e^{2\alpha} \mathbf{K} \frac{1}{2}(-2\alpha) \quad (\alpha < 0).$$

The formula (19-43) is introduced into the result of equation (19-45) to obtain the result as follows:

$$(19-46) \quad \mathbf{J}(\alpha) = \sqrt{-\alpha} e^{2\alpha} \cdot \sqrt{\frac{\pi}{2(-2\alpha)}} e^{2\alpha} = \frac{\sqrt{\pi}}{2} e^{4\alpha} \quad (\alpha < 0).$$

Because equations (19-44) and (19-46) show that the function $\mathbf{J}(\alpha)$ is continuous in the neighborhood of zero,

$$(19-47) \quad \mathbf{J}(\alpha) = \frac{\sqrt{\pi}}{2} \quad (\alpha \geq 0).$$

The above result is the same to that of the equation (19-40). Therefore, another proof of the formula (19-41) is completed.

When $p = -1$,

I consider the integral

$$\int_0^\infty x^{-2} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0).$$

For this integral, I perform the variable transformation $x = \alpha/y$.

$$(19-48) \quad \int_0^\infty x^{-2} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx =$$

$$\int_\infty^0 \left(\frac{\alpha}{y}\right)^{-2} e^{-\left(\frac{\alpha^2}{y^2} + y^2\right)} (-\alpha y^{-2}) dy = \frac{1}{\alpha} \int_0^\infty e^{-\left(y^2 + \frac{\alpha^2}{y^2}\right)} dy \quad (\alpha > 0).$$

Therefore

$$(19-49) \quad \int_0^\infty x^{-2} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2\alpha} e^{-2\alpha} \quad (\alpha > 0).$$

When $p = 1$,

For the recurrence formula (19-28), 1 is substituted for θ .

$$(19-50) \quad \int_0^\infty x^2 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \alpha^2 \int_0^\infty x^{-2} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx + \frac{1}{2} \int_0^\infty x^0 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx$$

$$= \frac{(2\alpha)^2}{4} \cdot \frac{\sqrt{\pi}}{2\alpha} e^{-2\alpha} + \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} e^{-2\alpha} = \frac{1}{4} ((2\alpha) + 1) \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0).$$

When $p = 2$,

For the recurrence formula (14-28), 3 is substituted for θ .

$$(19-51) \quad \int_0^\infty x^4 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \alpha^2 \int_0^\infty x^0 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx + \frac{3}{2} \int_0^\infty x^2 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx$$

$$= \frac{(2\alpha)^2}{4} \cdot \frac{\sqrt{\pi}}{2} e^{-2\alpha} + \frac{3}{2} \cdot \frac{(2\alpha)+1}{4} \sqrt{\pi} e^{-2\alpha} = \frac{1}{8} ((2\alpha)^2 + 3(2\alpha) + 3) \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0).$$

19-4. General representation for the integrals

$$\int_0^\infty x^{2k} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha \geq 0, k \in \mathbb{N})$$

Hereafter, I will use mathematical induction to find the general representation for the integrals. First, in order to prove, I prepared the following equation that I had found in advance:

$$(19-52) \int_0^\infty x^{2k} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, k \in \mathbb{N}).$$

Where

$$(19-53) a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \quad (k \in \mathbb{N}, \mu = 0, 1, \dots, k).$$

When $k = 1$,

For equation (19-52), 1 is substituted for k .

$$(19-54) \int_0^\infty x^2 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^2} \sum_{\mu=0}^1 \frac{(2-\mu)!}{2^{1-\mu} \mu! (1-\mu)!} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} = \frac{1}{4} ((2\alpha) + 1) \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0).$$

The results of equations (19-50) and (19-54) are the same, so in the case of $k = 1$, equation (19-52) is correct.

When $k = 2$,

For equation (19-52), 2 is substituted for k .

$$(19-55) \int_0^\infty x^4 e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^3} \sum_{\mu=0}^2 \frac{(4-\mu)!}{2^{2-\mu} \mu! (2-\mu)!} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} = \frac{1}{8} ((2\alpha)^2 + 3(2\alpha) + 3) \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0).$$

The results of equations (19-51) and (19-55) are the same, so in case of $k = 2$, equation (19-52) is correct.

When $k = m$, I assume that equation (19-52) is correct. The following equation holds under the assumption:

$$(19-56) \int_0^\infty x^{2m} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{m+1}} \sum_{\mu=0}^m \frac{(2m-\mu)!}{2^{m-\mu} \mu! (m-\mu)!} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, m \in \mathbb{N}).$$

When $k = m + 1$, I assume that equation (19-52) is also correct. The following equation holds under the assumption:

$$(19-57) \int_0^\infty x^{2(m+1)} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{m+2}} \sum_{\mu=0}^{m+1} \frac{(2m+2-\mu)!}{2^{m+1-\mu} \mu! (m+1-\mu)!} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, m \in \mathbb{N}).$$

When $k = m + 2$, I apply the recurrence formula (19-28) to the integral.

$$\begin{aligned}
 (19-58) \quad & \int_0^\infty x^{2(m+2)} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \\
 & \alpha^2 \int_0^\infty x^{2m} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx + \frac{2m+3}{2} \int_0^\infty x^{2(m+1)} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \\
 & = \frac{(2\alpha)^2}{4} \cdot \frac{1}{2^{m+1}} \sum_{\mu=0}^m \frac{(2m-\mu)!}{2^{m-\mu} \mu! (m-\mu)!} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \\
 & \quad + \frac{2m+3}{2} \cdot \frac{1}{2^{m+2}} \sum_{\mu=0}^{m+1} \frac{(2m+2-\mu)!}{2^{m+1-\mu} \mu! (m+1-\mu)!} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \\
 & = \frac{1}{2^{m+3}} \left(\sum_{\mu=2}^{m+2} \frac{(2m+2-\mu)!}{2^{m+2-\mu} (\mu-2)! (m+2-\mu)!} (2\alpha)^\mu \right. \\
 & \quad \left. + (2m+3) \sum_{\mu=0}^{m+1} \frac{(2m+2-\mu)!}{2^{m+1-\mu} \mu! (m+1-\mu)!} (2\alpha)^\mu \right) \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, m \in \mathbb{N}).
 \end{aligned}$$

The coefficient of the highest degree of the polynomial of the variable (2α) in the sum is simplified as follows:

$$(19-59) \quad \left. \frac{(2m+2-\mu)!}{2^{m+2-\mu} (\mu-2)! (m+2-\mu)!} \right|_{\mu=m+2} = \frac{m!}{2^0 m! 0!} = 1 \quad (m \in \mathbb{N}).$$

This polynomial is called the monic polynomial since it is unity.

The coefficient of the first degree of the polynomial is simplified as follows:

$$(19-60) \quad (2m+3) \left. \frac{(2m+2-\mu)!}{2^{m+1-\mu} \mu! (m+1-\mu)!} \right|_{\mu=1} = \frac{(2m+3)(2m+1)!}{2^m m!} = (2m+3)!! \quad (m \in \mathbb{N}).$$

The constant term of the polynomial is simplified as follows:

$$(19-61) \quad (2m+3) \left. \frac{(2m+2-\mu)!}{2^{m+1-\mu} \mu! (m+1-\mu)!} \right|_{\mu=0} = \frac{(2m+3)(2m+2)!}{2^{m+1} (m+1)!} = (2m+3)!! \quad (m \in \mathbb{N}).$$

Thus, the first-degree coefficient and the constant term are the same.

The above results are introduced into the result of equation (19-58) to obtain the general term.

$$\begin{aligned}
 (19-62) \quad & \int_0^\infty x^{2(m+2)} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{m+3}} \left((2\alpha)^{m+2} + \sum_{\mu=2}^{m+1} \left(\frac{(2m+2-\mu)!}{2^{m+2-\mu} (\mu-2)! (m+2-\mu)!} \right. \right. \\
 & \quad \left. \left. + (2m+3) \frac{(2m+2-\mu)!}{2^{m+1-\mu} \mu! (m+1-\mu)!} \right) (2\alpha)^\mu + (2m+3)!! (2\alpha) + (2m+3)!! \right) \sqrt{\pi} e^{-2\alpha} \\
 & = \frac{1}{2^{m+3}} \left((2\alpha)^{m+2} + \sum_{\mu=2}^{m+1} \left(\frac{\mu(\mu-1)}{2^{m+2-\mu} \mu! (m+2-\mu)!} + \frac{2(m+2-\mu)(2m+3)}{2^{m+2-\mu} \mu! (m+2-\mu)!} \right) (2m+2-\mu)! (2\alpha)^\mu \right. \\
 & \quad \left. + (2m+3)!! (2\alpha) + (2m+3)!! \right) \sqrt{\pi} e^{-2\alpha} \\
 & = \frac{1}{2^{m+3}} \left((2\alpha)^{m+2} + \sum_{\mu=2}^{m+1} \frac{(2m+4-\mu)!}{2^{m+2-\mu} \mu! (m+2-\mu)!} (2\alpha)^\mu \right. \\
 & \quad \left. + (2m+3)!! (2\alpha) + (2m+3)!! \right) \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, m \in \mathbb{N}).
 \end{aligned}$$

Therefore

$$(19-63) \int_0^\infty x^{2(m+2)} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{m+3}} \left((2\alpha)^{m+2} + \sum_{\mu=2}^{m+1} a_{m+2,\mu} (2\alpha)^\mu + (2m+3)!! (2\alpha) + (2m+3)!! \right) \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, m \in \mathbb{N}).$$

For the coefficient of the general term of the polynomial, $m + 2$ is substituted for μ .

$$(19-64) a_{m+2,m+2} = \frac{(2m+4-\mu)!}{2^{m+2-\mu} \mu! (m+2-\mu)!} \Big|_{\mu=m+2} = \frac{(m+2)!}{2^0 (m+2)! 0!} = 1 \quad (m \in \mathbb{N}).$$

Also, 1 is instituted for μ .

$$(19-65) a_{m+2,1} = \frac{(2m+4-\mu)!}{2^{m+2-\mu} \mu! (m+2-\mu)!} \Big|_{\mu=1} = \frac{(2m+3)!}{2^{m+1} (m+1)!} = (2m+3)!! \quad (m \in \mathbb{N}).$$

And also, 0 is instituted for μ .

$$(19-66) a_{m+2,0} = \frac{(2m+4-\mu)!}{2^{m+2-\mu} \mu! (m+2-\mu)!} \Big|_{\mu=0} = \frac{(2m+4)!}{2^{m+2} (m+2)!} = (2m+3)!! \quad (m \in \mathbb{N}).$$

The results of equations (19-64), (19-65), and (19-66) are the same to that of equation (19-63), so in the case of $k = m + 2$, equation (19-52) is also correct.

$$(19-67) \int_0^\infty x^{2(m+2)} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{m+3}} \sum_{\mu=0}^{m+2} a_{m+2,\mu} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, m \in \mathbb{N}).$$

Where

$$(19-68) a_{m+2,\mu} = \frac{(2m+4-\mu)!}{2^{m+2-\mu} \mu! (m+2-\mu)!} \quad (m \in \mathbb{N}, \mu = 0, 1, 2, \dots, m+2).$$

I can obtain the general representation for the integrals as a result of mathematical induction. At this stage, the proof is complete. Therefore, the following equation holds true:

$$(19-69) \int_0^\infty x^{2k} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^k \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, k \in \mathbb{N}).$$

Finally, the general representation for the integrals is given by using the coefficients.

$$(19-70) \int_0^\infty x^{2k} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \frac{1}{2^{k+1}} \sum_{\mu=0}^k a_{k,\mu} (2\alpha)^\mu \sqrt{\pi} e^{-2\alpha} \quad (\alpha \geq 0, k \in \mathbb{N}).$$

Where

$$(19-71) a_{k,\mu} = \frac{(2k-\mu)!}{2^{k-\mu} \mu! (k-\mu)!} \quad (k \in \mathbb{N}, \mu = 0, 1, 2, \dots, k).$$

And the coefficients deserving special mention are shown as follows:

$$(19-72) a_{k,k} = 1 \quad (k \in \mathbb{N}).$$

$$(19-73) a_{k,1} = a_{k,0} = (2k-1)!! \quad (k \in \mathbb{N}).$$

Table of the coefficients $a_{k,\mu}$

k	$a_{k,10}$	$a_{k,9}$	$a_{k,8}$	$a_{k,7}$	$a_{k,6}$	$a_{k,5}$	$a_{k,4}$	$a_{k,3}$	$a_{k,2}$	$a_{k,1}$	$a_{k,0}$
1										1	1
2									1	3	3
3								1	6	15	15
4							1	10	45	105	105
5						1	15	105	420	945	945
6					1	21	210	1260	4725	10395	10395
7				1	28	378	3150	17325	62370	135135	135135
8			1	36	630	6930	51975	270270	945945	2027025	2027025
9		1	45	990	13860	135135	945945	4729725	16216200	34459425	34459425
10	1	55	1485	25740	315315	2837835	18918900	91891800	310134825	654729075	654729075

Fig. 19-3

19-5. Relation between the modified Bessel function of the second kind and the integral

$$\int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C})$$

For the integral in the subtitle, the variable transformation

$$x = \sqrt{\alpha y} \quad (\alpha, y > 0)$$

is performed.

$$\begin{aligned} (19-74) \quad \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx &= \int_0^\infty (\sqrt{\alpha y})^{\theta-1} e^{-\left(\alpha y + \frac{\alpha^2}{\alpha y}\right)} \frac{\sqrt{\alpha}}{2} y^{-\frac{1}{2}} dy \\ &= \alpha^{\frac{\theta}{2}} \cdot \frac{1}{2} \int_0^\infty y^{\frac{\theta}{2}-1} e^{-\frac{2\alpha}{y}\left(y + \frac{1}{y}\right)} dy = \alpha^{\frac{\theta}{2}} K_{\frac{\theta}{2}}(2\alpha) \quad (\alpha > 0, \theta \in \mathbb{C}). \end{aligned}$$

Therefore

$$(19-75) \quad K_{\frac{\theta}{2}}(2\alpha) = \alpha^{-\frac{\theta}{2}} \int_0^\infty x^{\theta-1} e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx \quad (\alpha > 0, \theta \in \mathbb{C}).$$

Thus, when α is a positive real number, the above modified Bessel function of the second kind absolutely converges.

19-6. Laurent series expansions of the infinite series

$$\sum_{n=1}^\infty n^k e^{-2nx} \quad (x > 0, k \in \mathbb{N})$$

The hyperbolic cotangent of the real variable x is a convergent function in the intervals of $-\pi < x < 0$ and $0 < x < \pi$.

And it has the Laurent series expansion described by the Bernoulli numbers “[12].”

$$(19-76) \quad \text{Coth}(x) = \frac{1}{x} + \sum_{\mu=1}^\infty \frac{2^{2\mu} B_{2\mu}}{(2\mu)!} x^{2\mu-1} \quad (0 < |x| < \pi).$$

The infinite series,

$$\sum_{n=1}^{\infty} e^{-2 n x}$$

can be written using the hyperbolic cotangent of the positive variable x as follows:

$$(19-77) \quad \sum_{n=1}^{\infty} e^{-2 n x} = \frac{e^{-2 x}}{1-e^{-2 x}} = \frac{e^{-x}}{e^x - e^{-x}} = \frac{\cosh(x) - \sinh(x)}{2 \sinh(x)} = \frac{1}{2} (\coth(x) - 1) \quad (x > 0).$$

By combining the above two equations,

$$(19-78) \quad \sum_{n=1}^{\infty} e^{-2 n x} = \frac{1}{2} \left(\frac{1}{x} - 1 + \sum_{\mu=1}^{\infty} \frac{2^{2 \mu} B_{2 \mu}}{(2 \mu)!} x^{2 \mu - 1} \right) \quad (0 < x < \pi).$$

For the right side of equation (19-78), I show several terms of the series in order.

$$(19-79) \quad \sum_{n=1}^{\infty} e^{-2 n x} = \frac{1}{2} \left(x^{-1} - 1 + \frac{x}{3} - \frac{x^3}{45} + \frac{2 x^5}{945} - \frac{x^7}{4725} + \frac{2 x^9}{93555} + \sum_{\mu=6}^{\infty} \frac{2^{2 \mu} B_{2 \mu}}{(2 \mu)!} x^{2 \mu - 1} \right) \quad (0 < x < \pi).$$

The infinite series,

$$\sum_{n=1}^{\infty} n^k e^{-2 n x} \quad (x > 0, k \in \mathbb{N})$$

has a recurrent formula based on differentiation.

$$(19-80) \quad \sum_{n=1}^{\infty} n^k e^{-2 n x} = -\frac{1}{2} \frac{d}{dx} \sum_{n=1}^{\infty} n^{k-1} e^{-2 n x} \quad (x > 0, k \in \mathbb{N}).$$

When $k = 1$,

$$\begin{aligned} (19-81) \quad \sum_{n=1}^{\infty} n e^{-2 n x} &= -\frac{1}{2} \frac{d}{dx} \sum_{n=1}^{\infty} e^{-2 n x} = -\frac{1}{2} \frac{d}{dx} \frac{1}{2} \left(\frac{1}{x} - 1 + \sum_{\mu=1}^{\infty} \frac{2^{2 \mu} B_{2 \mu}}{(2 \mu)!} x^{2 \mu - 1} \right) \\ &= -\frac{1}{4} \left(-\frac{1}{x^2} + \sum_{\mu=1}^{\infty} (2 \mu - 1) \frac{2^{2 \mu} B_{2 \mu}}{(2 \mu)!} x^{2 \mu - 2} \right) \\ &= \frac{1}{4} \left(\frac{1}{x^2} - (2 \mu - 1) \frac{2^{2 \mu} B_{2 \mu}}{(2 \mu)!} x^{2 \mu - 2} \Big|_{\mu=1} - \sum_{\mu=2}^{\infty} (2 \mu - 1) \frac{2^{2 \mu} B_{2 \mu}}{(2 \mu)!} x^{2 \mu - 2} \right) \\ &= \frac{1}{4} \left(\frac{1}{x^2} - \frac{1}{3} - \sum_{\mu=1}^{\infty} (2 \mu + 1) \frac{2^{2 \mu + 2} B_{2 \mu + 2}}{(2 \mu + 2)!} x^{2 \mu} \right) \quad (0 < x < \pi). \end{aligned}$$

Therefore

$$(19-82) \quad \sum_{n=1}^{\infty} n e^{-2 n x} = \frac{1}{4} \left(x^{-2} - \frac{1}{3} + \frac{x^2}{15} - \frac{2 x^4}{189} + \frac{x^6}{675} - \frac{2 x^8}{10395} - \sum_{\mu=5}^{\infty} (2 \mu + 1) \frac{2^{2 \mu + 2} B_{2 \mu + 2}}{(2 \mu + 2)!} x^{2 \mu} \right) \quad (0 < x < \pi).$$

The following Pochhammer symbol will be used for convenience hereafter “[2]”:

$$(19-83) \quad (a)_k := a \times (a + 1) \times \dots \times (a + k - 1) \quad (k \in \mathbb{N}).$$

When $k = 2$,

$$\begin{aligned}
(19-84) \quad & \sum_{n=1}^{\infty} n^2 e^{-2nx} = \\
& -\frac{1}{2} \frac{d}{dx} \sum_{n=1}^{\infty} n e^{-2nx} = -\frac{1}{8} \left(-\frac{2}{x^3} - \sum_{\mu=1}^{\infty} (2\mu)(2\mu+1) \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-1} \right) \\
& = \frac{1}{4} \left(\frac{1}{x^3} + \frac{1}{2} \sum_{\mu=1}^{\infty} (2\mu)_2 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-1} \right) \quad (0 < x < \pi).
\end{aligned}$$

Therefore

$$\begin{aligned}
(19-85) \quad & \sum_{n=1}^{\infty} n^2 e^{-2nx} = \\
& \frac{1}{4} \left(x^{-3} + 0 - \frac{x}{15} + \frac{4x^3}{189} - \frac{x^5}{225} + \frac{8x^7}{10395} + \frac{1}{2} \sum_{\mu=5}^{\infty} (2\mu)_2 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-1} \right) \quad (0 < x < \pi).
\end{aligned}$$

When $k=3$,

$$\begin{aligned}
(19-86) \quad & \sum_{n=1}^{\infty} n^3 e^{-2nx} = -\frac{1}{8} \left(-\frac{3}{x^4} + \frac{1}{2} \sum_{\mu=1}^{\infty} (2\mu-1)_3 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-2} \right) \\
& = \frac{3}{8} \left(\frac{1}{x^4} - \frac{1}{3!} (2\mu-1)_3 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-2} \Big|_{\mu=1} - \frac{1}{3!} \sum_{\mu=2}^{\infty} (2\mu-1)_3 \frac{2^{2\mu+2} B_{2\mu+2}}{(2\mu+2)!} x^{2\mu-2} \right) \\
& = \frac{3}{8} \left(\frac{1}{x^4} + \frac{1}{45} - \frac{1}{3!} \sum_{\mu=1}^{\infty} (2\mu+1)_3 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu} \right) \quad (0 < x < \pi).
\end{aligned}$$

Therefore

$$\begin{aligned}
(19-87) \quad & \sum_{n=1}^{\infty} n^3 e^{-2nx} = \\
& \frac{3}{8} \left(x^{-4} + \frac{1}{45} - \frac{4x^2}{189} + \frac{x^4}{135} - \frac{8x^6}{4455} - \frac{1}{3!} \sum_{\mu=4}^{\infty} (2\mu+1)_3 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu} \right) \quad (0 < x < \pi).
\end{aligned}$$

When $k=4$,

$$\begin{aligned}
(19-88) \quad & \sum_{n=1}^{\infty} n^4 e^{-2nx} = -\frac{3}{16} \left(-\frac{4}{x^5} - \frac{1}{3!} \sum_{\mu=1}^{\infty} (2\mu)_4 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-1} \right) \\
& = \frac{3}{4} \left(\frac{1}{x^5} + \frac{1}{4!} \sum_{\mu=1}^{\infty} (2\mu)_4 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-1} \right) \quad (0 < x < \pi).
\end{aligned}$$

Therefore

$$\begin{aligned}
(19-89) \quad & \sum_{n=1}^{\infty} n^4 e^{-2nx} = \\
& \frac{3}{4} \left(x^{-5} + 0 + \frac{2x}{189} - \frac{x^3}{135} + \frac{4x^5}{1485} + \frac{1}{4!} \sum_{\mu=4}^{\infty} (2\mu)_4 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-1} \right) \quad (0 < x < \pi).
\end{aligned}$$

When $k=5$,

$$\begin{aligned}
(19-90) \quad & \sum_{n=1}^{\infty} n^5 e^{-2nx} = -\frac{3}{8} \left(-\frac{5}{x^6} + \frac{1}{4!} \sum_{\mu=1}^{\infty} (2\mu-1)_5 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-2} \right) \\
& = \frac{15}{8} \left(\frac{1}{x^6} - \frac{1}{5!} (2\mu-1)_5 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-2} \Big|_{\mu=1} - \frac{1}{5!} \sum_{\mu=2}^{\infty} (2\mu-1)_5 \frac{2^{2\mu+4} B_{2\mu+4}}{(2\mu+4)!} x^{2\mu-2} \right)
\end{aligned}$$

$$= \frac{15}{8} \left(\frac{1}{x^6} - \frac{2}{945} - \frac{1}{5!} \sum_{\mu=1}^{\infty} (2\mu+1)_5 \frac{2^{2\mu+6} B_{2\mu+6}}{(2\mu+6)!} x^{2\mu} \right) \quad (0 < x < \pi).$$

Therefore

$$(19-91) \quad \sum_{n=1}^{\infty} n^5 e^{-2nx} = \frac{15}{8} \left(x^{-6} - \frac{2}{945} + \frac{x^2}{225} - \frac{4x^4}{1485} - \frac{1}{5!} \sum_{\mu=3}^{\infty} (2\mu+1)_5 \frac{2^{2\mu+6} B_{2\mu+6}}{(2\mu+6)!} x^{2\mu} \right) \quad (0 < x < \pi).$$

19-7. Formulae using the hyperbolic functions for the infinite series

$$\sum_{n=1}^{\infty} n^k e^{-2nx} \quad (x > 0, k \in \mathbb{N})$$

When $k = 1$, taking derivative and multiplying by $-1/2$ for the result of equation (19-77),

$$(19-92) \quad \sum_{n=1}^{\infty} n e^{-2nx} = -\frac{1}{2} \frac{d}{dx} \left(\frac{\cosh(x)}{2 \sinh(x)} - \frac{1}{2} \right) = -\frac{1}{4} \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = \frac{1}{4 \sinh^2(x)} \quad (x > 0).$$

When $k = 2$, in the same way,

$$(19-93) \quad \sum_{n=1}^{\infty} n^2 e^{-2nx} = -\frac{1}{2} \frac{d}{dx} \left(\frac{1}{4 \sinh^2(x)} \right) = -\frac{1}{8} \frac{-2 \sinh(x) \cosh(x)}{\sinh^4(x)} = \frac{\coth(x)}{4 \sinh^2(x)} \quad (x > 0).$$

When $k = 3$,

$$(19-94) \quad \sum_{n=1}^{\infty} n^3 e^{-2nx} = -\frac{1}{2} \frac{d}{dx} \left(\frac{\cosh(x)}{4 \sinh^3(x)} \right) = -\frac{1}{8} \frac{\sinh^4(x) - 3 \cosh^2(x) \sinh^2(x)}{\sinh^6(x)} = \frac{1}{8} \frac{3 \cosh^2(x) - \sinh^2(x)}{\sinh^4(x)} = \frac{1}{8} \frac{3(\sinh^2(x)+1) - \sinh^2(x)}{\sinh^4(x)} = \frac{1}{8} \frac{2 \sinh^2(x) + 3}{\sinh^4(x)} \quad (x > 0).$$

Therefore

$$(19-95) \quad \sum_{n=1}^{\infty} n^3 e^{-2nx} = \frac{3}{8} \left(\frac{2}{3} + \frac{1}{\sinh^2(x)} \right) \frac{1}{\sinh^2(x)} \quad (x > 0).$$

When $k = 4$, I apply the same operations to the result of equation (19-94),

$$(19-96) \quad \sum_{n=1}^{\infty} n^4 e^{-2nx} = -\frac{1}{2} \frac{d}{dx} \left(\frac{1}{8} \frac{2 \sinh^2(x) + 3}{\sinh^4(x)} \right) = -\frac{1}{16} \frac{4 \sinh^5(x) \cosh(x) - 4(2 \sinh^2(x) + 3) \sinh^3(x) \cosh(x)}{\sinh^8(x)} = \frac{1}{4} \frac{(2 \sinh^2(x) + 3) \cosh(x) - \sinh^2(x) \cosh(x)}{\sinh^5(x)} = \frac{1}{4} \frac{(\sinh^2(x) + 3) \cosh(x)}{\sinh^5(x)} \quad (x > 0).$$

Therefore

$$(19-97) \quad \sum_{n=1}^{\infty} n^4 e^{-2nx} = \frac{3}{4} \left(\frac{1}{3} + \frac{1}{\sinh^2(x)} \right) \frac{\coth(x)}{\sinh^2(x)} \quad (x > 0).$$

Hereafter, I show some results in order.

$$(19-98) \quad \sum_{n=1}^{\infty} n^5 e^{-2nx} = \frac{15}{8} \left(\frac{2}{15} + \frac{1}{\sinh^2(x)} + \frac{1}{\sinh^4(x)} \right) \frac{1}{\sinh^2(x)} \quad (x > 0).$$

$$(19 - 99) \quad \sum_{n=1}^{\infty} n^6 e^{-2 n x} = \frac{45}{8} \left(\frac{2}{45} + \frac{2}{3 \sinh^2(x)} + \frac{1}{\sinh^4(x)} \right) \frac{\coth(x)}{\sinh^2(x)} \quad (x > 0).$$

$$(19 - 100) \quad \sum_{n=1}^{\infty} n^7 e^{-2 n x} = \frac{315}{16} \left(\frac{4}{315} + \frac{2}{5 \sinh^2(x)} + \frac{4}{3 \sinh^4(x)} + \frac{1}{\sinh^6(x)} \right) \frac{1}{\sinh^2(x)} \quad (x > 0).$$

$$(19 - 101) \quad \sum_{n=1}^{\infty} n^8 e^{-2 n x} = \frac{315}{4} \left(\frac{1}{315} + \frac{1}{5 \sinh^2(x)} + \frac{1}{\sinh^4(x)} + \frac{1}{\sinh^6(x)} \right) \frac{\coth(x)}{\sinh^2(x)} \quad (x > 0).$$

$$(19 - 102) \quad \sum_{n=1}^{\infty} n^9 e^{-2 n x} = \frac{2835}{8} \left(\frac{2}{2835} + \frac{17}{189 \sinh^2(x)} + \frac{7}{9 \sinh^4(x)} + \frac{5}{3 \sinh^6(x)} + \frac{1}{\sinh^8(x)} \right) \frac{1}{\sinh^2(x)} \quad (x > 0).$$

20. Conclusion

I obtained three types of general representations for the zeta function for any odd number of either 3 or 7, or more; one has the leading term, while the others do not.

Unfortunately, because the general representations lack essential information on the zeta function at these points, finding exact values (closed representations) such as

$$\zeta(2) = \pi^2 / 6$$

would be extremely difficult.

I discovered that some infinite series represented using the divisor sigma function give transcendental numbers with fixed values. The Riemann hypothesis was proved by applying reduction to absurdity using the representation containing the leading term of the zeta function for any complex number. In addition, I obtained the new explicit formulae for the zeta and the Riemann Xi functions. And the Riemann hypothesis was again proved using the deductive method. I believe that the Riemann hypothesis is a symmetry problem unrelated to prime numbers. Moreover, the Hadamard product representation for the zeta function for any integer of 2 or more was obtained.

Furthermore, I confirmed the explicit formulae for each of the Riemann-Siegel $Z(t)$ and the Riemann-Siegel theta functions.

Conversely, I obtained a new function, named the Chi function, for the left side of the origin symmetric functional equation that includes corrective terms. The Chi function is similar to the Riemann Xi function and exhibits origin symmetry. Furthermore, I obtained a new function, the eta function, which is similar to the zeta function. The eta function's pole and trivial zeros are the same as those of the zeta function. Furthermore, the Chi and the eta functions have the same non-trivial zeros on the imaginary axis. Here, the imaginary axis corresponds to the critical line of the eta function. And I proposed a generalized Riemann hypothesis for the eta function that states that all non-trivial zeros lie on the imaginary axis. Since I was able to discover the explicit formulae for the eta and the Chi functions, the deductive method was used to prove the generalized Riemann hypothesis for the eta function. In addition, I discovered that an infinite series represented using the divisor sigma function gives the algebraic number 0 anywhere in the whole complex plane. I also found an infinite series of the same form which gives the algebraic number 1/4, albeit only one.

The following are the new core conjectures:

1. Pythagorean primes, squares of non-Pythagorean primes, and even prime 2 would be generated as the main spectrum using information on the imaginary parts of the non-trivial zeros of the eta function.
2. Almost half of the imaginary parts of the non-trivial zeros of the modified eta function $\eta(2\theta)$ would be generated as a spectrum from prime number information.
3. A spectrum based on all odd primes would generate almost half of the imaginary parts of the non-trivial

zeros of the eta function $\eta(\theta)$.

4. Twice the imaginary part of any non-trivial zero of the zeta function would belong to the subset of imaginary parts of the non-trivial zeros of the eta function. It supports the existence of an infinite number of non-trivial zeros of the eta function.

5. I could predict that the number of non-trivial zeros of the eta function within a certain distance ρ_m (the positive imaginary part of the m -th non-trivial zero of the zeta function) from the origin would be $m-1$, m , or $m+1$ on the imaginary axis.

6. Considering where the non-trivial zeros of the sum function of the zeta and eta functions lie, they would all be in the habitable zone $0 < \text{Re}(\theta) < 1/2$.

7. There is a reciprocal series of the conditional Pythagorean primes (twice the conditional non-Pythagorean primes minus 1 are Pythagorean primes), it would diverge slower than the reciprocal series of the Pythagorean primes.

However, those were found using the functions defined as analogies in previous studies of the zeta function. I could not provide any mathematical evidence for these. Hence, the knowledge obtained in subsection 16-4 remains within the anticipated range. Moreover, studies on Dirichlet series representation and Euler product representation for the eta function have not yet started.

I hope you will be able to solve the new problems someday!

When interpreting that the functional equation (2-2) stipulates the relation between whole positive real number and all squares of natural numbers, I must conclude that the essence of the numbers is condensed here.

By the way, the security of the current information society depends on the difficulty of factorizing the product of two huge prime numbers. In contrast, we know some people believe that when the Riemann hypothesis is solved, safety will be violated. Because the explicit formula for the zeta function has the same difficulty, I believe the safety will be maintained in the future as well.

21. Thanks

21-1. Letter to Professor Hardy “[18]”

Dear Professor G. H. Hardy,

I'm delighted to inform you that you and I have proved the Riemann hypothesis, which was proposed by Bernhard Riemann himself in 1859. You demonstrated (together with Professor J. E. Littlewood) in 1914 that there are an infinite number of non-trivial zeros of the Riemann zeta function $\zeta(s)$ on the critical line.

Recently, I was able to demonstrate that the non-trivial zeros of the zeta function don't lie off the critical line. Combining these two results, we arrived at the conclusion that all non-trivial zeros of the zeta function lie on the critical line. Problems with symmetry in general, not just prime numbers, were raised by the Riemann hypothesis. I want to sincerely commend you for your great achievement.

I sincerely appreciate you, Hardy-san.

From future Japan 106 years after your time, on 30 Apr, 2020.

Hideharu Maki · · · was born on 30 Jun, 1959.

21-2. To those who supported and encouraged me

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