A Proof of the Legendre Conjecture

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Abstract: If Legendre conjecture does not hold all integers in the interior of $(n^2, (n+1)^2)$ are composed numbers. The composite integers counting shown that the rate of the number of the odd composites to the number of odd integers in the interior of $(n^2, (n+1)^2)$ is smaller than one. Consequently, the Legendre conjecture holds.

Key words: Legendre, residual set, prime integer, composite integer, fundamental theorem of arithmetic, integer and rational counting, prime counting function, harmonic function, mean value theorem.

Introduction

The problem set by Adrien-Marie Legendre (1752-1833), known as the Legendre conjecture, states that *between the squares of any two consecutive integers there is a prime number*. Exactly, for any integer n, there is a prime p such that $a = n^2 < p < (n+1)^2 = b$. The conjecture is proven numerically for all integers up to $9 \cdot 10^9$. The conjecture is one of the unsolved problems.

The integer interior of the interval (a, b) contains $2n$ integers, half are even composites, and half are odd integers containing the primes if there are any therein. The idea is simple, the interior of the interval (a, b) tests on the primes, and the testing device is the multiplication function $f : (x, y) \to xy$. Without the unit integer in the multiplication function domain the prime numbers do not produce in the interval (a, b) .

Corollary 1. *The Legendre conjecture holds if the sets of all odd composites and all odd integers in the integer interior of the set (a, b) are not identical.*

$\square$ The even integers are already composites, and it is sufficient to focus on the odd integers only. When the unit integer is excluded from the odd integers domain the multiplication function will not produce composites in the interior of (a, b). Thus, if the set of the odd multiples in the interior of (a, b) is not the set of all odd numbers there in there is at least one prime in the interior of the (a, b). $\blacksquare$

The multiplication function is essential in this work, and the next section will be devoted to the multiplication matrix.

Multiplication Matrix

The multiplication matrix $A = |y\rangle\langle x|$ of the integer row $|y\rangle$ and the column $\langle x|$ axes is a collection of the entries, multiples $A^y_x = f(x, y) = xy$, collected in the rows $\mathbf{a}_x = x|y\rangle$, or the columns $\mathbf{a}^y = \langle x|y$. Table 1. represents the multiplication matrix of the row $y = 1, 2, 3, \cdots, 19$ and the column $x = 1, 2, 3, \cdots, 10$. The matrix entries A_x^y and A_y^x are identical, the matrix A is diagonally symmetric, and we may exclude the matrix entries below the main diagonal. Since the unit integer is not in the domain of the multiplication function, all required multiples are in the matrix rows $\{\mathbf{a}^y : y = 2, 3, 4, \cdots, n\}$. The multiple restriction to the interior of (a, b) imposes the following inverse function conditions on the multiplication function domain in the $x \otimes y$,

$$\begin{aligned}
&\forall x \therefore\ 3 \le x < n, \quad 3 \le y_* \le y \le y^* < (n+1)^2, \\
&y_* = \min_y \{y : n^2 < xy < (n+1)^2\}, \\
&y^* = \max_y \{y : n^2 < xy < (n+1)^2\}..
\end{aligned}$$

The boundaries of the variable y impose the limits $m(x) \in \mathbf{m}$ and $M(x) \in \mathbf{M}$ on the matrix entries,

$$\begin{aligned}
&\mathbf{m} = \{m(x) = xy_*\} \ \ \mathbf{M} = \{M(x) = xy^*\}, \\
&\forall x \in (3, n) \therefore\ \ n^2 \le m(x) \le A_x^y \le M(x) < (n+1)^2.
\end{aligned}$$

An illustration is the multiplication matrix of the odd integers only presented in Table 2.

Table 1. Multiplication matrix at $n = 9$

$x \backslash y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\cdots$
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	$\cdots$
3	3	6	**9**	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	$\cdots$
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	$\cdots$
5	5	10	15	20	**25**	30	35	40	45	50	55	60	65	70	75	80	85	90	95	$\cdots$
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	$\cdots$
7	7	14	21	28	35	42	**49**	56	63	70	77	84	91	98	105	112	119	126	133	$\cdots$
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	$\cdots$
9	9	18	27	36	45	54	63	72	**81**	90	99	108	117	126	135	144	153	162	171	$\cdots$
10	10	20	30	40	50	60	70	80	90	**100**	110	120	130	140	150	160	170	180	190	$\cdots$

Corollary 1. restricts variable x and y to the odd integers greater than 3. The main-diagonal matrix symmetry restricts the variable x to the odd integers $\langle x| = \langle 3, 5, 7, 9, \cdots, n'|$ only, where $n' = n$ if n is odd or $n' = n - 1$ if n is even. The variable y takes odd integer values $|\varpi\rangle = |3, 5, 7, \cdots, \Omega\rangle$, where Ω is the largest $y : 3 \le y < (n+1)^2$, for an illustration see Table 2. Thus $\mathsf{A} = |\varpi\rangle\langle\mathsf{X}|$. Notice that all minimums $m(x)$ and all maximums $M(x)$ almost satisfy the $xm(x) \approx n^2 = const$ and $x\Omega(x) \approx n^2 = const$ condition at each row $\mathbf{a}_x$, so that the vector entries of $\mathbf{m}$ increase and that of $\mathbf{M}$ decrease by the variable x.

Table 2. Odd Integer Multiplication Matrix at$n = 9$

$\langle\mathsf{X}\| \ \|\varpi\rangle$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
3	3	9	15	21	27	33	39	45	51	57	63	69	75	**81**	87	93	99	105	111	**117**
5	5		25	35	45	55	65	75	**85**	95	105	**115**								
7	7			49	63	77	**91**	105	**119**											
9	9				81	**99**	**117**													
11	11					121														

Redundancy Reduction

The matrix over-diagonal symmetry and domain reduction to the odd integers only reduced the composite repetition significantly, still leaving the composite redundancy high. The following Corollary will reduce the composite redundancy while preserving the completeness of the multiplication matrix.

Corollary 2. *The multiplication matrix of the composites is complete if the odd integer* $\langle \mathrm{x}|$ *axes reduces to the primes only, the multiplication matrix is* $\mathbf{A} = \mathbf{p} \otimes \varpi$.

□ For this purpose, we let the integers from the column vector $\mathsf{X} = \langle 3, 5, 7, 9, \cdots, n'|$ to be the operators $\hat{x}$ acting on the row vector $|\varpi\rangle = |3, 5, 7, \cdots, \Omega\rangle$ of the odd integers as the multiplication functions, $\hat{x}|\varpi\rangle = x|\varpi\rangle_x = \mathbf{a}_x$ for all $x \in \mathsf{X}$. Such n operators create n rows $\mathbf{a}_x$ of the composed integers left-right restricted to the $[\,\mathbf{m}, \mathbf{M}\,]$ of the matrix A. The right limits satisfy $M(x') \leq M(x)$ for all $x' \geq x$.
For the next follow the mapping diagram

$$\begin{array}{ccc} \hat{p} & \xrightarrow{\mathrm{x}} & p(a\omega) = \omega(pa) = A_p^{a\omega} \equiv A_{\mathrm{x}}^{\omega} \\ \hat{a}\big\downarrow & & \updownarrow \\ \hat{\mathrm{x}} & \xrightarrow{\omega} & \mathrm{x}\omega = A_{\mathrm{x}}^{\omega}. \end{array}$$

The integer $\mathrm{x} \in \mathsf{X}$ is either prime or a composed integer. Suppose that x is a multiple ap of an integer $a < n$ and the prime $p < \mathrm{x}$. The integer x creates the matrix row vector $\mathbf{a}_{\mathrm{x}} = \mathrm{x}|\varpi\rangle$ in the range $[m(x), m(x) + \mathrm{x}, m(x) + 2\mathrm{x}, \cdots, M(\mathrm{x})]$. The row $p\varpi\rangle$ contains all p-creations of the odd integers and must contain the composite $\hat{p}\mathrm{x} = \hat{p}(a\omega) \equiv A_{\mathrm{x}}^{\omega} \leq M(x)$, which must be smaller than $M(p)$. Hence, the composite $\mathrm{x}\omega$ had been already created in the row $\mathbf{a}_p$ by the prime p, and discards. Such is the destiny of all $\mathrm{x}|\varpi\rangle_{\mathrm{x}}$. Since x is an arbitrarily composed integer in X, the same holds for all odd composed integers in X. Hence, the primes $\mathbf{p} \subset \mathsf{X}$ span composed odd integers in the interior of (a, b), and $\mathbf{p}\ \otimes \varpi$ is the domain of the multiplication function. ■

Integer and Rational, Global and Local Counting

It is already clear that counting the composed integers in the interior of (a, b) is essential. In this section, we will specify the meaning and use of already-known concepts, such as additive or integer and multiplicative or rational measurements of an integer by another integer, and we will consider the concepts of local and global integer counting and their relations.

Definition: *An integer* N, $\hat{N}_m\colon\ (m, y) \to z = m\mathrm{N} + r$ *is the* m *integer measure of the integer* z *by the integer* m *measurement* $\hat{N}_m$. *The integer* r *is the division reminder in the residual set* $\mathrm{Res}_m = \{0, 1, 2, 3, \cdots, m-1\} \equiv \big|0\ \ m-1\big)$.

A number ξ, $\hat{R}_m\colon (m, z) \to z = \xi m$, *is the* m *rational measure of the integer* z *by the integer* m *rational measurement* $\hat{R}_m$.

Remark: Further, the symbol $|0\ \ \mathrm{z})$ is the set of a variable z which takes values zero to Z but not z. However, in an equation, it means that z takes a value from the set $|0\ \ \mathrm{z})$. In addition, the integer and the rational measurements are the z preserving operators, relating the integer and rational counting. For,

$$\hat{N}_m z = \mathrm{N}m + r = \ z \ = \xi m = \hat{R}_m z \ \Rightarrow \ \xi = \mathrm{N} + r_{:m} \ \Leftrightarrow \ \mathrm{N} = \xi - r_{:m}, \quad r_{:m} = \frac{r}{m}.$$

While the measurement between two free integers is unique, the integer counting on an integer interval requires little consideration.

The first, for $m = 1$ the integer measure of the interior of (a, b) is the number of the units m from the point a to and including the point $b - 1$. Therefore, the consistent measure of the interior of (a, b) is the measure of the interval $(a, b - 1]$, and further, the interior of (a, b) is the interval $(a, b - 1]$.

Second, the counting measure of the interior of isolated interval (a, b) and the same one placed on the integer number axes may differ. Table 3., with $n = 8$ and $m = 4$, 5 and $m = 7$, gives an illustration. The integer interval $(a = 64, 65, 66, \cdots 80 = b_*]$ contains $2n = 16$ integer points. In the local measurement, the first integer counted point in the interval is $a + m$, the rest of them follow in the natural order, and the numbers $4, 4, 2$ of m-multiples are found. However, when the interval $(a, b - 1]$ is in the integer set $m\mathbb{N}$ of the global m -multiples, one finds numbers $m = 4, 4, 3$ of the m-multiples respectively in its interior. Consequently, the composite numbers count in the local and global integer counting measurements may not be identical.

Table 3. Composed Integers in the interior of (a, b) for $n = 8$

MEASURE	GLOBA								LOCAL						
$m = 4$	60	**64**	68	72	76	80	**81**	$\Rightarrow \mathrm{N}^* = 4$	$b_*^* - a = 16$	0	**4**	8	12	**16**	$\Rightarrow \mathrm{N} = 4$
$m = 5$	60	**64**	65	70	75	80	**81**	$\Rightarrow \mathrm{N}^* = 4$	$b_*^* - a = 16$	0	**5**	10	15	**16**	$\Rightarrow \mathrm{N} = 3$
$m = 7$	56	**64**		63	70	78	**81**	$\Rightarrow \mathrm{N}^* = 3$	$b_*^* - a = 16$	0	**7**	14		**16**	$\Rightarrow \mathrm{N} = 2$

Definition: *The local integer counting measure* N' *of the interior of* (a, b) *is the number of the* m *- multiples in the interior of the isolated interval* $(a, b]$, *and*

$$\hat{N}_m(a, b) \quad = \hat{N}_m(a, b - 1] = (b - 1) - a = m\mathrm{N}' + r, \quad r \in \mathrm{Res}_m.$$

The global integer measure N^* *of the same interval is the number of m-multiples from* $m\mathbb{N}$ *found in the interval* $(a, b - 1]$, *and*

$$\hat{N}_m^*(a, b_*) \quad = \hat{N}_m(a, b_*] = \hat{N}' b_* - \hat{N}' a = \mathrm{N}'_{b_*} - \mathrm{N}'_a = \mathrm{N}^*.$$

Definition: *The local rational counting measure of the interior of* (a, b) *is a rational number* ξ *such that* $\hat{R}_m(a, b) = (b - 1) - a = \xi m$. *Its global rational counting measure is a number* ξ^* *such that* $\hat{R}_m^*(a, b) = \hat{R}_m(b - 1, a) - \hat{R}_m a = m\xi_{b-1} - m\xi_a = \xi^* m$.

Corollary 4. *The global and local numbers of the composite integers in the interior of* (a, b) *are equal, or the global integer counting is for one greater,* $\mathrm{N} = \mathrm{N}' + |0 \ \ 1\rangle$. *However, the local and global rational counts on the interior of* (a, b) *are identical. Each rational counting is greater than or equal to the corresponding integer counting. The exact* N, *the local* N' *and the rational*

ξ counting numbers are ordered according to

$$\mathrm{N}' \leq \mathrm{N} \leq \xi.$$

The exact N *composite integer counting on the interior of (a, b) is identical to its global counting* N^*. *Further, the measure of the integer interior of (a, b) is invariant under measurements, and*

$$2n \quad = p\mathrm{N}'_p + r_p = p\mathrm{N}^*_p - p|0 \;\; 1) + r_p \equiv \quad p\mathrm{N}_p + r_p - p\,|0 \;\; 1)\,, \tag{1}$$

□ Local m-integer counting of the composite starts at $a + m$ and ends at $a + m\mathrm{N}'$ with the residual r'_b, while the global m-integer counting starts at $a + x,\; 1 < x \leq m$ and ends at $a + m\mathrm{N}$ with residual $r_b = r'_b + (m - x) = m + r'_b - x$. If $x = m$ the local and global m-partitions of (a, b) are identical, and the local and global numbers of the m- composites in (a, b) are the same. Else, if $x = 1$ the residual $r_b \to r'_b + (m - 1) = m + (r'_b - 1) = m + r''_b \geq m$, and in this case the global integer partition gains one more point, and $\mathrm{N}^* = \mathrm{N}' + 1$. We introduce the vector $|0 \;\; 1\rangle$ to write $\mathrm{N}^* = \mathrm{N}' + |0 \;\; 1\rangle$.

The rational local and global integer counting are in the rational numbers and must be identical.
Any two integers are commensurable in the real numbers so that

$$\hat{R}^*_m(a, b) \quad = \xi_{b-1}(b - 1) - \xi_a a \sim m(\,\xi_{b-1} - \xi_a) = \quad \hat{R}_m(a, b).$$

However, they may not be commensurable in the integers, so the integer counting number is smaller or equal to the rational one, and the counting numbers ordering above holds.
The global counting is the natural composite count, and the exact number of the composites in the interior of (a, b) is their global counting.
Equation (1) is the exact statement of the invariance of the integer size of the integer interior (a, b) measurements. ■

Remark: *We introduce the following notation*

$$\pi(n) \simeq \tfrac{n}{\ln n}, \;\; p \leq n$$
$$\Pi = 3 + 5 + \cdots + \tfrac{1}{p} + \cdots + p_{\pi(n)},$$
$$H_n = \tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{p} + \cdots + \tfrac{1}{P}, \;\; p_{\pi(n)} = P.$$

Corollary 5. *The average and exact number of the composites in the interior of (a, b) obey the following equations,*

$$\overline{\mathsf{N}}\Pi \quad = 2n\pi(n) + \Pi\big| - 1 \;\; 0 \;\; 1\big), \tag{2}$$
$$\mathsf{N} \quad = 2nH_n \quad + \pi(n)\big| - 1 \;\; 0 \;\; 1\big) \tag{3}$$

□ Equation (2) is a solution of equation (1) for pN_p, summed over all $\pi(n)$ primes with the use of the integral average theorem. Hence

$$pN_p \quad = 2n + p|0 \;\; 1\rangle - r_p \quad = \quad 2n + p\,|{-1} \;\; 0 \;\; 1)$$
$$\Rightarrow \qquad \sum_3^{\mathrm{P}} pN_p = \quad \overline{\mathsf{N}}\Pi = 2n\pi(n) + \Pi\,|{-1} \;\; 0 \;\; 1)\,.$$

Equation (3) is the sum of the exact number of odd composites over all $\pi(n)$ primes, and H_n is the prime harmonic function. Thus

$$\mathrm{N}_p \;=\; \frac{2n}{p} + \big| -1\ \ 0\ \ 1\big) \quad \therefore \quad \mathsf{N} = 2nH_n \;\; + \pi(n)\,|{-1}\ \ 0\ \ 1)\,.$$

In the next section, we will conclude the Legendre conjecture problem by estimating the average and exact numbers of the composites in the interior of (a, b) .

The Average Composite Number Proof

Even though the average composite number confirmation of the Legendre conjecture may not be its proof, it may be an indication of the validity of the Legendre conjecture.
The residuals $r_p \in |0\ \ p)$ satisfy the inequality $0 \le r_p < p$, and their sum over all $\pi(n)$ primes $R = 2n\pi(n) - \sum_p p\mathrm{N}_p + |0\ \ 1\rangle\Pi$, the inequality $0 \le R < \Pi$, see equation (1). Further

$$\begin{gathered} 0 < 2n\pi(n) - \overline{\mathsf{N}}\Pi + |0\ \ 1\rangle\Pi < \Pi \\ 0 < \tfrac{2\pi(n)}{\Pi} - \tfrac{\overline{\mathsf{N}}}{n} + \tfrac{|0\ \ 1\rangle}{n} < \tfrac{1}{n} \\ -\frac{|1\ \ 0\rangle}{n} + \frac{2\pi(n)}{\Pi} < [\,\overline{\mathsf{N}} : n\,] < \frac{2\pi(n)}{\Pi} + \frac{|0\ \ 1\rangle}{n}. \end{gathered} \tag{4}$$

The final decision requires well-defined bounds on the number and the sum of the primes smaller or equal to a large integer n. According to Havil, J.Gamma [1], page 186, for function $\pi(n)$,

$$0.922 < \tfrac{\pi(n)}{n/\ln n} < 1.105 \quad \Leftrightarrow \quad 0.922\,n/\ln n < \pi(n) < 1.105\,n/\ln n \quad \Rightarrow \quad \pi(n) \sim n/\ln n.$$

$\pi(n)$ is very well bounded function , so that for our purpose we may take that its exact value is $\pi(n) = n/\ln n$. The sum Π of the first $\pi(n)$ primes estimates are those of Dusart, presented in the Christian Axler [2] paper. There, $f(n) = \ln n + \ln\ln n - 3/2$ and $g(n) = (\ln\ln n - 5/2)/\ln n$, and the Dusart bounds on the sum of the first $\pi(n)$ primes are

$$\frac{n^2}{2} f(n) \le\; \Pi \;< \frac{n^2}{2}(f(n) + g(n)).$$

The right boundary is valid for $n \ge 115149$ and the left boundary, obtained from the condition of the positivity of the numerical function $g(n)$, for $n \ge 305494$. Both numerical bounds are in the range $n = 9 \cdot 109$ of the numerically proven validity of the Legendre conjecture. Further, the reciprocal of the sum Π of the primes is needed, and we calculate

$$\begin{array}{rcl} \frac{2}{n^2}\,\frac{1}{f(n)+g(n)} & < \frac{1}{\Pi} \le & \frac{2}{n^2}\,\frac{1}{f(n)} \\ \frac{4\pi(n)}{n^2}\,\frac{1}{f(n)+g(n)} & < \frac{2\pi(n)}{\Pi} \le & \frac{4\pi(n)}{n^2}\,\frac{1}{f(n)} \\ \frac{4}{n\ln n}\,\frac{1}{f(n)+g(n)} & < \frac{2\pi(n)}{\Pi} \le & \frac{4}{n\ln n}\,\frac{1}{f(n)} \end{array} \tag{5}$$

Now we have all we need to use equation (4) to construct the composed to odd integers rate in the interior of (a, b) . The composed to odd integers rate obeys

$$\begin{aligned}[\overline{\mathsf{N}}:n] &> \tfrac{2\pi(n)}{\Pi} - \tfrac{|1\ \ 0\rangle}{n} \quad > \tfrac{4}{n\ln n}\tfrac{1}{f(n)+g(n)} - \tfrac{|1\ \ 0\rangle}{n} \\ [\overline{\mathsf{N}}:n] \quad &< \tfrac{2\pi(n)}{\Pi} + \tfrac{|0\ \ 1\rangle}{n} \leq \quad \tfrac{4}{n\ln n}\tfrac{1}{f(n)} + \tfrac{|0\ \ 1\rangle}{n},\end{aligned}$$

inequalities, and the following bounds for the rate of composed to odd integers apply

$$\frac{1}{n}\Big(- |1\ \ 0\rangle + \frac{4}{\ln n}\frac{1}{f(n)+g(n)}\Big) < \quad [\overline{\mathsf{N}}:n] \quad \leq \Big(\frac{4}{\ln n}\frac{1}{f(n)} + |0\ \ 1\rangle\Big)$$

We substitute the left limiting function by the natural bound zero, and

$$0 \leq \ [\overline{\mathsf{N}}:n] \ \leq \frac{1}{n}\Big(\frac{4}{\ln n}\frac{1}{f(n)} + 1\Big) \tag{6}$$

The upper limiting function $F_{\rm O}(k)$ is decreasing to zero by n. Numerical calculation at the points $n = e^k\, k = 1, 2, 3, \cdots$, given in Table 4, shows that for all $n > e^k \approx 20$ function $F_{\rm O}(k)$ is smaller than one.

Table 4. Function $F_{\rm O}(k) = \frac{1}{n}\Big(\frac{4}{\ln n}\frac{1}{f(n)} + 1\Big)$

k	1	2	3	4	5	6	7	$\cdots$	
$F_{\rm O}(k)$	-2.575	2.667	0.546	0.084	0.0.021	0006	0.002	$\cdots$	$\longrightarrow 0$

Corollary 6. *The average composite counting affirms the Legendre conjecture.*

□ An existing numerical proof holds for all $n \leq 9 \cdot 10^9$. The function $F_{\rm O}(k)$ is smaller than one for all $n > 20$, and the limitations of the used boundaries are valid beyond $n = 115149$. Hence, the Legendre conjecture holds everywhere. ■

Conclusion

The affirmative statement of the Legendre conjecture in the case of average composite counting is a good indication but not proof of the Legendre conjecture. The exact number of composite counts case presents in this section. We start with the sum of all composites in equation (3) to calculate the odd composite to odd integers rate in the interior of (a, b)

$$\begin{aligned} \mathsf{N} \quad &= 2nH_n + \pi(n)\big| - 1\ \ 0\ \ 1\big) \\ \Rightarrow \quad \mathsf{N}:n \quad &= 2H_n + \frac{\pi(n)}{n}\big| - 1\ \ 0\ \ 1\big). \end{aligned} \tag{7}$$

The rate calculation/estimation relies essentially on the prime harmonic function H_n, closely related to the more convenient and explicitly defined logarithmic function. Exactly, $A = \ln x + const = \int \frac{dx}{x}$ is the area function of the $\frac{1}{x}$. The prime partition $3, 5, 7, 11, \cdots, \mathrm{P} \leq n$ of the interval

$[3, n]$ creates $\pi(n) - 1$ segments, each of a measure $\mu_p = p_{next} - p$, $p = 1, 2, 3, \cdots, \pi(n) - 1$, see Picture 1. The functions H_n and H'_n,

$$\begin{aligned} H_n &= \frac{1_3}{3} + \frac{1_5}{5} + \frac{1_7}{7} + \cdots + \frac{1_{\mathrm{P}}}{\mathrm{P}} \sim \quad \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{\mathrm{P}} = \bar{H}_n \\ H'_n &= \qquad\;\; \frac{1_5}{5} + \frac{1_7}{7} + \cdots + \frac{1_{\mathrm{P}}}{\mathrm{P}} \sim \qquad\;\; \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{\mathrm{P}} = \bar{H}'_n. \end{aligned}$$

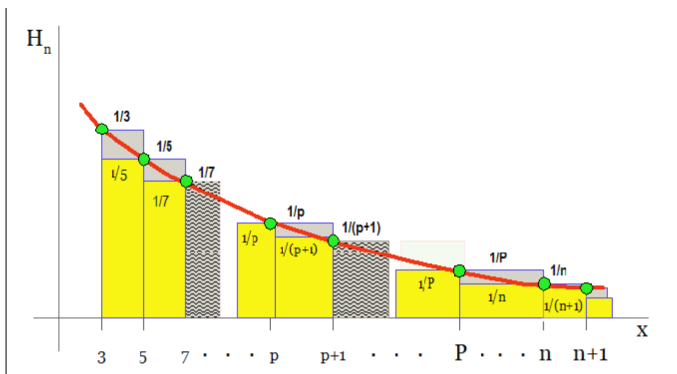
defined on this partition are the upper and lower simple functions of the $\frac{1}{x}$ function. We distinguish these functions from their unit, or the counting measure functions $\bar{H}_n$ at the right side of equations.

The explicit area under the upper simple function on the prime partition in Picture 1. satisfies

$$A \quad = \frac{\mu_3}{3} + \frac{\mu_5}{5} + \frac{\mu_7}{7} + \cdots + \frac{\mu_{\mathrm{P}}}{\mathrm{P}} > \int_3^{\mathrm{P}} \frac{dx}{x} > \int_3^n \frac{dx}{x} = \quad \ln \frac{n}{3}.$$

By the integral average theorem, there is a measure $\overline{\mu} : 2 \leq \overline{\mu} \leq sup\{\mu_p\} \leq \mathrm{P} < n$ such that

$$A \quad = \overline{\mu}\Big(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{\mathrm{P}}\Big) = \quad \overline{\mu} H_n > \ln \frac{n}{3}. \tag{8}$$



Picture 1: Harmonic Function

Figure 1: Harmonic Function

However, the case of the $\bar{H}'_n$ function requires little more work. The area below that function is limited above by the area below the $\frac{1}{x}$ function, and

$$A' \quad = \frac{\mu_3}{5} + \frac{\mu_5}{7} + \frac{\mu_7}{11} + \cdots + \frac{\mu_{\mathrm{P}-1}}{\mathrm{P}} < \quad \int_3^{\mathrm{P}} \frac{dx}{x} < \int_3^n \frac{dx}{x} = \quad \ln \frac{n}{3}.$$

Again, by the integral average theorem, there is a measure $\overline{\mu} : 1 \leq \overline{\mu} \leq sup\{\mu_p\} \leq \mathrm{P} < n\}$ such that

$$A' \quad = \overline{\mu} \sum_5^{\mathrm{P}} \frac{1}{p} = \quad \overline{\mu}\Big(\sum_5^{\mathrm{P}} \frac{1}{p} + \frac{1}{3}\Big) - \frac{\overline{\mu}}{3} = \overline{\mu} \sum_3^{\mathrm{P}} \frac{1}{p} - \frac{\overline{\mu}}{3} = \quad \overline{\mu}\, H_n - \frac{\overline{\mu}}{3} < \quad \ln \frac{n}{3} \tag{9}$$

The average $\overline{\mu}$ is part of n, there is an $s \geq 0$ such that $\overline{\mu} = n - s = n(1 - \frac{s}{n})$. We use the equation (8) and (9) to impose the following lower and upper limits on the harmonic function H_n ,

$$\begin{aligned} \tfrac{1}{\mu} \ln \tfrac{n}{3} &< H_n < \tfrac{1}{\mu} \ln \tfrac{n}{3} + \tfrac{1}{3}, \\ \mathrm{G}\left(\tfrac{s}{n}\right) \tfrac{1}{n} \ln \tfrac{n}{3} &< H_n < \mathrm{G}\left(\tfrac{s}{n}\right) \tfrac{1}{n} \ln \tfrac{n}{3} + \tfrac{1}{3} \\ \tfrac{1}{3}\mathrm{G}\left(\tfrac{s}{n}\right) \tfrac{3}{n} \ln \tfrac{n}{3} &< H_n < \tfrac{1}{3}\mathrm{G}\left(\tfrac{s}{n}\right) \tfrac{3}{n} \ln \tfrac{n}{3} + \tfrac{1}{3} \qquad \{(s,n) \to 3(t,x)\} \\ \tfrac{1}{3}\mathrm{G}\left(\tfrac{t}{x}\right) \tfrac{1}{x} \ln x &< H_n < \tfrac{1}{3}\mathrm{G}\left(\tfrac{t}{x}\right) \tfrac{1}{x} \ln x + \tfrac{1}{3}. \end{aligned}$$

Here, $\mathrm{G}(x) = \frac{1}{1-x})$, $x = \frac{s}{n} = \frac{t}{x} < 1$, is the geometric series function. The function $\pi(n)$ contributes the function $\frac{\pi(n)}{n} = \frac{1}{\ln n}$, monotonically decreasing to zero. Further, the function $F = \mathrm{G}\left(\frac{t}{x}\right) \frac{1}{\pi(x)} = \mathrm{G}\left(\frac{t}{x}\right) \frac{\ln x}{x}$ is of the most interest and we make some suitable rearrangements of it.

$$\begin{aligned} F &= \left(1 + \tfrac{t}{x} + \tfrac{t^2}{x^2} + \tfrac{t^3}{x^3} \ldots\right) \tfrac{\ln x}{x} \\ &= \left(\tfrac{1}{x} + \tfrac{t}{x^2} + \tfrac{t^2}{x^3} + \tfrac{t^3}{x^4} \ldots\right) \ln x, \\ \theta &= \tfrac{t}{x} < 1. \\ F &= \left(\tfrac{1}{x} + \tfrac{\theta}{x} + \tfrac{\theta^2}{x^2} + \tfrac{\theta^3}{x^3} \ldots\right) \ln x \\ &= \left(\tfrac{1}{x} + \tfrac{\theta}{x} + \tfrac{\theta^2}{x^2} + \tfrac{\theta^3}{x^3} \ldots\right) \ln x \\ &= \tfrac{\ln x}{x} + \tfrac{\theta}{x}\left(1 + \tfrac{\theta}{x} + \tfrac{\theta^2}{x^2} + \tfrac{\theta^3}{x^3} \ldots\right) \ln x, \end{aligned}$$

$$\therefore \quad F = \quad \tfrac{\ln x}{x}\left(1 + \tfrac{\theta}{1 - \frac{\theta}{x}}\right) \quad = \frac{\ln x}{x} \frac{(1+\theta)x - \theta}{x - 1}, \quad 1 > \theta = o(x).$$

The variable θ, is an unknown implicit function of the harmonic function and variable x, takes values between zero and one and oscillates between its θ -infimum F_* and θ- supremum F^*,

$$\begin{aligned} F_* &= \inf_\theta \tfrac{\ln x}{x} \tfrac{(1+\theta)x - \theta}{x-1} = \tfrac{\ln x}{x} \tfrac{x-1}{x} = \quad \tfrac{\mathbf{(x-1)\ln x}}{\mathbf{x^2}} \\ &\rightsquigarrow \tfrac{1}{2}\left(\tfrac{\ln x}{x} + \tfrac{x-1}{x^2}\right) \\ &\rightsquigarrow \tfrac{1}{2}\left(\tfrac{1}{x} + \tfrac{1}{2x}\right) = \tfrac{5}{4x} \xrightarrow{x \to \infty} 0, \\ F^* &= \sup_\theta \tfrac{\ln x}{x} \tfrac{(1+\theta)x - \theta}{x-1} = \tfrac{\ln x}{x} \tfrac{2x}{x-1} = \quad \mathbf{2}\tfrac{\mathbf{\ln x}}{\mathbf{x-1}} \\ &\rightsquigarrow \tfrac{2}{x} \xrightarrow{x \to \infty} 0. \end{aligned}$$

Numerical values of the $F's$ and $\frac{1}{\ln x}$ functions calculated at the sequence $x = e^k,\ k = 1, 2, 3, \cdots$, $F^*(k;1) = \frac{2k}{e^k - 1}$ and $F^*(k;1) = \frac{k(e^k-1)}{e^{2k}}$. are presented in Table 5. F functions are smaller than one at $x = e$ and further decrease monotonically to zero; function $\frac{\ln x}{x} \rightsquigarrow \frac{1}{x} : 1 \to\ 0$. At $k \geq 3$, or the $n \geq 3e^3$, Finally, the harmonic function is bounded as follows,

$$\frac{1}{3}F_*(x) \quad < H_n < \quad \frac{2}{3}F^*(x) + \frac{1}{3},$$

Table 5.Functions $F_*(k)$ and $F^*(k)$

k	1	2	3	4	5	6	7	$\cdots$	
$F^*(k)$	1.164	0.626	0.314	0.149	0.068	0.030	0.013	$\cdots$	$\longrightarrow 0$
$F_*(k)$	0.233	0.234	0.142	0.072	0.033	0.015	0.006	$\cdots$	$\longrightarrow 0$
$\frac{1}{\ln 3\mathrm{N}}$	1.00	0.50	0.333	0.250	0.200	0.167	0.143	$\cdots$	$\longrightarrow 0$
$[\mathsf{N} : n]_*$	2.804	1.575	0.873	0.495	0.300	0.201	0.149	$\cdots$	$\longrightarrow 0$
$[\mathsf{N} : n]^*$	1.608	1.457	1.195	1.007	0.898	0.837	0..803	$\cdots$	$\longrightarrow 0$

Notice that $\frac{1}{3}$ originates from the messing prime $p = 3$ segment in the area of the lower bounding simple function.

Corollary 7. *Between the squares of any two consecutive integers there is at least one prime number.*

□ After finding the limits of the harmonic function, we are constructing step by step the composite to odd integers ratio and its lower and upper bounding functions

$$0 \quad \le \tfrac{2}{3}F_*(n;\theta) < \ 2H_n \ < \tfrac{2}{3}F_*(k;\theta) + \tfrac{1}{3} \le \quad \tfrac{2}{3}$$

$$\frac{\pi(\mathbf{n})}{\mathbf{n}} \Big| -\mathbf{1} \ \ \mathbf{0} \ \ \mathbf{1}\Big) \quad \le \tfrac{2}{3}F_*(n;\theta) + \frac{\pi(n)}{n} \Big| -1 \ \ 0 \ \ 1\Big)$$
$$< \ 2H_n + \frac{\pi(n)}{n} \Big| -1 \ \ 0 \ \ 1\Big) \ <$$
$$\tfrac{2}{3}F^*(k;\theta) + \frac{\pi(n)}{n} \Big| -1 \ \ 0 \ \ 1\Big) + \tfrac{2}{3} \le \quad \boldsymbol{\tfrac{2}{3}} + \frac{\pi(\mathbf{n})}{\mathbf{n}} \Big| -\mathbf{1} \ \ \mathbf{0} \ \ \mathbf{1}\Big)$$

$$\frac{\mathbf{1}}{\ln \mathbf{n}} \Big| -\mathbf{1} \ \ \mathbf{0} \ \ \mathbf{1}\Big) \quad \le \tfrac{2}{3}F_*(n;\theta) + \frac{1}{\ln n} \Big| -1 \ \ 0 \ \ 1\Big)$$
$$< \ \mathsf{N} : \mathbf{n} \ <$$
$$\tfrac{2}{3}F^*(k;\theta) + \frac{1}{\ln n} \Big| -1 \ \ 0 \ \ 1\Big) + \boldsymbol{\tfrac{1}{3}} \le \quad \boldsymbol{\tfrac{2}{3}} + \frac{\mathbf{1}}{\ln \mathbf{n}} \Big| -\mathbf{1} \ \ \mathbf{0} \ \ \mathbf{1}\Big)$$

Contributions of the $\frac{1}{\ln x}$ are smaller than one, and the function decreases to zero at infinity. The numerical values $[\mathsf{N} : n]_*$ of the lower and The numerical values $[\mathsf{N} : n]_*$ of the lower and $[\mathsf{N} : n]^*$ of upper composite to odd integers ratios on the interior of (a, b). are shown in Table 5. For all $k \ge 5$, which is $n \ge 3e^5 \approx 445$, the bounding ratio functions are smaller than one and monotonically decrease to zero by the increasing n. Numerical calculation confirms that the Legendre conjecture holds up to $n = 9 \cdot 10^9$. Hence, the Legendre conjecture is true for all integers. ■

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