Reformulation of Syracuse Function and its Convergence

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Abstract

This paper presents a geometrical approach to tackle the infamous Collatz conjecture. In this approach, we represent odd natural numbers as points in 2-D space. We then define a iterative geometrical algorithm and prove that this algorithm is equivalent to the Collatz function (more precisely, Syracuse function). Using the monotone convergence theorem, we prove the sequence generated by this algorithm always converges to 1. Since, this is same as saying Collatz (Syracuse) sequence converges to 1, we prove that the Collatz conjecture is true.

1 Introduction

The Collatz conjecture, also known as 3n+1 problem, was conjectured in 1937 by mathematician Lothar Collatz. This is one of the most famous unsolved problems in mathematics. The conjecture states that if we apply Collatz funtion to positive integer, and continuously iterate this function with previous result, we will eventually reach 1.

1.1 Collatz, Syracuse sequence, and Collatz conjecture

Collatz sequence for $N > 0, N \in \mathbb{N}$ can be generated by iterating following function:

$$T(N) = \begin{cases} 3N+1, & \text{if } N \text{ is odd.} \\ \dfrac{N}{2}, & \text{if } N \text{ is even.} \end{cases}$$

Example Find the Collatz sequence for number 9.
We begin with 9 as first element of the sequence. Since, its an odd number, the second element is,

$$T(9) = 3 * 9 + 1 = 28$$

Since, 28 is an even number, next element is

$$T(T(9)) = T^2(9) = T(28) = \frac{28}{2} = 14$$

If we iterate this process of $3N + 1$ for odd N and dividing by 2 for even, we get following sequence $9 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow 2 \rightarrow 1$

We can see that $\mathrm{T}^{19}(9) = 1$,and after 19th iteration, the sequence enters into to a loop of $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$.

Conjecture 1.1 Collatz conjecture states that $\forall N \in \mathbb{N}, N > 0$, $T^{\theta}(N) = 1$.
$\theta \in \mathbb{N}$ and denotes minimum number of iterations of Collatz function to enter the loop $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$.

Syracuse sequence is the sequence of odd elements of the Collatz sequence. For $n \in 2\mathbb{N}+1$, the next element in the sequence can be calculated using following function

$$S(n) = \frac{3n+1}{2^{\beta}} \quad , \beta > 0, \beta \in \mathbb{N} \tag{1}$$

For example, Syracuse sequence of 9 is as following:
$9 \rightarrow 7 \rightarrow 11 \rightarrow 17 \rightarrow 13 \rightarrow 5 \rightarrow 1 \rightarrow 1$

We find that, $\mathrm{S}^6(9) = 1$. and notice that any more iterarations will only generate 1.

Conjecture 1.2 Syracuse formulation of Collatz conjecture states that $\forall n \in 2\mathbb{N}+1$, $S^{\theta}(n) = 1$. $\theta \in \mathbb{N}$ and denotes minimum number of iterations of Syracuse function to reach 1.

2 Geometrical Algorithm equivalent to Syracuse Function

2.1 Construction of $\mathbb{P}$: Positive odd integers in 2-D space

Definition 1.1 Let $\mathbb{Z}$ be the set of integers, $\mathbb{N} := \{0, 1, 2, ...\}$ be the natural numbers, and $2\mathbb{N}+1 := \{1, 3, 5, ...\}$ be the positive odd integers. In a 2-D Cartesian plane, let us plot all points $(x, y) = (n * 2^{\alpha}, 2^{\alpha})$, where $n \in 2\mathbb{N}+1$ and, $\alpha \in \mathbb{Z}$.We define this infinite collection of points as $\mathbb{P}$ and label each point $P_i = (n * 2^{\alpha}, 2^{\alpha})$ as n, as shown in Figure 1. In this paper, whenever P_i **corresponds to** n, or $P_i = n$ is mentioned, we mean point P_i is labeled as n.

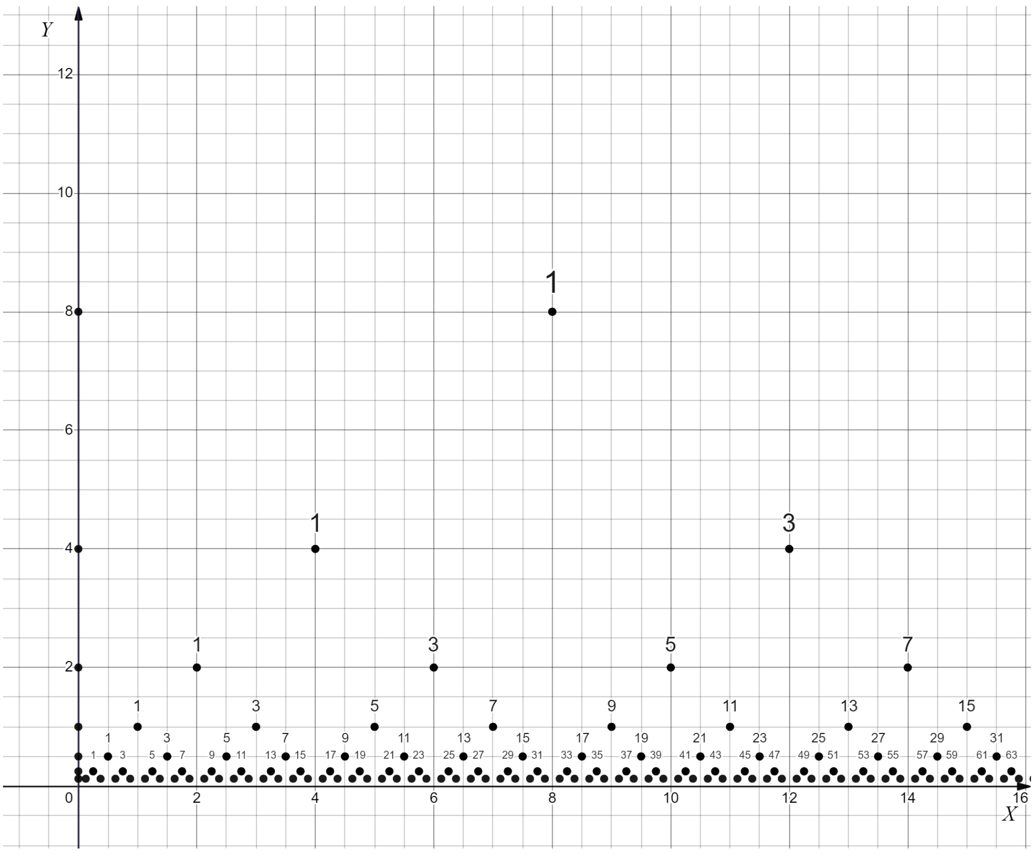

Figure 1: Odd positive intergers represented as points on 2-D plane.

In each horizontal line at $y = 2^{\alpha}$, $\alpha \in \mathbb{Z}$, the points (positive odd integers) are evenly spaced and increases as we move right. Each of these horizontal lines at $y = 2^{\alpha}$ is equivalent to a number line with unit length 2^{α}. Hence, we can think of $\mathbb{P}$ as the collection of postitive odd integers on number lines that are positioned at $y = 2^{\alpha}$, and scaled by 2^{α}.

2.2 Properties of $\mathbb{P}$

Property 1.1 All points that correspond to n, lie on the line $y = \frac{1}{n}x$.

The coordinates (x, y) for any point P_i that corresponds to n is defined as $(n * 2^\alpha, 2^\alpha)$, which always satisfies the equation $y = \frac{1}{n}x$. In Figure 2, we demostrate this property with some examples. We can also see that when $n > 1$, all points are located below the line $y = x$.

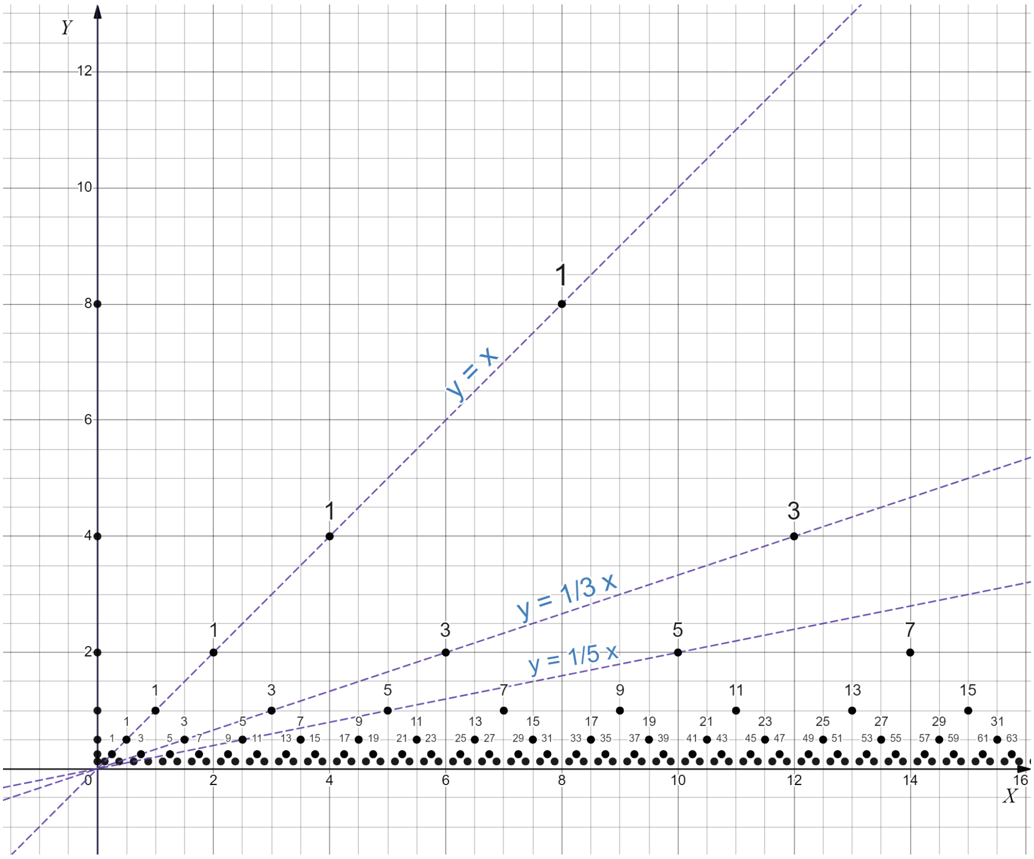

Figure 2: Example of $y = \frac{1}{n}x$ passing through all n in $\mathbb{P}$

Property 1.2 A straight line with slope(m)= -3 that passes through point n in $\mathbb{P}$, also passes through $4n + 1$.
Let us consider two points: $(x_1, y_1) = (n * 2^\alpha, 2^\alpha)$ and $(x_2, y_2) = ((4n + 1) * 2^{\alpha-2}, 2^{\alpha-2})$. These points correspond to n and $4n + 1$ respectively in $\mathbb{P}$. We can calculate the slope using the formula,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{2^{\alpha-2} - 2^\alpha}{(4n + 1) * 2^{\alpha-2} - n * 2^\alpha} = -3$$

Using point-slope line equation formula, we can write the equation of this line with slope(m) $= -3$, and passing through $(n * 2^\alpha, 2^\alpha)$ as:

$$y - 2^\alpha = -3(x - n * 2^\alpha)$$

Simplifying the equation we get,

$$y + 3x = (3n + 1) * 2^\alpha \tag{2}$$

The y-intercept part in Equation 2 is $(3n + 1)$ and powers of 2. We shall utilize this fact to connect with Collatz conjecture.

2.3 Syracuse algorithm $Syr_{\mathbf{algo}} : \mathbb{P} \to \mathbb{P}$

We have a point, $P_i \in \mathbb{P}$ that corresponds to n, that is located in coordinates $(n * 2^{\alpha}, 2^{\alpha}), \alpha \in \mathbb{Z}$. To find a point that corresponds to next element in Syracuse sequence $S(n)$, we need to follow the algorithm defined below.

Definition 2.1
We define following 2-step geometrical algorithm $Syr_{\mathbf{algo}} : \mathbb{P} \to \mathbb{P}$, that maps P_i to P_{i+1}.

Step 1. From the point P_i, draw a line with slope(m) = -3, until it meets $y = x$. We define this intersection point as A_i
Step 2. Draw a verticle line (perpendicular to X-axis) from pointA_i, until it meets P_{i+1}. P_{i+1} is the next point/element in the sequence.

This 2-step algorithm is demonstrated on Figure 3. Here, $Syr_{\mathbf{algo}}$ maps point $P_1 = 9$ to point $P_2 = 7$

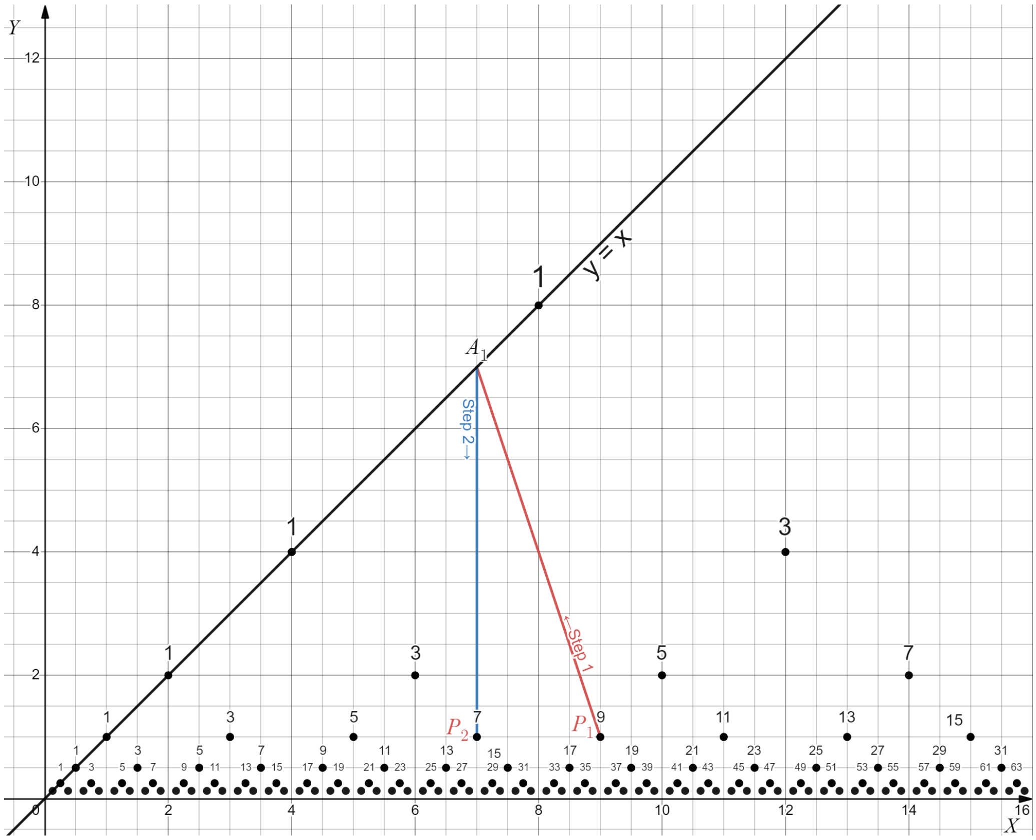

Figure 3: Algorithm steps to find next element in the Syracuse sequence

In order to generate a sequence of points, we need to iterate $Syr_{\mathbf{algo}}$ from the new point. Demonstration of iteration is shown in Figure 4, where red lines represent Step 1, and blue lines represent Step 2 of the algorithm. In this example we apply this algorithm iteratively starting from point P_1 that corresponds to 9. We see that the sequence reaches P_7 which corresponds to 1.

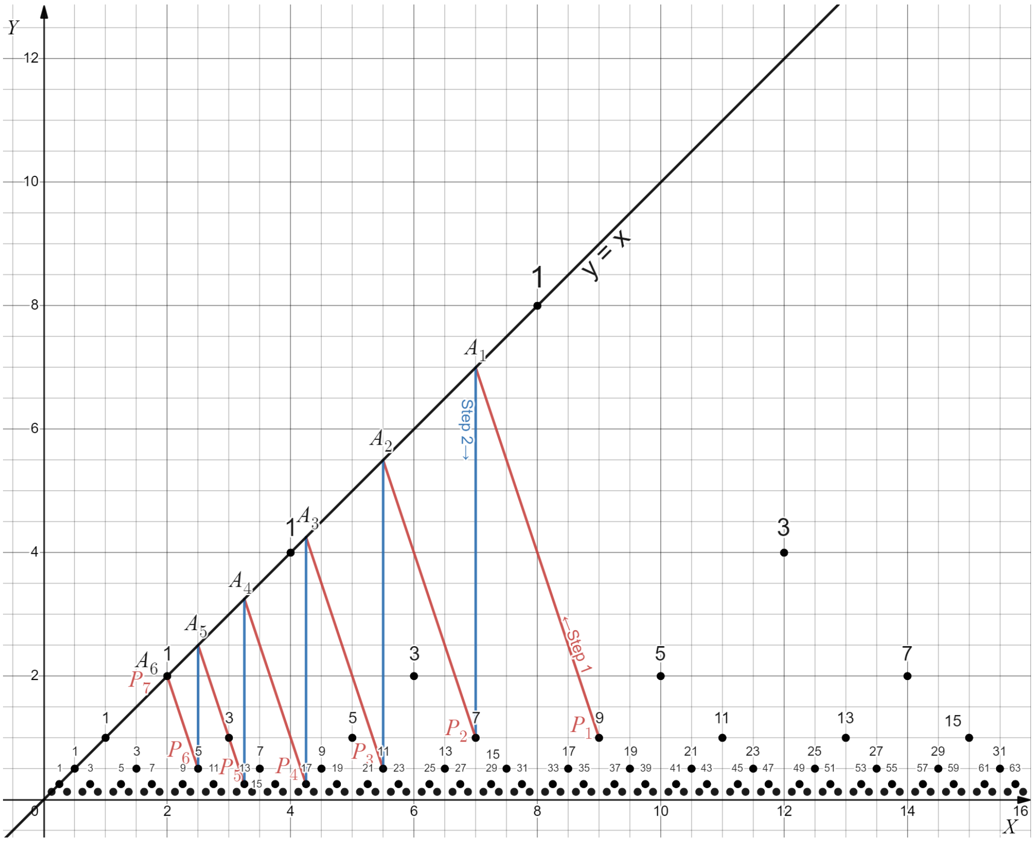

Figure 4: Example of iterations of algorithm defined above

The sequence of points in Figure 4 is
$9 \to 7 \to 11 \to 17 \to 13 \to 5 \to 1$

This sequence is exactly same as the sequence generated by the Syracuse function for number 9. In next section we prove that this Syracuse algorithm in $\mathbb{P}$ is equivalent to Syracuse function for all positive odd integers.

Remark: In the figure we stop at $P_7 = 1$, but we can keep going. However, when we apply the algorithm at $P_i = 1$, we get $P_i = A_i$=P_{i+1} =1. This is because point correspoinding to 1 lies on the line $y = x$. This means after a sequence reaches 1, just like Syracuse function, further iterations of this algorithm only generate 1 (or in this case, it stays on the same point).

2.4 Proving equivalence between $Syr_{\textbf{algo}}$ and Syracuse function$S(n)$

Proposition 1.1 $Syr_{\textbf{algo}}$ is equivalent to Syracuse function $S(n)$.

Proof. In Step 1 of the $Syr_{\textbf{algo}}$ we draw a line slope(m) $= -3$ passing through point P_i located in $(n * 2^\alpha, 2^\alpha)$. Using point and slope, we find the line equation, which is shown in Equation 2:

$$y + 3x = (3n + 1) * 2^\alpha \tag{2}$$

To find the intersection point A_i, we substitute $y = x$ in Equation 2

$$\begin{aligned} x + 3x &= (3n + 1) * 2^\alpha \\ 4x &= (3n + 1) * 2^\alpha \\ x &= (3n + 1) * 2^{\alpha-2} \end{aligned}$$

Since, $n \in 2\mathbb{N} + 1$, $3n + 1$ is always even, i.e. it shall always have a factor 2^β, where $\beta > 0, \beta \in \mathbb{N}$.

$$x = \frac{3n + 1}{2^\beta} * 2^\beta * 2^{\alpha-2} \tag{3}$$

Substituting, Equation 1, Syracuse function, $S(n) = \dfrac{3n+1}{2^{\beta}}$, $\beta > 0, \beta \in \mathbb{N}$ into Equation 3, we get,

$$x = S(n) * 2^{\alpha-2+\beta} \tag{4}$$

For A_i, $y = x$

$$\therefore \textbf{Coordinates of the point} A_i, (x, y) = (S(n) * 2^{\alpha-2+\beta}, S(n) * 2^{\alpha-2+\beta}) \tag{5}$$

In Step 2 of the $Syr_{\textbf{algo}}$, we draw a line from A_i perpendicular to the X-axis. This step identifies the next point P_{i+1} in the sequence. For this step to be valid, following conditions must be met.
(1) There must be a point in $\mathbb{P}$ with same x-coordinate as A_i, and
(2) The point must correspond to $S(n)$.

From Equation 4, we have,

x-coordinate of A_i, $x = S(n) * 2^{\alpha-2+\beta}$, $S(n) \in 2\mathbb{N}+1$ and $\alpha - 2 + \beta \in \mathbb{Z}$

By definition, $\mathbb{P}$ contains all points in 2-D cartesian space that is of form $(x, y) = ((2N + 1) * 2^Z, 2^Z)$, where $2N + 1 \in 2\mathbb{N} + 1$, and $Z \in \mathbb{Z}$.

Therefore, Point P_{i+1} with same x-coordinate as A_i exists, and is located in

$$P_{i+1} : (x, y) = (S(n) * 2^{\alpha-2+\beta}, 2^{\alpha-2+\beta}) \tag{6}$$

Similarly we have, from **Property1.1** of $\mathbb{P}$, that all points that correspond to $2N + 1$, lie on the line $y = \frac{1}{2N+1}x$.
Point $(S(n) * 2^{\alpha-2+\beta}, 2^{\alpha-2+\beta})$ corresponds to $S(n)$, because this point satisfies $y = \frac{1}{S(n)}x$.

Thus, we have proved that $\forall n \in 2\mathbb{N} + 1$, $Syr_{\textbf{algo}} : \mathbb{P} \to \mathbb{P}$ maps point that corresponds to n to point that corresponds to $S(n)$.

Hence, the Syracuse algorithm $Syr_{\textbf{algo}}(n)$ defined in $\mathbb{P}$ is equivalent to the Syracuse function $S(n)$

As a consequence, we can reformulate Collatz conjecture using the Syracuse Algorithm as :

Conjecture 1.3 For any point that correspond to n in $\mathbb{P}$, $n \in 2\mathbb{N} + 1$, $Syr^{\theta}_{\textbf{algo}}(n) = 1$.

$\theta \in \mathbb{N}$ and denotes minimum number of iterations of Syracuse algorithm to reach a point that corresponds to 1.

Since we have established equivalence between Syracuse (and thus Collatz) function and the Syracuse algorithm in $\mathbb{P}$, in next section we will study convergence of this algorithm.

□

3 Convergence of Syracuse Algorithm $Syr_{\mathsf{algo}}(n)$ in $\mathbb{P}$

Figure 5 shows the geometry of the Syracuse Algorithm, with red lines representing Step 1 and blue lines representing Step 2 of the algorithm. Black dots are the points in $\mathbb{P}$, and dot on the line $y = x$ corresponds to 1.

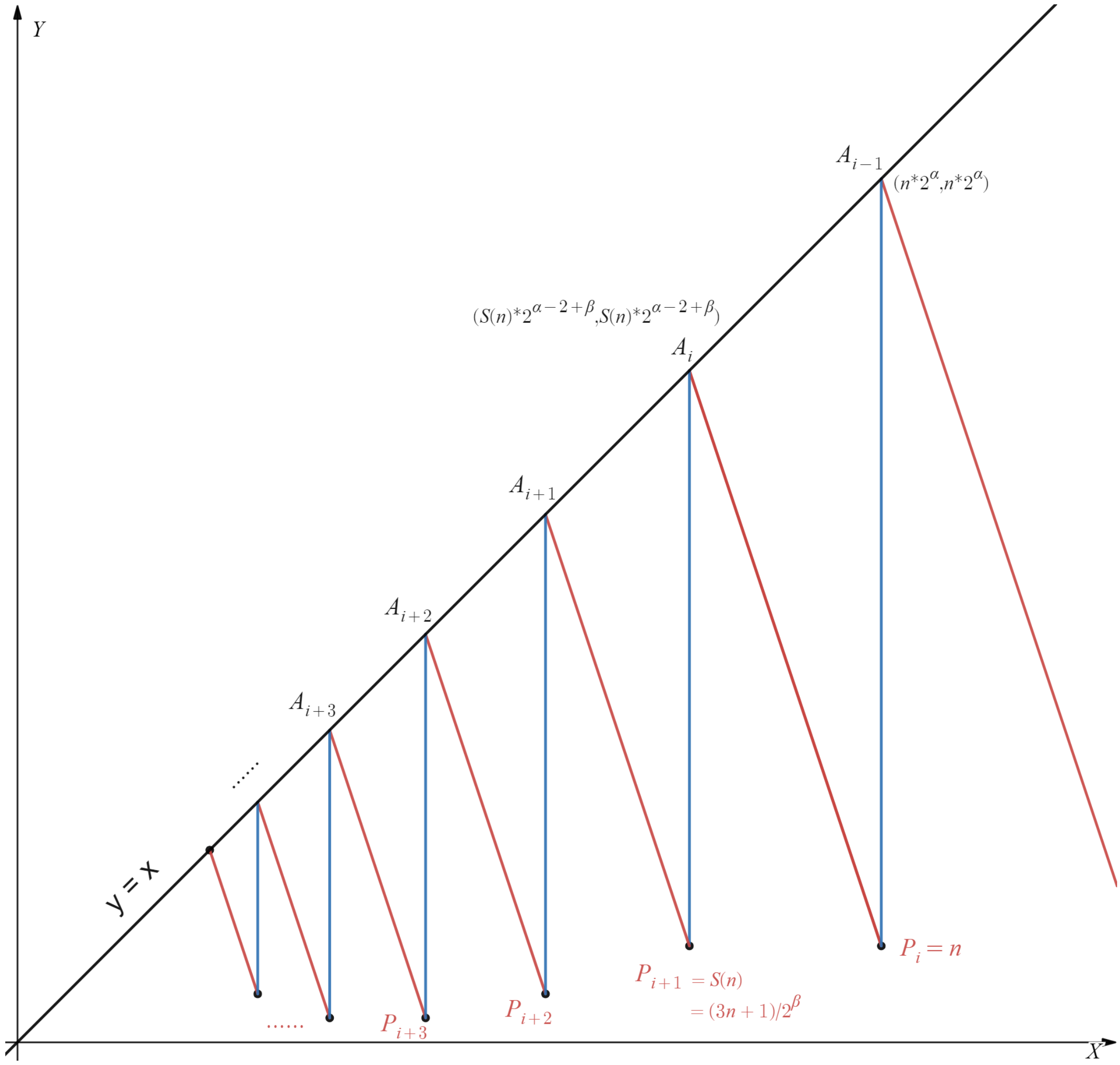

Figure 5: Example of iterations of algorithm defined above

In Figure 5, we notice that, the red line shifts towards left by some distance δ as we iterate the algorithm, (e.g. $\mathrm{P}_{i+1}A_{i+1}$ is left of P_iA_i), and when the red line passes through point $1 \in \mathbb{P}$, it stops shifting and halts.

Proposition 1.2 The red line halts if and only if the line passes through 1.

Proof. Let $P_i = 1$ and δ_i be distance (shift) between consecutive red lines, P_iA_i and line $P_{i+1}A_{i+1}$. and

From Figure 5, following holds true for any i: $\delta_i \propto$ length $(\overline{\mathrm{A_iA_{i+1}}}) \propto$ length $(\overline{\mathrm{A_iP_{i+1}}})$

Point $P_i = 1$ in $\mathbb{P}$ lies on $y = x$, So the length $(\overline{\mathrm{P_iA_i}})$ = length $(\overline{\mathrm{A_iP_{i+1}}}) = 0$, therefore, $\delta_i = 0$. Hence, the red line stops shifting if it passes through 1.

Similarly, if the red line does not shift after an iteration ($\delta_i = 0$), then the length $(\overline{\mathrm{A_iP_{i+1}}}) = 0$, which means point lies on $y = x$. Therefore, the point P_i must be 1. Hence, **Proposition 1.2** is proved.

Conversely, we can also say following is always true.
Corollary1.1 The red line shifts left by some distance δ if and only if it does not pass through point that corresponds to 1 in $\mathbb{P}$.

Therefore, to prove **Conjecture 1.3**, Reformulated Collatz conjecture using the Syracuse alogrithm, we need to prove that all δ converges to 0 (i.e. length of line segment $\mathrm{A}_i P_{i+1}$ converges to 0.)

□

3.1 Proof of convergence

Theorem 1.1 $\forall n \in 2\mathbb{N}+1\ Syr^{\theta}_{\textbf{algo}}(n) : \mathbb{P} \to \mathbb{P}$ always converges, and converges to 1.

Proof. Let us consider a sequence of lengths of line segments, $L_i, L_{i+1}, L_{i+2}.....$, generated by the Syracuse algorithm.

Where, $L_i = \overline{\mathrm{A_{i-1}P_i}}, L_{i+1} = \overline{\mathrm{A_iP_{i+1}}}, L_{i+2} = \overline{\mathrm{A_{i+1}P_{i+2}}}$, and so on. P_i corresponds to n, P_{i+1} corresponds to $S(n)$ and so on.

We have, coordinates of P_i and A_{i-1}:

$$P_i = (n * 2^{\alpha}, 2^{\alpha})$$
$$A_{i-1} = (n * 2^{\alpha}, n * 2^{\alpha})$$
$$\therefore \mathrm{L}_i = \overline{\mathrm{A_{i-1}P_i}} = n * 2^{\alpha} - 2^{\alpha} = 2^{\alpha} * (n-1) \tag{7}$$

For L_{i+1}, from Equation 5 and 6 we have

$$A_i = (S(n) * 2^{\alpha-2+\beta}, S(n) * 2^{\alpha-2+\beta})$$
$$P_{i+1} = (S(n) * 2^{\alpha-2+\beta}, 2^{\alpha-2+\beta})$$
$$\therefore \mathrm{L}_{i+1} = \overline{\mathrm{A_{i+1}P_{i+2}}} = S(n) * 2^{\alpha-2+\beta} - *2^{\alpha-2+\beta} = 2^{\alpha-2+\beta} * (S(n)-1) \tag{8}$$

Substutuing Equation 1 in Equation 8, $S(n) = \dfrac{3n+1}{2^{\beta}} \quad , \beta >= 1, \beta \in \mathbb{N}$

$$L_{i+1} = 2^{\alpha-2+\beta} * (\frac{3n+1}{2^{\beta}} - 1)$$
$$L_{i+1} = 2^{\alpha-2+\beta} * (\frac{3n+1-2^{\beta}}{2^{\beta}})$$
$$L_{i+1} = 2^{\alpha} * (\frac{3n-3+4-2^{\beta}}{2^2})$$
$$L_{i+1} = 2^{\alpha} * (\frac{3}{4}(n-1) + \frac{4-2^{\beta}}{2^2})$$
$$L_{i+1} = \frac{3}{4} * 2^{\alpha} * (n-1) + 2^{\alpha} * (1-2^{\beta-2})$$
$$\text{Substituting Equation 7, } L_{i+1} = \frac{3}{4} * L_i + 2^{\alpha} * (1-2^{\beta-2}) \tag{9}$$

We have, $\beta >= 1, \beta \in \mathbb{N}$

Case1: When $\beta > 1, 2^{\alpha} * (1-2^{\beta-2}) <= 0$ Therefore, from Equation 9 we get, $L_{i+1} < L_i$

Case 2:When $\beta = 1$, $2^{\alpha} * (1-2^{\beta-2}) = 2^{\alpha-1}$

Substituting in Equation 9 we get, $\mathrm{L}_{i+1} = \dfrac{3}{4} * L_i + 2^{\alpha-1}$

Lets check for the condition, when $L_{i+1} =< L_i$

$$\begin{aligned}
&\frac{3}{4} * L_i + 2^{\alpha-1} =< Li \\
&2^{\alpha-1} =< \frac{1}{4} * L_i \\
\text{From Equation 7, } &2^{\alpha-1} =< \frac{1}{4} * 2^{\alpha} * (n-1) \\
&2 =< n-1 \\
&3 =< n
\end{aligned} \tag{10}$$

from Case 1 and Case 2, when $n >= 3$, then, $L_{i+1} =< L_i$
Also, we have from **Proposition 1.2**, when $n = 1, L_{i+1} = L_i = 0$

Therefore, $L_{i+1} =< L_i$, is true $\forall n \in 2\mathbb{N} + 1$, where, $L_i = length(\overline{\mathrm{A_{i-1}P_i}}), L_{i+1} = length(\overline{\mathrm{A_iP_{i+1}}})$
This means the sequence, L_i, L_{i+1}, L_{i+2}..... is a Monotone non-increasing sequence, with a lower bound (infimum) of 0.

According to the Monotone convergence theorem, If a sequence of real numbers is decreasing and bounded below, then it will converge to the infimum.

Since the lengths, L_i, L_{i+1}, L_{i+2}..... converges to 0, this means the distance between red lines(Step 1) will also converge to 0, meaning, stop shifting any further left.
From **Proposition 1.2** we have, distance between two consecutive red lines, $\delta = 0$, if and only if it passes through Point 1.

Therefore, $\forall n \in 2\mathbb{N} + 1$ $Syr^{\theta}_{\mathbf{algo}}(n) : \mathbb{P} \to \mathbb{P}$ **always converges, and converges to 1.**

□

3.2 Proof of Collatz Conjecture

Proof. We have,
1) Syracuse algorithm $Syr_{\mathbf{algo}}(n)$ defined in $\mathbb{P}$ is equivalent to the Syracuse function $S(n)$, which is equivalent to Collatz function.
2) $\forall n \in 2\mathbb{N} + 1$ $Syr^{\theta}_{\mathbf{algo}}(n) : \mathbb{P} \to \mathbb{P}$ always converges to 1.
Hence, **Collatz conjecture is true.**

□

References

[1] Collatz Conjecture: https://en.wikipedia.org/wiki/Collatz_conjecture

[2] Monotone Convergence Theorem: https://en.wikipedia.org/wiki/Monotone_convergence_theorem