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THE INTEGRAL SEEN AS THE (-1)st DERIVATIVE AND THE HALF-INTEGRAL

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Abstract: The integral seen as the (-1)st derivative and the Half-Integral.

1-(-1)st derivative of x

$$\boxed{\frac{d^{(-1)}}{dx^{(-1)}}x=\frac{1}{2}x^{2}}=\int xdx$$

Proof:

$$\frac{d}{dx}x^{k}=\frac{d^{1}}{dx^{1}}x^{k}=kx^{k-1}\ ,\ \frac{d^{2}}{dx^{2}}x^{k}=k(k-1)x^{k-2}\ ,\ \frac{d^{n}}{dx^{n}}x^{k}=k(k-1)...(k-n+1)x^{k-n}=\frac{k!}{(k-n)!}x^{k-n}\quad(1.1)$$

and with k=1 and n=-1, we have: $\frac{d^{(-1)}}{dx^{(-1)}}x^{1}=\frac{1!}{[1-(-1)]!}x^{1-(-1)}=\frac{1!}{2!}x^{2}=\frac{1}{2}x^{2}=\int xdx$.

<u>Check:</u>

as it must be: $(\frac{d^{(1)}}{dx^{(1)}}\frac{d^{(-1)}}{dx^{(-1)}})(\frac{1}{2}x^{2})=\frac{d^{1-1}}{dx^{1-1}}(\frac{1}{2}x^{2})=\frac{1}{2}\frac{d^{0}}{dx^{0}}x^{2}=\frac{1}{2}x^{2}$, then:

$$(\frac{d^{(1)}}{dx^{(1)}}\frac{d^{(-1)}}{dx^{(-1)}})(\frac{1}{2}x^{2})=\frac{d^{(1)}}{dx^{(1)}}[\frac{d^{(-1)}}{dx^{(-1)}}(\frac{1}{2}x^{2})]=\frac{1}{2}\frac{d^{(1)}}{dx^{(1)}}(\frac{d^{(-1)}}{dx^{(-1)}}x^{2})=\frac{1}{2}\frac{d^{(1)}}{dx^{(1)}}\{\frac{2!}{[2-(-1)]!}x^{[2-(-1)]}\}=$$

$$=\frac{1}{2}\frac{d^{(1)}}{dx^{(1)}}(\frac{2!}{3!}x^{3})=\frac{1}{2}\frac{d}{dx}(\frac{2}{6}x^{3})=\frac{1}{6}\frac{d}{dx}x^{3}=\frac{1}{6}3x^{2}=\frac{1}{2}x^{2}=(\frac{d^{(1)}}{dx^{(1)}}\int \_\, dx)(\frac{1}{2}x^{2})\quad \text{qed.}$$

To solve the objection that $\int xdx=\frac{1}{2}x^{2}+C$ and not just $\int xdx=\frac{1}{2}x^{2}$, we could redefine in this way:

$$\frac{d^{(-1)}}{dx^{(-1)}}x=\frac{1}{2}x^{2}=\int_{0}^{x}\bar{x}d\bar{x}\qquad(1.2)$$

2-Half-Integral of x ($\oint_{1/2}$ x)

$$\oint_{1/2} x=\int_{\frac{1}{2}}x=\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}x^{1}=\frac{1!}{[1-(-\frac{1}{2})]!}x^{1-(-\frac{1}{2})}=\frac{1}{(\frac{3}{2})!}x^{\frac{3}{2}}=\frac{1}{\Gamma(\frac{3}{2}+1)}x^{\frac{3}{2}}=\frac{1}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}}=\frac{1}{\frac{3}{2}\Gamma(\frac{3}{2})}x^{\frac{3}{2}}=$$

$$=\frac{1}{\frac{3}{2}\frac{\sqrt{\pi}}{2}}x^{\frac{3}{2}}=\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}$$

. To have the value for Γ(3/2), see the (A3.2) in the Appendix.

So: $$\oint_{1/2} x = \frac{4}{3\sqrt{\pi}} x^{\frac{3}{2}} \qquad \text{(Half-Imtegral)} \qquad (2.1)$$

As a check, by applying twice the (2.1) to x, we get the classic integral of x, that is (1/2)x^2:

$$\oint_{1/2}\oint_{1/2} x = \int_{\frac{1}{2}}\int_{\frac{1}{2}} x = \int_{1} x = [\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}]x^{1} = \frac{d^{(\frac{-1}{2}-\frac{1}{2})}}{dx^{(\frac{-1}{2}-\frac{1}{2})}}x = \frac{d^{(-1)}}{dx^{(-1)}}x = \frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}[\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}x] =$$

$$= \frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\{\frac{1!}{[1-(-\frac{1}{2})]!}x^{1-(-\frac{1}{2})}\} = \frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\{\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}\} = \frac{4}{3\sqrt{\pi}}\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\{x^{\frac{3}{2}}\} = \frac{4}{3\sqrt{\pi}}\frac{\frac{3}{2}!}{[\frac{3}{2}-(-\frac{1}{2})]!}x^{\frac{3}{2}-(-\frac{1}{2})} =$$

$$= \frac{4}{3\sqrt{\pi}}\frac{\frac{3}{2}\frac{\sqrt{\pi}}{2}}{2!}x^{2} = \frac{1}{2}x^{2} = \int x\,dx$$

On the contrary, if we start from the definition of integral, that is:

$\int x^{k}dx = \frac{1}{k+1}x^{k+1} + C$ and by considering that, out of simplicity, for the time being C is always zero (or you can use the definition (1.2), if you don't tolerate that C=0), we have:

$$\int\int x^{k} = \frac{1}{(k+1)(k+2)}x^{k+2} \quad , \quad \int\int\int x^{k} = \frac{1}{(k+1)(k+2)(k+3)}x^{k+3} \quad \text{and}$$

$$\int\int .....(n-times).......\int x^{k} = \frac{1}{(k+1)(k+2)(k+3).....(k+n)}x^{k+n} = \frac{k!}{(k+n)!}x^{k+n} = \int_{n} x^{k} \qquad (2.2)$$

In fact, by applying, as an example,the (2.2) in order to integrate three times x^4 :

(n=3 and k=4) $\int_{3} x^{4} = \frac{4!}{(4+3)!}x^{4+3} = \frac{1}{210}x^{7}$, that is the same value we get by integrating three times, indeed, one after the other, x^4. Well then, according to the (2.2) with n=1/2 and applied to x (k=1), we get:

$$\int_{\frac{1}{2}} x = \frac{1!}{(1+\frac{1}{2})!}x^{1+\frac{1}{2}} = \frac{1}{(\frac{3}{2})!}x^{\frac{3}{2}} = \frac{1}{\Gamma(\frac{3}{2}+1)}x^{\frac{3}{2}} = \frac{1}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}} = \frac{1}{\frac{3}{2}\Gamma(\frac{3}{2})}x^{\frac{3}{2}} = \frac{1}{\frac{3}{2}\frac{\sqrt{\pi}}{2}}x^{\frac{3}{2}} = \frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}} , \qquad (2.3)$$

that is the same result as that of the (2.1).
As a check for the (2.3), let's apply again the (2.2), still with n=1/2, to the (2.3) itself, so obtaining a double half-integral, that is the classic integral of x, that is (1/2)x^2:

$$\int_{\frac{1}{2}}\int_{\frac{1}{2}} x = \int_{(\frac{1}{2}+\frac{1}{2})} x = \int_{(1)} x = \frac{1}{2}x^{2} = \int_{\frac{1}{2}}[\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}] = \frac{4}{3\sqrt{\pi}}\int_{\frac{1}{2}}[x^{\frac{3}{2}}] = \frac{4}{3\sqrt{\pi}}\frac{(3/2)!}{(3/2+1/2)!}x^{3/2+1/2} =$$

$$= \frac{4}{3\sqrt{\pi}}\frac{(3/2)!}{2!}x^{2} = \frac{4}{3\sqrt{\pi}}\frac{(3\sqrt{\pi}/4)}{2!}x^{2} = \frac{1}{2}x^{2} \qquad !!!!!!!!!!!$$

3-(-1)st derivative of cosx

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ... \quad (3.1)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ... \quad (3.2)$$

We have above proved the following obvious identity (1.1):

$\frac{d^n}{dx^n}x^k = \frac{k!}{(k-n)!}x^{k-n}$. For n=-1, it becomes:

$$\frac{d^{-1}}{dx^{-1}}x^k = \frac{k!}{(k+1)!}x^{k+1} \quad (3.3)$$

Let's apply now the (3.3) to the (3.1):

$$\frac{d^{-1}}{dx^{-1}}\cos x = \frac{d^{-1}}{dx^{-1}}(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ...) = \frac{d^{-1}}{dx^{-1}}x^0 - \frac{1}{2!}\frac{d^{-1}}{dx^{-1}}x^2 + \frac{1}{4!}\frac{d^{-1}}{dx^{-1}}x^4 - \frac{1}{6!}\frac{d^{-1}}{dx^{-1}}x^6 + ... =$$

$$= \frac{0!}{(0+1)!}x^{0+1} - \frac{1}{2!}\frac{2!}{(2+1)!}x^{2+1} + \frac{1}{4!}\frac{4!}{(4+1)!}x^{4+1} - \frac{1}{6!}\frac{6!}{(6+1)!}x^{6+1} + ... =$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ... = \sin x = \int \cos x dx \qquad !!!!!$$

APPENDIX

A1-EULER GAMMA FUNCTION

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \quad (A1.1)$$

Of course $\Gamma(1) = 1$. Then,

$$\Gamma(n+1) = n\Gamma(n); \quad (A1.2)$$

in fact, after an integration by parts:

$\Gamma(z) = e^{-t}\left.\frac{t^z}{z}\right|_0^\infty + \frac{1}{z}\int_0^\infty e^{-t}t^z dt = 0 + \frac{1}{z}\Gamma(z+1)$ and by iterating the $\Gamma(n+1) = n\Gamma(n)$, we get:

$$\Gamma(n+1) = n! \ . \quad (A1.3)$$

A2-GAUSS INTEGRAL

We have the following two integrals (identical):

$$I = \int_0^\infty e^{-\alpha x^2}dx \qquad I = \int_0^\infty e^{-\alpha y^2}dy \quad (A2.1)$$

After multiplying them each other: $I^2 = \int_0^\infty \int_0^\infty e^{-\alpha(x^2+y^2)}dxdy$; now, in polar coordinates: ($x^2 + y^2 = r^2$, $dS = dxdy = rdrd\theta$), we have:

$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-\alpha r^2} rdrd\theta = \frac{\pi}{2}\int_0^\infty e^{-\alpha r^2} rdr = -\frac{\pi}{4\alpha}\left|e^{-\alpha r^2}\right|_0^\infty = \frac{\pi}{4\alpha}$, so:

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \ . \quad (A2.2)$$

A3-PARTICULAR VALUES OF THE EULER GAMMA FUNCTION

The (A1.3) with n=0 yields Γ(1)=0!=1, while with n=1, gives Γ(2)=1!=1.
Moreover, the (A1.1) with z=1/2 becomes:

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$; now, by saying that t=w^2, we have: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \int_0^\infty e^{-w^2} w^{-1} 2w dw = 2\int_0^\infty e^{-w^2} dw \qquad \text{(A3.1)}$$

And according to the (A2.2) with α=1, we have: $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-w^2} dw = \sqrt{\pi}$.

At last, according to the (A1.2): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\frac{\sqrt{\pi}}{2}$

Results: Γ(1)=1, Γ(2)=1, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$. (A3.2)

L'INTEGRALE INTESO COME DERIVATA (-1)esima E IL SEMINTEGRALE

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Abstract: L'Integrale visto come derivata (-1)esima e il Semintegrale.

1-Derivata (-1)esima di x

$$\boxed{\frac{d^{(-1)}}{dx^{(-1)}}x=\frac{1}{2}x^2}=\int xdx$$

Dimostrazione:

$$\frac{d}{dx}x^k=\frac{d^1}{dx^1}x^k=kx^{k-1}\ ,\ \frac{d^2}{dx^2}x^k=k(k-1)x^{k-2}\ ,\ \frac{d^n}{dx^n}x^k=k(k-1)...(k-n+1)x^{k-n}=\frac{k!}{(k-n)!}x^{k-n}\quad(1.1)$$

e per k=1 ed n=-1, si ha: $\frac{d^{(-1)}}{dx^{(-1)}}x^1=\frac{1!}{[1-(-1)]!}x^{1-(-1)}=\frac{1!}{2!}x^2=\frac{1}{2}x^2=\int xdx$.

<u>Controprova:</u>

visto che deve essere: $(\frac{d^{(1)}}{dx^{(1)}}\frac{d^{(-1)}}{dx^{(-1)}})(\frac{1}{2}x^2)=\frac{d^{1-1}}{dx^{1-1}}(\frac{1}{2}x^2)=\frac{1}{2}\frac{d^0}{dx^0}x^2=\frac{1}{2}x^2$, allora:

$$(\frac{d^{(1)}}{dx^{(1)}}\frac{d^{(-1)}}{dx^{(-1)}})(\frac{1}{2}x^2)=\frac{d^{(1)}}{dx^{(1)}}[\frac{d^{(-1)}}{dx^{(-1)}}(\frac{1}{2}x^2)]=\frac{1}{2}\frac{d^{(1)}}{dx^{(1)}}(\frac{d^{(-1)}}{dx^{(-1)}}x^2)=\frac{1}{2}\frac{d^{(1)}}{dx^{(1)}}\{\frac{2!}{[2-(-1)]!}x^{[2-(-1)]}\}=$$

$$=\frac{1}{2}\frac{d^{(1)}}{dx^{(1)}}(\frac{2!}{3!}x^3)=\frac{1}{2}\frac{d}{dx}(\frac{2}{6}x^3)=\frac{1}{6}\frac{d}{dx}x^3=\frac{1}{6}3x^2=\frac{1}{2}x^2=(\frac{d^{(1)}}{dx^{(1)}}\int\_\,dx)(\frac{1}{2}x^2)\quad \text{cvd.}$$

Per rispondere all'obiezione secondo cui $\int xdx=\frac{1}{2}x^2+C$ e non semplicisticamente $\int xdx=\frac{1}{2}x^2$, si potrebbe ridefinire come segue:

$$\frac{d^{(-1)}}{dx^{(-1)}}x=\frac{1}{2}x^2=\int_0^x \bar{x}d\bar{x}\qquad(1.2)$$

2-Semintegrale di x ($\int_{(1/2)}$ **x**)

$$\int_{(1/2)} x=\int_{\frac{1}{2}}x=\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}x^1=\frac{1!}{[1-(-\frac{1}{2})]!}x^{1-(-\frac{1}{2})}=\frac{1}{(\frac{3}{2})!}x^{\frac{3}{2}}=\frac{1}{\Gamma(\frac{3}{2}+1)}x^{\frac{3}{2}}=\frac{1}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}}=\frac{1}{\frac{3}{2}\Gamma(\frac{3}{2})}x^{\frac{3}{2}}=$$

$$=\frac{1}{\frac{3}{2}\frac{\sqrt{\pi}}{2}}x^{\frac{3}{2}}=\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}\ .$$

Per il valore di Γ(3/2), si veda la (A3.2) in Appendice.

Dunque: $$\oint_{1/2} x = \frac{4}{3\sqrt{\pi}} x^{\frac{3}{2}} \qquad \text{(Semintegrale)} \tag{2.1}$$

Come controprova, applicando due volte la (2.1) ad x, si ottiene l'integrale normale di x, ossia $(1/2)x^2$:

$$\oint_{1/2}\oint_{1/2} x = \int_{\frac{1}{2}}\int_{\frac{1}{2}} x = \int_{1} x = [\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}]x^1 = \frac{d^{(\frac{-1}{2}-\frac{1}{2})}}{dx^{(\frac{-1}{2}-\frac{1}{2})}}x = \frac{d^{(-1)}}{dx^{(-1)}}x = \frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}[\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}x] =$$

$$= \frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\{\frac{1!}{[1-(-\frac{1}{2})]!}x^{1-(-\frac{1}{2})}\} = \frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\{\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}\} = \frac{4}{3\sqrt{\pi}}\frac{d^{(\frac{-1}{2})}}{dx^{(\frac{-1}{2})}}\{x^{\frac{3}{2}}\} = \frac{4}{3\sqrt{\pi}}\frac{\frac{3}{2}!}{[\frac{3}{2}-(-\frac{1}{2})]!}x^{\frac{3}{2}-(-\frac{1}{2})} =$$

$$= \frac{4}{3\sqrt{\pi}}\frac{\frac{3}{2}\frac{\sqrt{\pi}}{2}}{2!}x^2 = \frac{1}{2}x^2 = \int x dx$$

Partendo invece dalla definizione di integrale, ossia dalla:

$\int x^k dx = \frac{1}{k+1}x^{k+1} + C$ e considerando ora, per semplicità, che la C sia sempre zero (oppure si usi la definizione (1.2), se non si tollera che C=0), si ha:

$$\int\int x^k = \frac{1}{(k+1)(k+2)}x^{k+2} \quad , \quad \int\int\int x^k = \frac{1}{(k+1)(k+2)(k+3)}x^{k+3} \quad \text{e}$$

$$\int\int .....(n-times).......\int x^k = \frac{1}{(k+1)(k+2)(k+3).....(k+n)}x^{k+n} = \boxed{\frac{k!}{(k+n)!}x^{k+n} = \int_n x^k} \tag{2.2}$$

Infatti, applicando ad esempio la (2.2) per integrare tre volte x^4 :

(n=3 e k=4) $\int_3 x^4 = \frac{4!}{(4+3)!}x^{4+3} = \frac{1}{210}x^7$, ossia lo stesso valore che si ottiene integrando appunto tre volte, una dopo l'altra, x^4. Bene, per la (2.2) con n=1/2 ed applicata ad x (k=1), si ottiene:

$$\int_{\frac{1}{2}} x = \frac{1!}{(1+\frac{1}{2})!}x^{1+\frac{1}{2}} = \frac{1}{(\frac{3}{2})!}x^{\frac{3}{2}} = \frac{1}{\Gamma(\frac{3}{2}+1)}x^{\frac{3}{2}} = \frac{1}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}} = \frac{1}{\frac{3}{2}\Gamma(\frac{3}{2})}x^{\frac{3}{2}} = \frac{1}{\frac{3}{2}\frac{\sqrt{\pi}}{2}}x^{\frac{3}{2}} = \frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}} \ , \tag{2.3}$$

ossia lo stesso risultato della (2.1).

Come controprova per la (2.3), applichiamo ancora la (2.2), sempre con n=1/2, alla (2.3) stessa, ottenendo un doppio semintegrale, ossia l'integrale classico di x, che è $(1/2)x^2$:

$$\int_{\frac{1}{2}}\int_{\frac{1}{2}} x = \int_{(\frac{1}{2}+\frac{1}{2})} x = \int_{(1)} x = \frac{1}{2}x^2 = \int_{\frac{1}{2}}[\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}] = \frac{4}{3\sqrt{\pi}}\int_{\frac{1}{2}}[x^{\frac{3}{2}}] = \frac{4}{3\sqrt{\pi}}\frac{(3/2)!}{(3/2+1/2)!}x^{3/2+1/2} =$$

$$= \frac{4}{3\sqrt{\pi}}\frac{(3/2)!}{2!}x^2 = \frac{4}{3\sqrt{\pi}}\frac{(3\sqrt{\pi}/4)}{2!}x^2 = \frac{1}{2}x^2 \qquad !!!!!!!!!!!$$

3-Derivata (-1)esima di cosx

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ... \quad (3.1)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ... \quad (3.2)$$

Più sopra abbiamo dimostrato la seguente e, peraltro, nota identità (1.1):

$\frac{d^n}{dx^n}x^k = \frac{k!}{(k-n)!}x^{k-n}$. Per n=-1, essa diventa:

$$\frac{d^{-1}}{dx^{-1}}x^k = \frac{k!}{(k+1)!}x^{k+1} \quad (3.3)$$

Applichiamo ora la (3.3) alla (3.1):

$$\frac{d^{-1}}{dx^{-1}}\cos x = \frac{d^{-1}}{dx^{-1}}(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ...) = \frac{d^{-1}}{dx^{-1}}x^0 - \frac{1}{2!}\frac{d^{-1}}{dx^{-1}}x^2 + \frac{1}{4!}\frac{d^{-1}}{dx^{-1}}x^4 - \frac{1}{6!}\frac{d^{-1}}{dx^{-1}}x^6 + ... =$$

$$= \frac{0!}{(0+1)!}x^{0+1} - \frac{1}{2!}\frac{2!}{(2+1)!}x^{2+1} + \frac{1}{4!}\frac{4!}{(4+1)!}x^{4+1} - \frac{1}{6!}\frac{6!}{(6+1)!}x^{6+1} + ... =$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ... = \sin x = \int \cos x dx \qquad !!!!!$$

APPENDICE

A1-FUNZIONE GAMMA DI EULERO

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \quad (A1.1)$$

Naturalmente, $\Gamma(1) = 1$. Poi,

$$\Gamma(n+1) = n\Gamma(n); \quad (A1.2)$$

infatti, dopo una integrazione per parti:

$\Gamma(z) = e^{-t}\frac{t^z}{z}\Big|_0^\infty + \frac{1}{z}\int_0^\infty e^{-t}t^z dt = 0 + \frac{1}{z}\Gamma(z+1)$ ed iterando la $\Gamma(n+1) = n\Gamma(n)$, otteniamo:

$$\Gamma(n+1) = n! \quad . \quad (A1.3)$$

A2-INTEGRALE DI GAUSS

Consideriamo i due integrali (uguali tra loro):

$$I = \int_0^\infty e^{-\alpha x^2}dx \qquad I = \int_0^\infty e^{-\alpha y^2}dy \quad (A2.1)$$

Moltiplicandoli tra loro: $I^2 = \int_0^\infty \int_0^\infty e^{-\alpha(x^2+y^2)}dxdy$ e, in coordinate polari:

($x^2 + y^2 = r^2$, $dS = dxdy = rdrd\theta$), si ha:

$I^2 = \int_0^{\pi/2}\int_0^\infty e^{-\alpha r^2} rdrd\theta = \frac{\pi}{2}\int_0^\infty e^{-\alpha r^2} rdr = -\frac{\pi}{4\alpha}\left|e^{-\alpha r^2}\right|_0^\infty = \frac{\pi}{4\alpha}$, perciò:

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad . \quad (A2.2)$$

A3-VALORI PARTICOLARI DELLA FUNZIONE GAMMA DI EULERO

La (A1.3) con n=0 fornisce Γ(1)=0!=1, mentre con n=1, fornisce Γ(2)=1!=1.
Inoltre, la (A1.1) con z=1/2 diventa:

$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$; adesso, ponendo t=w^2, si ha: (→ dt/dw=2w)

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = \int_0^\infty e^{-w^2} w^{-1} 2w dw = 2\int_0^\infty e^{-w^2} dw \qquad \text{(A3.1)}$$

e per la (A2.2) con α=1, si ha: $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-w^2} dw = \sqrt{\pi}$.

Infine, per la (A1.2): Γ(3/2)=Γ(1/2 + 1)=1/2 Γ(1/2)=(1/2)!= $\frac{\sqrt{\pi}}{2}$

Per riassumere: Γ(1)=1, Γ(2)=1, $\Gamma(1/2) = \sqrt{\pi}$, Γ(3/2) = $\frac{\sqrt{\pi}}{2}$. (A3.2)